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A Computational Method for Solving Optimal Control Problems and Their Applications

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Abstract. In order to obtain a solution to an optimal control problem, a numerical technique based on state-control parameterization method is presented. This method can be facilitated by the computation of performance index and state equation via approximating the control and state variable as a function of time. Several numerical examples are presented to confirm the analytical findings and illustrate the efficiency of the proposed method.

Keywords. Optimal control, State-control parameterization, Basis polynomials.

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1 Introduction

To identify the best solution to optimal control problems, many numerical methods have been introduced. These methods are classified as direct methods by which many disadvantages of indirect methods can be eliminated. Direct methods are based on the transformation of the original optimal control problem into a nonlinear programming problem (NLP) by discretizing or parameterizing the state and/or control variables and then solving the resulting NLP problem Also, they can be classified into three different approaches. The first approach is based on state parameterization only. The second approach is control parameterization and its idea is to approximate the control variables and obtain the state variables by integrating the state equations. The third approach is based on state and control variable parameterization.

Vlassenbrock and Van Dooren in [11]-[12], used Chebyshev polynomials to parameterize the state and control variables to solve the constrained and unconstrained nonlinear control problems. Jaddu in [3] proposed a numerical method that is based on parameterizing the system variables via Chebyshev polynomials to solve the nonlinear quadratic optimal control problems. In [1], a state-control parameterization method based on Chebyshev wavelets for solving the optimal control of linear time-varying systems was used. Also, Kafash et. al. [6]-[5] proposed a method that is based on state parameterization which solved the optimal control problems using iteration technique. An efficient recursive shooting method for the optimal control of time-varying systems with time delay in state variables was introduced in [4]. Mirhosseini et al. in [7] proposed an iterative method for the control of linear time delay systems. In [8], the Chebyshev wavelet method was used for solving various optimal control problems.

In this paper, a state-control parameterization method based on using basis polynomials to approximate the state and control variables is presented. Although the number of unknowns are increased by using this parameterization method, the approximated optimal state and control variables can be obtained at the same time. Also this parameterization does not need to integrate the system state equations as in control parameterization. In comparison with other works, the proposed method does not attempt to use operational matrix to transform the optimal control problem in to optimization problem. The focus of this paper is on introducing an applicable numerical method to find an optimal solution. For this goal, first, we present a brief description of optimal control problem and state-control parameterization method for solving this problem. Finally, several numerical examples are presented to illustrate efficiency of this method.

2 Optimal Control Problems

Optimal control problem can be considered as a generalization of the classical calculus of variation. The essential parts of an optimal control problem are, a mathematical system to be controlled, a desired output of the system, a set of admissible inputs and a performance index or a cost functional that measures the effectiveness of a given control operation. There are three equivalent optimization problems, which are called Mayer, Lagrange and Bolza [2]. The performance index in them is of the form, $J(x_0,u) = \phi(t_1, x(t_1))$, $J(x_0,u) = \int_{t_0}^{t_1} L(t, x(t), u(t))dt$ and $J(x_0,u) = \phi(t_1, x(t_1) + \int_{t_0}^{t_1} L(t, x(t), u(t))dt$ respectively. where, $L(t, x(t), u(t))$ is the running cost and $\phi(t, x)$ is the terminal cost. Generally, in order to get the simplest mathematical description that predicts the response of the systems to all inputs, a system described by ordinary differential equation can be considered. Thus, if we let $x(t) \in R^l$ as the state vector of the system and $u(t) \in R^q$ as the control vector, then we can write the state equation in the form of $\dot{x} = f(t, x(t), u(t))$ on the time interval $[t_0, t_1]$ with initial condition $x(t_0) = x_0$. Usually, two different ways of choosing the control variable exist. First, choosing $u(t)$ as a function of time t , namely open loop. Second, choosing $u(t)$ as a function of state variable $x(t)$, namely closed loop or feedback. Finally, in general form, the optimal control problem can be written as:

$$min \qquad J(x,u) = \phi(t_1, x(t_1)) + \int_{t_0}^{t_1} L(t, x(t), u(t))dt, \tag{1}$$

subject to:

$$\dot{x}(t) = f\left(t, x(t), u(t)\right), \tag{2}$$

$$x(t_0) = x_0. \tag{3}$$

3 The Proposed State-Control Parameterization

In this section, we describe our proposed method to determine the solution to optimal control problems. The method is based on applying the state-control parameterization technique and uses basis polynomials. Let $Q \subset C^1([t_0, t_1])$ be a set of all functions satisfying initial condition (3), and then consider Q_n as a subset of Q consisting of basis polynomials of degree at most n $(Q_0 = 1, Q_1(t) = t, Q_2(t) = t^2, ...)$ and consider the minimization of J on Q_n with $\{a_k\}_{k=0}^n$ and $\{b_k\}_{k=0}^n$ as unknowns. In this manner, the state and control variables can be considered as follows:

$$\hat{x}_n(t) = \sum_{k=0}^{n} a_k t^k, \tag{4}$$

$$\hat{u}_n(t) = \sum_{k=0}^{n} b_k t^k. \tag{5}$$

By substituting (4) and (5) in to (1) and then integrating over the time interval $[t_0, t_1]$, the performance index can be computed easily.

$$\hat{J} = \phi(t_1, \sum_{k=0}^{n} a_k t_1^k) + \int_{t_0}^{t_1} L(t, \sum_{k=0}^{n} a_k t^k, \sum_{k=0}^{n} b_k t^k) dt, \tag{6}$$

Also, we obtain the following system of equations by conditions (2) and (3):

$$k \sum_{k=1}^{n} a_k t^{k-1} = f(t, \sum_{k=0}^{n} a_k t^k, \sum_{k=0}^{n} b_k t^k), \tag{7}$$

$$\hat{x}(t_0) = \sum_{k=0}^{n} a_k t^k |_{t=t_0} = x_0. \tag{8}$$

This process yields the solution to problem (1)-(3) via the following problem:

$$\min_{(a_0,a_1,...,a_n,b_0,b_1,...,b_n)\in R^{2n+2}} \hat{J}(a_0, a_1, ..., a_n, b_0, b_1, ..., b_n) \tag{9}$$

subject to:

$$\begin{cases} k \sum_{k=1}^{n} a_k t^{k-1} = f(t, \sum_{k=0}^{n} a_k t^k, \sum_{k=0}^{n} b_k t^k), \\ \hat{x}(t_0) = \sum_{k=0}^{n} a_k t^k |_{t=t_0} = x_0. \end{cases} \tag{10}$$

If we let $\alpha = (a_0, a_1, ..., a_n)$ and $\beta = (b_0, b_1, ..., b_n)$, then problem (9) subject to constraints (10) can be written as the following optimization problem:

$$\min_{(\alpha,\beta)\in R^{2n+2}} \hat{J}(\alpha, \beta) \tag{11}$$

subject to:

$$P[\alpha, \beta] = h, \tag{12}$$

where P is the coefficient matrix of constraints (10).

In fact, problem (1)-(3) are converted to optimization problems (11)-(12) with $2n+2$ unknowns $(\alpha, \beta) = (a_0, a_1, ..., a_n, b_0, b_1, ..., b_n)$. Solving this problem is easier than the original one by using well-developed algorithms. The following algorithm shows the process of transforming optimal control problems (1)-(3) into optimization problems (11)-(12).

Algorithm

Input: Optimal control problem (1)-(3).

Output: The approximate optimal trajectory, approximate optimal control and approximate performance index $\hat{J}$.

Step 0: Choose $\epsilon > 0$ and let $n = 2$.

Step 1: Approximate the state and control, variables by the n^{th} basis polynomials from equations (4)-(5).

Step 2: Find an expression of $\hat{J}_n$ from equation (6).

Step 3: Determine the set of equality constraints due to conditions (7)-(8) and find matrix P.

Step 4: Determine optimal parameters (α^*, β^*) by solving optimization problem (11)-(12) and substitute these parameters into equations (4)-(6) to find the approximate optimal trajectory, approximate optimal control and approximate performance index $\hat{J}_n$.

Step 5: Let $n + 1 \to n$ and go to step 1.

Step 6: If $|J_{n+1} - J_n| \leq \epsilon$ then stop otherwise, return to Step 5.

4 Convergence Analysis

Theorem 1. Let $f \in (C[a, b], R)$. Then there is a sequence of polynomials $P_n(x)$ that converges uniformly to $f(x)$ on $[a, b]$.

Proof. See [9]. □

Lemma 1. If $\gamma_n = \inf_{Q_n} J$; $\quad (n = 1, 2, 3, ...)$ then $\lim_{n\longrightarrow\infty}(\gamma_n) = \gamma$ where $\gamma = \inf_Q J$.

Proof. If we define:

$$\gamma_n = \min_{(\alpha_n,\beta_n)\in\mathbb{R}^{2n+2}} J(\alpha_n, \beta_n),$$

then

$$\gamma_n = J(\alpha_n^*, \beta_n^*),$$

where

$$(\alpha_n^*, \beta_n^*) \in argmin\{J(\alpha_n, \beta_n) : (\alpha_n, \beta_n) \in \mathbb{R}^{2n+2}\}.$$

Now, let $(x_n^*(t), u_n^*(t)) \in argmin\{J(x(t), u(t)) : (x(t), u(t)) \in Q_n\}$, then

$$J(x_n^*(t), u_n^*(t)) = \min_{(x(t),u(t))\in Q_n} J(x(t), u(t)),$$

in which Q_n is a class of basis polynomials in t of degree n, so

$$\gamma_n = J(x_n^*(t), u_n^*(t)).$$

Furthermore, according to $Q_n \subset Q_{n+1}$, we have:

$$\min_{(x(t),u(t))\in Q_{n+1}} J(x(t), u(t)) \leq \min_{(x(t),u(t))\in Q_n} J(x(t), u(t)).$$

Thus, we will have $\gamma_{n+1} \leq \gamma_n$, which means γ_n is a non-increasing sequence. Also, this sequence is upper bounded, and therefore is convergent. Now, the proof is complete, that is:

$$\lim_{n\longrightarrow\infty} (\gamma_n) = \min_{(x(t),u(t))\in Q} J(x(t), u(t)). \qquad \square$$

5 Numerical Examples

In this section for illustrating the efficiency of our proposed method, four optimal control examples are considered. Three first examples have analytical solutions and thus are suitable for validating the proposed method by comparing the results of exact solutions. Also, because there is no analytical solution for fourth example (Van der Pol problem), the approximated solutions of states and control variables are compared by state parameterization method.

Example 1. Our goal is to find the optimal control which minimizes the following optimal control problem:

$$min \quad J = \frac{1}{2}\int_0^1 (x(t)^2 + u(t)^2)dt$$

subject to:

$$\begin{cases} \dot{x}(t) = -x(t) + u(t), \\ x(0) = 1. \end{cases}$$

The analytical solution of this example is [10]:

$$\begin{cases} x(t) = \cosh(\sqrt{2}t) + \delta\sinh(\sqrt{2}t), \\ u(t) = (1 + \sqrt{2}\delta)\cosh(\sqrt{2}t) + (\sqrt{2} + \delta)\sinh(\sqrt{2}t), \end{cases}$$

where,

$$\delta = -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})}.$$

The exact solution for the performance index is 0.1929092981.

Table 1: Comparison between exact and approximate solutions' performance indexes

n	J	Error
2	0.1942959002	$1.3e^{-3}$
3	0.1929316056	$2.2e^{-5}$
4	0.1929094450	$1.4e^{-7}$
5	0.1929092990	$9.0e^{-10}$

We use our proposed method for solving this problem and present the obtained results in Table 1.
As seen from Table 1, when we increase n, we can obtain the results of performance index J more close to the exact solution. The simulation curves of $x(t)$ and $u(t)$ in comparison by their exact solutions and the error functions $|x^*(t) - x(t)|$ and $|u^*(t) - u(t)|$ are shown in Figure 1.

Example 2. Our goal is to find the control which minimizes the following optimal control problem:

$$min \quad J = \int_0^1 (\frac{5}{8}x(t)^2 + \frac{1}{2}x(t)u(t) + \frac{1}{2}u(t)^2)dt,$$

subject to:

$$\begin{cases} \dot{x}(t) = \frac{1}{2}x(t) + u(t), \\ x(0) = 1. \end{cases}$$

The analytical solution in this example is:

$$\begin{cases} x(t) = \frac{\cosh(1-t)}{\cosh(1)}, \\ u(t) = -\frac{(\tanh(1-t)+0.5)\cosh(1-t)}{\cosh 1}, \end{cases}$$

where the exact solution for the performance index is $J = 0.3807970779$. The proposed method is used for solving this problem and obtained results for performance index J are shown in Table 2. Also, in Figure 2, the obtained solutions and the analytical solutions of state and control variables and their errors ($|x^*(t)-x(t)|$ and $|u^*(t)-u(t)|$) are plotted.

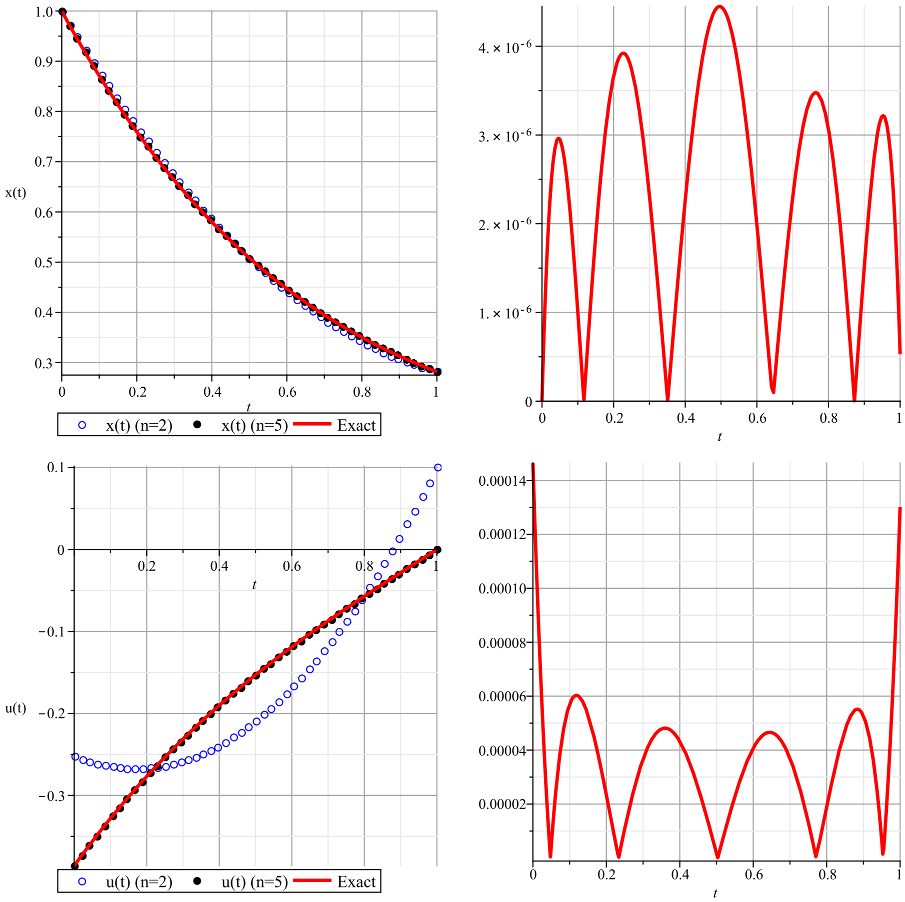

Figure 1: Plots of the numerical solutions compared by exact solutions and the absolute errors for Example1.

Table 2: Comparison between exact and approximate solutions' performance indexes

n	J	Error
2	0.3808837656	$8.6e^{-5}$
3	0.3807998336	$2.7e^{-6}$
4	0.3807970803	$2.3e^{-9}$
5	0.3807970780	$1.0e^{-10}$

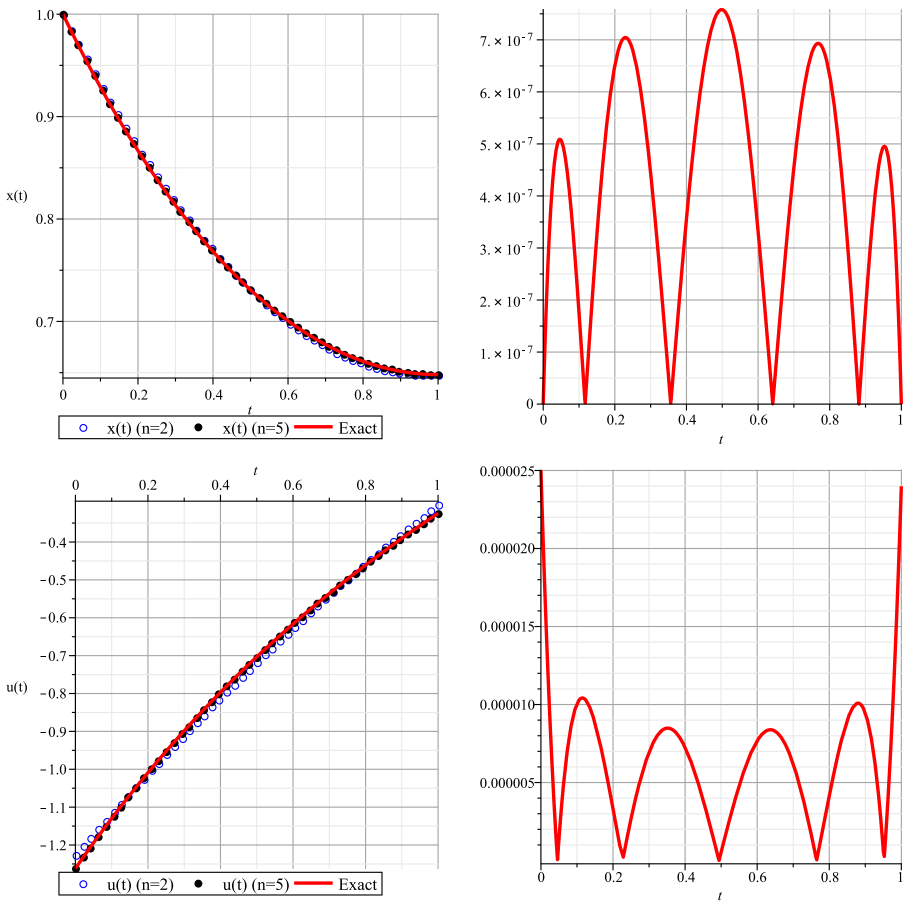

Figure 2: Plots of the numerical solutions compared by the exact solutions and the absolute errors for Example 2.

Example 3. Consider the optimal control of a linear oscillator:

$$min \quad J = \frac{1}{2}\int_{-2}^{0} u(t)^2 dt,$$

subject to:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -x(t) + u(t), \\ x(0) = 0, \\ y(0) = 0, \\ x(-2) = 0.5, \\ y(-2) = -0.5. \end{cases}$$

The analytical solutions of this example are:

$$\begin{cases} x(t) = \frac{1}{2}(At\sin t + B(\sin - t\cos t)), \\ y(t) = \frac{1}{2}(A(t\sin t + t\cos t) + Bt\sin t), \\ u(t) = A\cos t + B\sin t, \end{cases}$$

where

$$A = \frac{2(\sin 2 + 0.5(2\cos 2 - \sin 2))}{4 - \sin^2 2},$$

and

$$B = \frac{2(-\sin 2 + 0.5(2\cos 2 + \sin 2))}{4 - \sin^2 2}.$$

Also, the exact solution for the performance index is:

$$J = \frac{1}{8}(4(A^2 + B^2) + (A^2 - B^2)\sin 4 - 4AB\sin^2 2) = 0.184858542.$$

In Table 3 The optimal performance index J obtained by the proposed method is shown and the obtained solutions and the analytical solutions of state and control variables and their errors ($|x^*(t) - x(t)|$, $|y^*(t) - y(t)|$ and $|u^*(t) - u(t)|$) are plotted in Figure 3.

Table 3: Comparison between exact and approximate solutions' performance indexes

n	J	Error
4	0.1849168910	$5.8e^{-5}$
5	0.18487352976	$1.4e^{-5}$
6	0.1848585741	$3.2e^{-8}$
7	0.18485854440	$2.4e^{-9}$

Example 4. Consider the Van der Pol oscillator problem:

$$min \quad J = \frac{1}{2}\int_0^5 \left(x(t)^2 + y(t)^2 + u(t)^2\right) dt,$$

subject to:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -x(t) - (x(t)^2 - 1)y(t) + u(t), \\ x(0) = 1, \\ y(0) = 0, \\ x(5) = -1, \\ y(5) = 0. \end{cases}$$

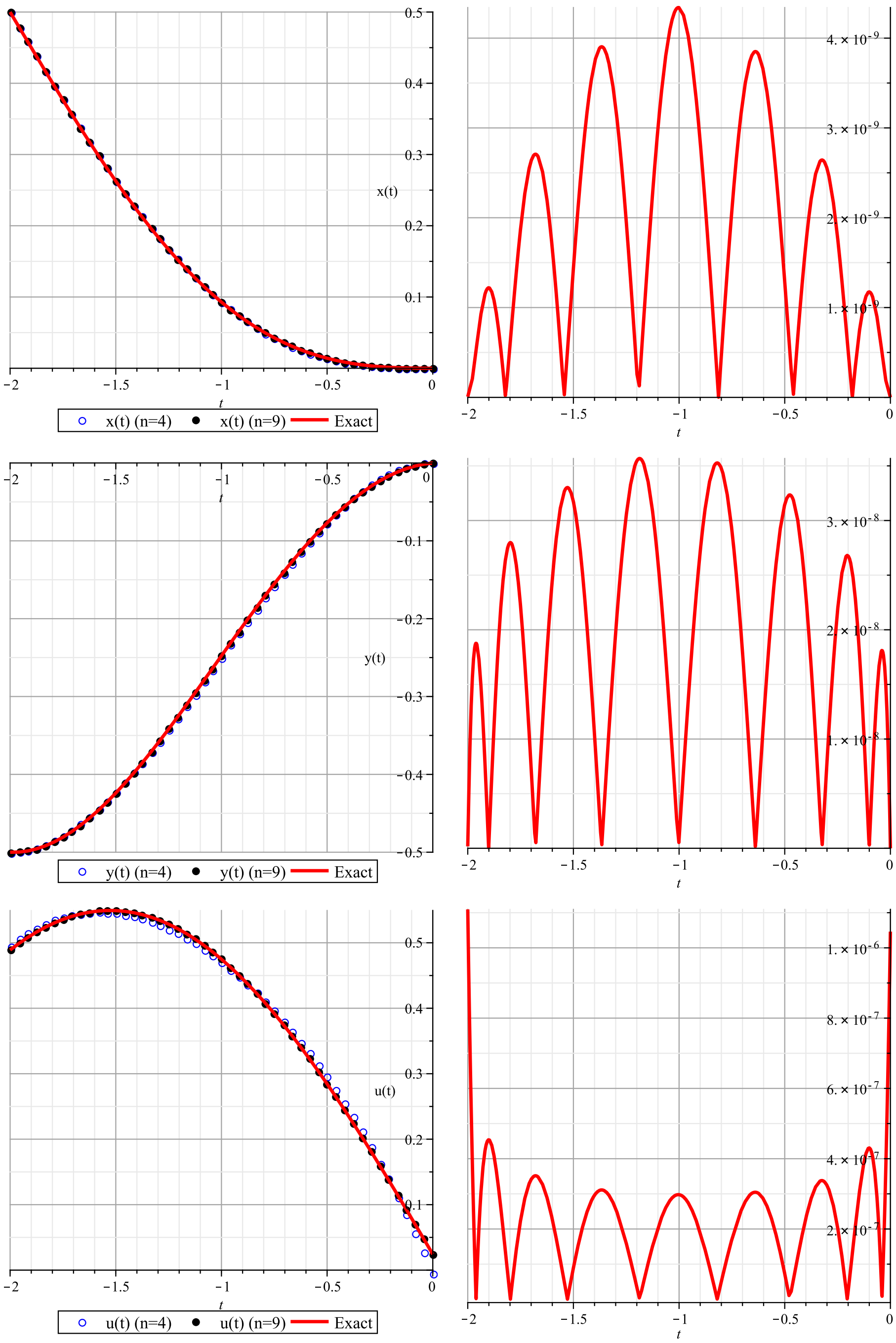

Figure 3: Plots of the numerical solutions compared by the exact solutions and the absolute errors for Example 3.

The optimal performance index J obtained by the proposed method is 2.140571429 for $n = 4$ which is in consonance with the solutions obtained by state parameterization (if we parameterize only state variables). Also, the approximate solution to the

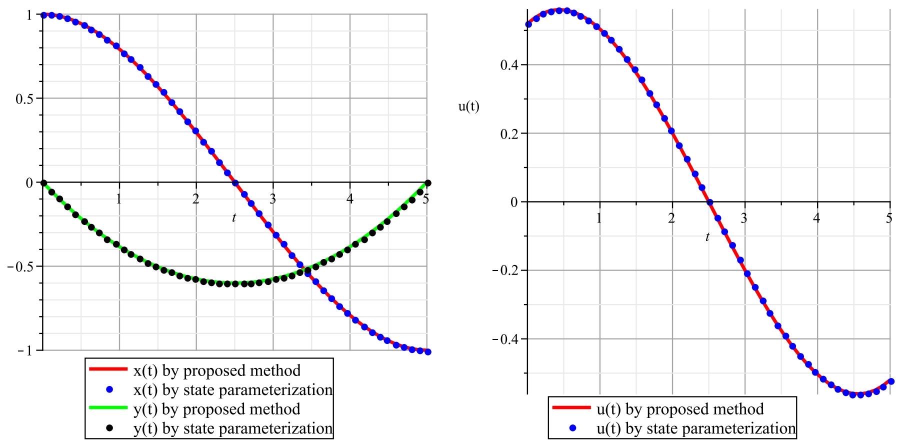

Figure 4: Plots of the numerical solutions compared by state parameterization for Example 4.

performance index given in [5] by three iterations is $J = 2.143904324$. In Figure 4, the obtained solutions by our proposed method compared by state parameterization method are plotted.

6 Conclusion

In this paper, an efficient numerical method based on state-control parameterization was presented. In comparison with other methods that use state-control parameterization, our method is more applicable and does not require complicated computations. Several illustrative examples were studied in details to show the efficiency and reliability of the presented method.

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یک روش محاسباتی برای حل مسایل کنترل بهینه و کاربردهای آن

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چکیده

به منظور به دست آوردن جوابی از یک مسئله کنترل بهینه، یک روش عددی بر پایه پارامتری سازی حالت-کنترل ارائه شده است. روش پیشنهادی باعث ساده‌سازی حل مساله کنترل بهینه با تقریب شاخص عملکرد، متغیرهای حالت و کنترل بر حسب تابعی از زمان می‌شود. هم‌چنین با حل چند مثال عددی، کارایی روش پیشنهادی تایید شده است.

کلمات کلیدی

کنترل بهینه، پارامتری سازی حالت-کنترل، چند جمله ای های پایه ای.