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Haar Matrix Equations for Solving Time-Variant Linear-Quadratic Optimal Control Problems

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Abstract. In this paper, Haar wavelets are performed for solving continuous time-variant linear-quadratic optimal control problems. Firstly, using necessary conditions for optimality, the problem is changed into a two-boundary value problem (TBVP). Next, Haar wavelets are applied for converting the TBVP, as a system of differential equations, in to a system of matrix algebraic equations, as Haar matrix equations using Kronecker product. Then the error analysis of the proposed method is presented. Some numerical examples are given to demonstrate the efficiency of the method. The solutions converge as the number of approximate terms increase.

Keywords. Time-variant linear-quadratic optimal control problems, Matrix algebraic equation, Haar wavelet.

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1 Introduction

Optimal control problems (OCP) are dynamic optimization problems with many applications in industrial processes such as airplane, robotic arm, bio-process system, biomedicine, electric power systems, plasma physics, etc., [1].

We consider a continuous time-variant linear-quadratic OCP, called TVLQ, in which a control function, u, is exerted over the planning horizon $[t_0, t_f]$. The particular problem considered is that of finding the control input $u(.) \in \mathbb{R}^r$ that minimize the cost functional with quadratic Bolza form with linear time-variant state equation, as following

$$\min \quad J = \frac{1}{2} x^T(t_f) \bar{F} x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (x^T(t) Q x(t) + u^T(t) R u(t)) dt \tag{1}$$

$s.t.$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad t \in [t_0, t_f] \tag{2}$$

$$x(t_0) = x_0, \tag{3}$$

where $A(t)$ and $B(t)$ are time-variant matrices with dimension $n \times n$ and $n \times r$, respectively, R is an $r-$square positive symmetric matrix, $\bar{F}$ and Q are $n-$square semi-positive symmetric matrices, $x(.) \in \mathbb{R}^n$ denotes the state vector for the system, $x_0 \in \mathbb{R}^n$ is the initial state, t_0 and t_f are constant initial and final times, respectively.

The classical numerical methods for solving OCPs are divided into two groups, direct [2] and indirect [3] methods. In indirect approaches, the problem, through the use of the Pontryagin's minimum principle (PMP), is converted into a TBVP, that can be solved by numerical methods such as shooting method [2]. These methods require the good initial guesses that lie within the domain of convergence. In direct methods, using control or/and state parametrization(s), convert the continuous problem to discrete problem. The quality of solution, in these methods depends on discretization resolution. However, the adaptive strategies [4, 5] can overcome these defects. Using orthogonal set of functions, as a basis for $L_2(\mathbb{R})$, is common approach in direct methods. In these methods, unknown functions in the problem are approximated as series of orthogonal functions with unknown coefficients. So, the dynamic equations in OCPs are converted to algebraic equations. There are three classes of the orthogonal functions: piecewise continuous functions such as Walsh [6], Block pulse [7], Haar wavelet [8] and etc. , the orthogonal polynomials such as Legendre [9], Chebyshev [10], Lagrange [11], sine and cosine functions as Fourier series [12].

Haar wavelets are applied to various engineering problems as [13, 14]. Also, they are used for solving a class of OCPs. Hsiao and Wang [15] applied Haar wavelets for solving optimal control time-variant systems. Dai and Cochran [16] applied wavelet collocation method for solving OCPs.

In this paper, we further extend the previous work on Haar wavelets for solving TVLQ, which was published by Hsiao and Wang [15]. At this point, using an indirect approach, the problem is converted to a TBVP, as a system of differential equations with boundary conditions. Next, using Haar wavelets, the unknown signals of the problem are approximated by Haar series, with Haar matrix coefficients. Moreover, the boundary conditions are approximated based on Haar coefficients. Finally, by replacing these series into TBVP, as differential equations system, the problem is converted to system of algebraic equations, which can be solved using Kronecker product of matrices.

The paper is organized as follows: in section 2, the Haar wavelets, required by the proposed approach, are described. In section 3, the proposed approach is presented. In section 4, the error analysis of the proposed algorithm is presented. In section 5, we provide some numerical examples. We conclude in section 6.

2 Haar Wavelets

In this section we review the basic properties of Haar wavelets [17], HWs, as the simplest class of Daubechies family of wavelets, with compact support. HWs are square waves with magnitude of ± 1 in some intervals and zeros elsewhere. The scale function for these wavelets is $h_0(t) = 1,\ 0 \leq t < 1$, which is applied for construct the mother wavelet as $h_1(t) = h_0(2t) - h_0(2t-1),\ 0 \leq t < 1$. The other HWs are made by the dilations and translations of $h_1(t)$ as following

$$h_n(t) = h_1(2^j t - k), \quad j \geq 0,\ k = 0, \ldots,\ 2^j - 1,\ n = 2^j + k \tag{4}$$

where j and k are the dilation and translation parameters. These functions construct a sequence of orthonormal functions. Each twice integrable function $y(t),\ 0 \leq t < 1$, in $L_2(0,1)$, can be expanded by the Haar series as following [15]

$$y(t) = \sum_{i=0}^{\infty} \alpha_i h_i(t), \quad 0 \leq t < 1 \tag{5}$$

where $\alpha_i = 2^j \int_0^1 y(t) h_i(t) dt,\ i \in \mathbb{N} \cup \{0\}$ are the Haar coefficients and are obtained by minimizing the following integral square error [15]

$$\varepsilon = \int_0^1 (y(t) - \sum_{i=0}^{m-1} \alpha_i h_i(t))^2 dt, \quad m = 2^j,\ j \in \mathbb{N} \cup \{0\} \tag{6}$$

For a piecewise constant approximation, eqn (6) can be considered as a finite series as following

$$y(t) \simeq y_{(m)}(t) = \sum_{i=0}^{m-1} \alpha_i h_i(t) = \alpha_{(m)}^T h_{(m)}(t), \quad 0 \le t < 1 \tag{7}$$

$$h_{(m)}(t) = [h_0(t), h_1(t), \ldots, h_{m-1}(t)]^T, \quad \alpha_{(m)} = [\alpha_0, \alpha_1, \ldots, \alpha_{m-1}]^T \tag{8}$$

where $h_{(m)}$ is the Haar function vector and $\alpha_{(m)}$ is the Haar coefficients vector. m is called the level of wavelet and equals to 2^j, j is the level resolution. Here, we perform (7) for approximating the derivations of the unknown signals in OCPs.

Theorem 1. Let $y(t) \in L_2[0,1)$. The square of the error norm for the Haar approximation, eqn (7), is given by

$$\|y(t) - y_m(t)\|^2 = \frac{K^3}{3}\frac{1}{2m^2}, \quad t \in (0,1) \tag{9}$$

where K is the upper bound of the first derivation for $y(t)$, that is $|y'(t)| \le K$, for all $t \in [0,1)$.

Proof. See [18]. □

From above theorem, it is obvious that $\|y(t) - y_m(t)\| = O(\frac{1}{m})$. Therefore, by increase the level of resolution, the HWs approximation will be more precisely.

The approximation for the integration of the Haar function vector is necessary for solving the differential or integral equations using HWs, which is done by following

$$\int_0^t h_{(m)}(\tau)d\tau \simeq P_{(m\times m)} h_{(m)}(t), \quad 0 \le t < 1 \tag{10}$$

where $P_{(m\times m)}$ is a $m-$square matrix and called Haar operational matrix. This matrix can be obtained by the following recursive equation [17]

$$P_{(m\times m)} = \frac{1}{2m}\begin{bmatrix} 2mP_{(m/2\times m/2)} & -H_{(m/2\times m/2)} \\ H^{-1}_{(m/2\times m/2)} & O_{(m/2\times m/2)} \end{bmatrix}, \qquad P_{(1\times 1)} = \frac{1}{2} \tag{11}$$

where, $O_{(m/2\times m/2)}$ is a $\frac{m}{2}-$square zero matrix and $H_{(m\times m)}$ is defined as a $m-$square Haar matrix as following

$$H_{(m\times m)} = [h_{(m)}(\frac{1}{2m}), h_{(m)}(\frac{3}{2m}), \ldots, h_{(m)}(\frac{2m-1}{2m})]^T \tag{12}$$

For $m = 2$, the Haar operational matrix in eqn (11), $P_{(2\times 2)}$, and Haar matrix in eqn (12), $H_{(2\times 2)}$, is as following:

$$P_{(2\times 2)} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & 0 \end{bmatrix}, \qquad H_{(2\times 2)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The operational matrix is applied for approximating the unknown signals and boundary conditions in next section. The square matrix, $M_{(m\times m)}$, can be defined as

$$M_{(m\times m)}(t) = h_{(m)}(t)h^T_{(m)}(t), \quad M_{(1\times 1)} = h_{(0)}(t), \qquad 0 \le t < 1 \tag{13}$$

which can be satisfied in following

$$M_{(m\times m)}(t)\alpha_{(m)} = \Omega_{(m\times m)}h_{(m)}(t) \qquad 0 \le t < 1 \tag{14}$$

where $\alpha_{(m)}$ is defined in (8) and the coefficient matrix $\Omega_{(m\times m)}$ is a $m-$square matrix which can be calculated recursively as following

$$\Omega_{(m\times m)} = \begin{bmatrix} \Omega_{(\frac{m}{2}\times\frac{m}{2})} & H_{(m/2\times m/2)}diag[\alpha_b] \\ diag[\alpha_{(b)}]H^{-1}_{(m/2\times m/2)} & diag[\alpha^T_{(a)}H_{(m/2\times m/2)}] \end{bmatrix}, \quad \Omega_{(1\times 1)} = c_0 \tag{15}$$

where

$$\alpha_{(a)} = \alpha_{(m/2)} = [\alpha_0, \alpha_1, \ldots, \alpha_{\frac{m}{2}-1}]^T, \quad \alpha_{(b)} = [\alpha_{m/2}, \alpha_{m/2+1}, \ldots, \alpha_{m-1}]^T \tag{16}$$

In the proposed method, we apply a linear operator, named *vec*, on matrices. Let $A = [a_1, a_2, \ldots, a_m] \in \mathbb{R}^{n\times m}$ be a matrix. Then the linear operator *vec* on A can be defined as $vec(A) = [a^T_1, a^T_2, \ldots, a^T_m]^T$. It can be show that

$$vec(AXB) = (B^T \otimes A)vec(X) \tag{17}$$

where $\otimes$ is the Kronecker product of matrices.

3 The Proposed Approach

In this section, we present the proposed method for solving TVLQ. Without loss the generality, we can assume that $[t_0, t_f] = [0, 1]$ and $r = 1$. Firstly, the problem is converted into a TBVP, using necessary optimality conditions, PMP principle. Next, the new problem is converted into a system of the matrix algebraic equations, using HWs.

3.1 Necessary Optimality Conditions

For the TVLQ problem, eqns (1)-(3), the necessary optimality conditions, which can be achieved by variational approach, are

$$\frac{\partial H}{\partial u} = Ru(t) + B^T(t)\lambda(t) = 0, \tag{18}$$

$$\dot{x} = \frac{\partial H}{\partial \lambda} = A(t)x(t) + B(t)u(t), \tag{19}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -Qx(t) - \lambda^T(t)A(t), \tag{20}$$

$$\lambda(t_f) = \bar{F}x(t_f), \quad x(t_0) = x_0. \tag{21}$$

where, the scaler function H, called Hamiltonian, is defined as following

$$H(x, u, \lambda) = \frac{1}{2}(x^T(t)Qx(t) + u^T(t)Ru(t)) + \lambda^T(t)(A(t)x(t) + B(t)u(t)) \tag{22}$$

where $\lambda(t)$ is an $n \times 1$ vector as the Lagrange multiplier also known as the costate or adjoint variable. From (18), the control signals can be represented by the costate vector as $u(t) = -R^{-1}B^T(t)\lambda(t)$. By replacing this equation in eqn (19), the following TBVP can be achieved as following

$$\dot{x} = A(t)x(t) + B(t)u(t) \tag{23}$$

$$= A(t)x(t) - B(t)R^{-1}B(t)^T\lambda(t)$$

$$= A(t)x(t) - \bar{B}(t)\lambda(t), \tag{24}$$

$$\dot{\lambda} = -A^T(t)\lambda(t) - Qx(t), \tag{25}$$

$$\lambda(t_f) = \bar{F}x(t_f), \quad x(t_0) = x_0. \tag{26}$$

where $\bar{B}(t) = B(t)R^{-1}B(t)^T$. The eqns (24)-(26), are constructed a TBVP of $2n$ dynamic equations with $2n$ unknowns, which are generality difficult for analytical solution. Therefore, we propose a numerical method based on HWs.

3.2 Changing TBVP Into a System of Algebraic Equations

Using HWs, the derivative of the control and costrate signals can be approximate by unknown Haar coefficients. It is done based on eqns (7) and (8), as following

$$\dot{x} = Fh_{(m)}, \tag{27}$$

$$\dot{\lambda} = Gh_{(m)}, \tag{28}$$

where F and G are unknowns $n \times m$ matrices, named Haar coefficients. Now, by the Haar operational matrix, $P_{(m\times m)}$, given in (11), and integrating of both sides of eqns (27) and (28), the state and costate signals can be calculated as following

$$x(t) = x(0) + \int_0^t \dot{x}(\tau)d\tau = x_0 + FPh_{(m)}(t) = (X_0 + FP)h_{(m)}(t), \tag{29}$$

$$\lambda(t) = \lambda(0) + \int_0^t \dot{\lambda}(\tau)d\tau = \lambda_0 + GPh_{(m)}(t) = (\Lambda_0 + GP)h_{(m)}(t) \tag{30}$$

where $X_0 = [x_0, 0, \ldots, 0]$ and $\Lambda_0 = [\lambda_0, 0, \ldots, 0]$, are $n \times m$ matrices. For the TBVP, given in (24)-(26), the initial condition of the costate variables, $\lambda_0 = \lambda(t_0)$, are not available. Therefore, they should be calculated based on the Haar coefficient matrices. From (26) and (28),

$$\begin{aligned} \lambda(t_f) = \bar{F}x(t_f) &= \lambda(0) + \int_0^{t_f} \dot{\lambda}(\tau)d\tau = \lambda(0) + G\int_0^{t_f} h_m(\tau)d\tau \\ &= \lambda(0) + G[1, \underbrace{0, \ldots, 0}_{m-1}]^T = (\Lambda_0 + GO)h_{(m)}(t) \end{aligned} \tag{31}$$

where O is a an $n \times m$ zero matrix which the first entire is one. Similarly, we have $x(t_f) = (X_0 + FO)h_{(m)}(t)$. Using this equation and eqn (31)

$$\Lambda_0 = \bar{F}(X_0 + FO) - GO \tag{32}$$

Now, we rewrite the state and costate equations, eqns (24) and (25), based on HWs, separately. Let $A(t) = \sum_{i=1}^{n^2} A_i\alpha_i(t)$ and $\bar{B} = \sum_{i=1}^{n} \bar{B}_i\beta_i(t)$, where $A_i,\ i = 1, 2, \ldots, n^2$ and $\bar{B}_i,\ i = 1, 2, \ldots, n$ are constant matrices, with appropriate dimensions. Also, $\alpha_i(t),\ i = 1, 2, \ldots, n^2$ and $\beta_i(t),\ i = 1, 2, \ldots, n$ are continuous scaler functions, in $L^2[0, 1]$. Now, by replacing eqns (27)-(30) and (32) in dynamic system (24) and (25), we can achieve the following matrix algebraic equations

$$\begin{cases} Fh_{(m)}(t) = \sum_{i=1}^{n^2} \alpha_i(t)A_i(X_0 + FP)h_{(m)}(t) - \sum_{i=1}^{n} \bar{B}_i\beta_i(t)(\Lambda_0 + GP)h_{(m)}(t) \\ Gh_{(m)}(t) = -Q(X_0 + FP)h_{(m)}(t) - \sum_{i=1}^{n^2} \alpha_i(t)A_i^T(t)(\Lambda_0 + GP)h_{(m)}(t) \end{cases} \tag{33}$$

By Haar approximation of each function, $\alpha_i(t),\ i = 1, 2, \ldots, n^2$ and $\beta_i(t),\ i = 1, 2, \ldots, n$, based on (7), we have $\alpha_i(t) = d_i^T h_{(m)}(t),\ \beta_i(t) = e_i^T h_{(m)}(t)$, where d_i and e_i are Haar coefficient vectors, defined in (8). Also, from (14), we use $M_{(m\times m)}d_i = D_i h_{(m)}$ and $M_{(m\times m)}e_i = E_i h_{(m)}$.
Now, by integrating from both side of eqns (33) and performing the linear operator vec, the system (33) reduced to a system of algebraic equation, which is called Haar matrix equations, as $Ax = b$ with

$$A = \begin{bmatrix} I - \sum_{i=1}^{n^2}(PD_i)^T \otimes A_i + \sum_{i=1}^{n}(OE_i)^T \otimes (\bar{B}_i\bar{F}) & \sum_{i=1}^{n}((O-P)E_i)^T \otimes \bar{B}_i \\ P^T \otimes Q + \sum_{i=1}^{n^2}(OD_i)^T \otimes (A_i^T\bar{F}) & I - \sum_{i=1}^{n^2}((P-O)D_i)^T \otimes A_i^T \end{bmatrix} \tag{34}$$

$$b = \begin{bmatrix} \sum_{i=1}^{n^2} A_iX_0D_i - \sum_{i=1}^{n} \bar{B}_i\bar{F}X_0E_i \\ -QX_0 - \sum_{i=1}^{n^2} A_i^T\bar{F}X_0D_i \end{bmatrix}, \qquad x = \begin{bmatrix} vec(F) \\ vec(G) \end{bmatrix} \tag{35}$$

Not that the above system is achieved after removing $Ph_{(m)}$ from both side of equations. The system (35) is a linear algebraic system contains of $2mn$ equations and $2mn$ unknowns, which can be solved using standard subordinate.
For a TVLQ problem with time-invariant system, TILQ, which the matrices $A(t)$ and $B(t)$ are replaced by constant matrices A and B, respectively, the linear system (34)-(35) can be simplified into the following linear system

$$\begin{bmatrix} I - P^T \otimes A + O^T \otimes (\bar{B}\bar{F}) & P^T \otimes \bar{B} \\ P^T \otimes Q + O^T(A^T \bar{F}) & (P-O)^T \otimes A^T \end{bmatrix} x = \begin{bmatrix} AX_0 - \bar{B}\bar{F}X_0 \\ -QX_0 - A^T \bar{F} X_0 \end{bmatrix} \tag{36}$$

The solution of the system (36), can be calculated, simplicity. By replacing the solutions of the above systems in the equations (27) and (28), the approximate signals can be calculated. The HWs approximation converges rapidly to the exact solutions by increasing the value of m.

4 Error Analysis

Let $(x^*(t), u^*(t))$ and $(\tilde{x}(t), \tilde{u}(t))$, $0 \leq t < 1$ be the exact and the approximate solutions of TVLQ problem. From Theorem 1, the norm of error for derivation of state and costate variables are bounded as following

$$\|\dot{x}^*(t) - Fh_{(m)}(t)\| = O(\frac{1}{m}), \quad \|\dot{\lambda}^*(t) - Gh_{(m)}(t)\| = O(\frac{1}{m}), \tag{37}$$

where m is the level of HWs. For the state signals, we have

$$\|x^*(t) - \tilde{x}(t)\| = \| \int_0^t (\dot{x}^*(\tau) - \dot{\tilde{x}}(\tau)) d\tau \| \leq \|\dot{x}^*(t) - \dot{\tilde{x}}(t)\| t = tO(\frac{1}{m}) \tag{38}$$

Because $0 \leq t < 1$, the right term is convergence to zero as $m \to \infty$. For control signals, from Section 3.1, we have $u(t) = -R^{-1}B^T(t)\lambda(t)$. Now, we have

$$\begin{aligned} \|u^*(t) - \tilde{u}(t)\| &= \|R^{-1}\tilde{B}^T(t)\tilde{\lambda}(t) - R^{-1}B^{*T}(t)\lambda^*(t)\| \\ &\leq \|R\|^{-1} \| \|B^{*T}(t)\lambda^*(t) - \tilde{B}^T(t)\tilde{\lambda}(t)\| \\ &\leq \alpha \|B^*(t)(\lambda^*(t) - \tilde{\lambda}(t))\| + \|\tilde{\lambda}(t)(B^*(t) - \tilde{B}(t))\| \\ &\leq \alpha(k_1 + k_2 O(\frac{1}{m}))O(\frac{1}{m}) \end{aligned} \tag{39}$$

where $k_1 = \|B^*(t)\|$, $\alpha = \|R\|^{-1}$ and $k_2 = \|\lambda^*(t)\|$. The right side of the (39) convergence to zero, as $m \to \infty$. So, by increasing the resolution, Haar approximate control signals converge to the exact signals.

5 Numerical Examples

In this section, for investigate the efficiency and simplicity of the proposed algorithm, three numerical examples are considered. The numerical results for Example 1 are compared with the exact solutions, which allows validation of the proposed method by comparison with the result of exact solutions. Moreover, for the cases of Example 2 and Example 3, which lack analytical solutions, the results are compared with the results of Matlab package *bvp4c*. The error of approximation for the control signal, u, and the state signals, $x_i,\ i = 1, 2, \ldots, n$, are calculated by the following integral square errors

$$E_{x_i} = \sqrt{\int_0^1 (\tilde{x}_i(t) - x_i^*(t))^2 dt}, \qquad E_u = \sqrt{\int_0^1 (\tilde{u}(t) - u^*(t))^2 dt} \tag{40}$$

where $x_i^*,\ i = 1, 2, \ldots, n$ and u^* are exact signals, if exist, and $\tilde{u}$ and $\tilde{x}_i, i = 1, 2, \ldots, n$ are the approximate signals, which is achieved by the proposed algorithm. The results are achieved with different level of wavelets. Also, we use $E_J = |J - J^*|$ as the absolute error for the performance index.

Example 1. Consider the following two-dimensional TILQ

$$\begin{aligned} \min \quad & J = \tfrac{1}{2}x_1^2(1) + \tfrac{1}{2}\textstyle\int_0^1 u^2(t)dt \\ s.t. \quad & \\ & \dot{x}_1 = x_2, \\ & \dot{x}_2 = u, \\ & x(0) = [-1, 0]^T. \end{aligned}$$

The exact solution of the problem is $x^*(t) = (3/8t^2 - 1/8t^3 - 1, 3/4t - 3/8t^2)^T$ with the exact optimal control as $u^*(t) = 3/4(1 - t)$. The solution of the corresponding linear system, (36), for $m = 1$ is equal to $x = [0.2, 0.4, 0, 0.8]^T$, and this solution for $m = 2$ is equal to

$$x = [0.2381, -0.9520, 0.3810, 0.1905, 0, 0, 0.7619, 0]^T$$

For more accurate solution, we used greater value of the parameter m. So, the numerical results of the proposed algorithm are summarized in Table 1, with $m = 2^j,\ j = 4, 5, 6, 7$. The graphical comparison of the exact and approximation for states and control signals are shown in Figures 1 and 2, respectively, with $m = 128$.

Table 1: Numerical results of Haar wavelet approximation solutions for Example 1, $m = 2^j$, $j = 4, 5, 6, 7$.

m	E_J	E_u	E_{x_1}	E_{x_2}	$Time$
16	8.60×10^{-2}	1.56×10^{-2}	6.24×10^{-3}	9.72×10^{-3}	1.06
32	9.17×10^{-2}	5.65×10^{-3}	3.74×10^{-3}	4.83×10^{-3}	4.19
64	9.29×10^{-2}	3.36×10^{-3}	3.41×10^{-3}	3.94×10^{-3}	17.86
128	9.35×10^{-2}	1.61×10^{-3}	3.25×10^{-3}	3.54×10^{-3}	72.27

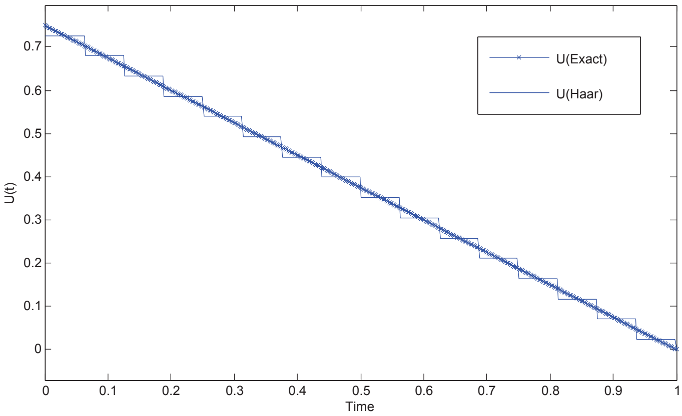

Figure 1: Comparison the Haar wavelet approximation and the exact signal for control signal, Example 1, $m = 128$.

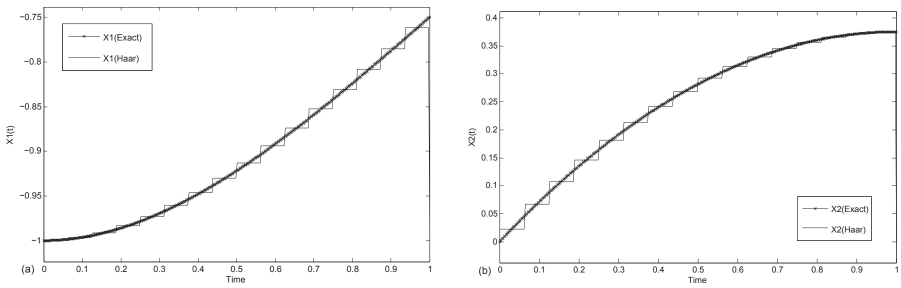

Figure 2: Comparison the Haar wavelet approximations and the exact signals for state signals, (a) x_1 and (b) x_2, Example 1, $m = 128$.

Example 2. Let the following one-dimensional TVLQ

$$J = \frac{1}{2} \int_0^1 (x^2 + u^2) dt$$
$$s.t.$$
$$\dot{x} = tx + u, \quad x(0) = 1.$$

The solution of the corresponding linear system, (36), for $m = 2$ is equal to $x = [-0.0945, -0.4551, \; -1.0287, -0.0252]^T$. For more accurate approximations, we used greater value of the parameter m. So, the numerical results of the proposed algorithm are summarized in Table 1, with $m = 2^j,\ j = 1, 2, \ldots, 5$. Exept the proposed algorithm, the problem was solved by the *bvp4c* commend of Matlab. The results of two methods are compared with different m in Table 2. Also, the graphical comparison between these methods are shown in Figures 3, with $m = 32$, for state and control signals.

Table 2: Numerical results of Haar wavelet approximation solutions for Example 2, $m = 2^j$, $j = 1, 2, 3, 4, 5$.

m	E_J	E_u	E_x	$Time$
2	2.49×10^{-1}	1.76	7.25×10^{-1}	0.3276
4	2.11×10^{-1}	1.21	3.87×10^{-1}	2.1684
8	2.02×10^{-1}	1.04	2.18×10^{-1}	15.60
16	2.0×10^{-1}	9.94×10^{-1}	1.42×10^{-1}	121.11
32	1.95×10^{-1}	9.30×10^{-1}	1.15×10^{-1}	1212.81

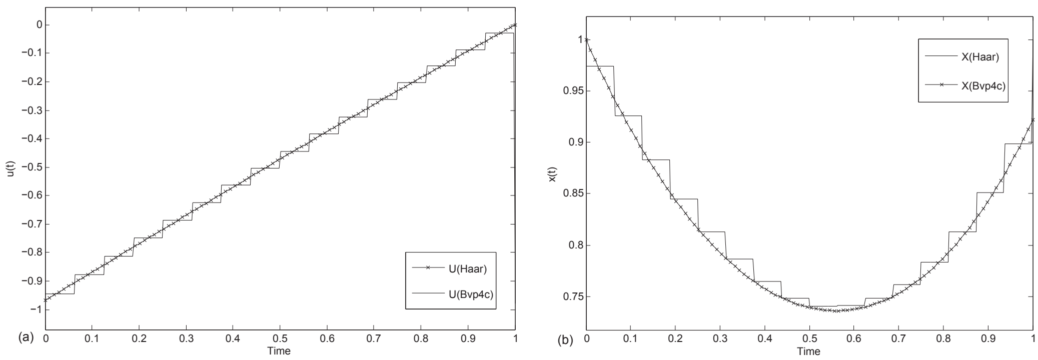

Figure 3: Comparison the Haar wavelet and the *bvp4c* approximations for control (a) and state (b) signals, Example 2, $m = 32$.

Example 3. Consider the following two dimensional TVLQ

$$\begin{array}{ll} \min & J = \frac{1}{2}\int_0^1 (x_1^2(t) + x_2^2(t) + u^2(t))dt, \\ s.t. & \\ & \dot{x} = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ & x(0) = [1, 1]^T. \end{array}$$

For this problem

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad R(t) = 1,$$
$$A = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The solution of the corresponding linear system, (36), for $m = 1$ is equal to $x = [-0.1539, 0.6666, -1.2307, -1.7777]^T$, and this solution for $m = 2$ is equal to

$$x = [-0.0945, 0.6564, -0.4551, -0.3897, -1.0287, -1.5417, -0.0252, 0.1750]^T$$

For more accurate solution, we used greater value of the parameter m. So, the numerical results of the proposed algorithm are summarized in Table 1, with $m = 2^j$, $j = 2, 3, 4, 5, 6$. Exept the Haar approximation approach, the problem us solved by the Matlab package *bvp4c*, and the results are compared in Table 3. The graphical comparison of these two methods, for states and control signals, are given in Figures 4, respectively, with $m = 64$.

Table 3: Numerical comparison of Haar wavelet and the *bvp4c* approximations for Example 3, $m = 2^j$, $j = 4, 5, 6, 7$.

m	E_J	E_u	E_{x_1}	E_{x_2}	$Time$
4	4.89×10^{-1}	3.91×10^{-1}	8.58×10^{-1}	3.87×10^{-1}	1.09
8	4.84×10^{-2}	3.72×10^{-1}	7.10×10^{-1}	2.18×10^{-1}	8.26
16	4.89×10^{-2}	3.67×10^{-1}	6.67×10^{-1}	1.41×10^{-1}	59.35
32	4.83×10^{-2}	3.65×10^{-1}	6.57×10^{-1}	1.15×10^{-1}	213.25
64	4.81×10^{-2}	3.61×10^{-1}	1.07×10^{-1}	1.15×10^{-1}	1225.35

6 Conclusions

Here, a new approach based on integration indirect methods and Haar wavelets was proposed for solving TVLQ. Using the necessary conditions for optimality, the problem could be converted in to a TBVP. Then, the control and state signals were approximated by using HWs with unknown coefficients. Also, the boundary conditions were presented based on Haar coefficients. So, the dynamic system was converted to a system of algebraic equations. Some numerical examples were given for investigating the proposed method.

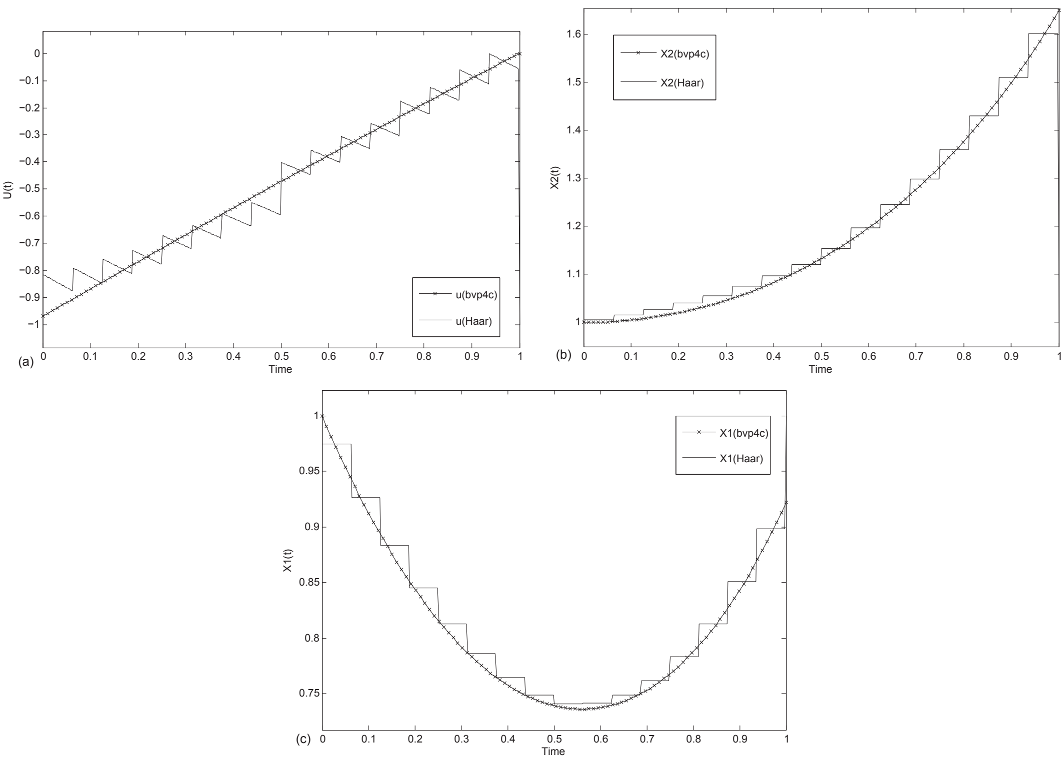

Figure 4: Comparison the Haar wavelet and the *bvp4c* approximations for control signal (a) and state signals, x_1 (b), x_2 (c) , Example 3, $m = 64$.

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معادلات ماتریسی هار برای حل مسائل کنترل بهینه درجه دوم خطی وابسته به زمان

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چکیده

در این مقاله موجک‌های هار برای حل مسائل کنترل بهینه وابسته به زمان درجه دوم خطی زمان پیوسته به‌کار رفته است. ابتدا با شرایط لازم بهینگی مساله به یک مساله مقدار مرزی دونقطه‌ای TBVP تبدیل می‌شود. سپس موجک‌های هار برای تبدیل TBVP، به‌عنوان یک سیستم از معادلات دیفرانسیلی، به یک سیستم معادلات جبری ماتریسی، به‌عنوان معادلات ماتریسی هار با ضرب کرونکر، تبدیل می‌شود. تحلیل خطای روش پیشنهادی ارائه شده است. جواب‌ها با افزایش تعداد جملات تقریب همگرا می‌شود.

کلمات کلیدی

مسائل کنترل بهینه درجه دوم خطی وابسته به زمان، معادله جبری ماتریسی، موجک هار.