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Two-Level Optimization Problems with Infinite Number of Convex Lower Level Constraints

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Abstract. This paper proposes a new form of optimization problem which is a two-level programming problem with infinitely many lower level constraints. Firstly, we consider some lower level constraint qualifications (CQs) for this problem. Then, under these CQs, we derive formula for estimating the subdifferential of its valued function. Finally, we present some necessary optimality conditions as Fritz-John type for the problem.

Keywords. Two-level programming, Constraint qualification, Optimality conditions, Lower level problem.

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1 Introduction

A generalized semi-infinite programming problem (GSIP in brief) is a two-level optimization problem, where the index set of the upper-level problem depends on the decision vector and the set of feasible solutions to the lower-level problem consists of finitely many inequality (and/or equality) constraints; see [14] for comprehensive discussion, various examples (including: reverse Chebyshev approximation, minimax problems, robust optimization, design centering and disjunctive programming), results and references.

GSIPs can be considered as bilevel problems which lower level problems have finite constraints. On the other hand, Dinh et al. in [5] considered bi-level problems which lower level problems' have infinite constraints. So it is natural that we generalize GSIPs such that lower level's problems have infinite constraints. In this paper we consider the following extension of GSIP:

$$\text{EGSIP} : \inf \varphi(x) \quad s.t. \quad x \in \mathcal{F}, \tag{1}$$

with the feasible set $\mathcal{F}$,

$$\mathcal{F} := \left\{ x \in \mathbb{R}^n \middle| \psi(x, y) \geq 0 \quad \text{for all} \quad y \in \Sigma(x) \right\},$$

and the index set

$$\Sigma(x) := \left\{ y \in \mathbb{R}^m \middle| \nu_i(x, y) \leq 0 \quad \text{for all} \quad i \in I \right\},$$

where φ, ψ and ν_is are real-valued convex functions respectively on $\mathbb{R}^n$ and $\mathbb{R}^{n+m}$ and I is an *arbitrary set*. The above assumptions are *standing* throughout the whole paper. Since the index set I in the lower-level inequality constraints is arbitrary, we generally refer (1) to *extended generalized semi-infinite programming problem* (shortly, EGSIP). Of course, EGSIP includes the GSIP when I is finite and if Σ does not depends on x, then the EGSIP reduces to a standard semi-infinite problem (SIP in brief).

In almost all existing literature on GSIP theory, the continuously differentiable (smoothness) assumption on the emerging functions is principle and restrictive. In order to establish optimality conditions for smooth GSIP, several kinds of lower level constraint qualifications are studied. Extensive references to optimality conditions and constraint qualifications for smooth GSIPs and their historical notes, can be found in the book by Stein [14].

It is well known that in the theory of convex minimization over the solution set of a finite system of convex inequalities the so-called basic constraint qualification plays an important role; (see e.g., [8]). For example, it is satisfied if and only if the Karush-Kahn-Tucker (KKT). necessary optimality conditions are also sufficient conditions. Another contribution to the field of constraint qualification in convex optimization can be found in [9], where a pair of constraint qualifications in convex system, namely the Farkas-Minkowski constraint qualification and closedness condition are studied. In this paper we develop these three constraint qualifications for EGSIP. Then, necessary optimality conditions of Fritz-John (FJ) type are established and an example is presented.

We organize the paper as follows. In Section 2, basic notations and results of convex analysis are reviewed. In Section 3, we are interested in three different lower level constraint qualifications for the EGSIP. Then, the results are used to establish efficient upper estimate of subdifferential of value functions. Finally, we apply the obtained subdifferential estimates to derive necessary optimality conditions for the problem.

2 Notations and Preliminaries

In this section we describe the notations used throughout this paper and present some preliminary results on convex analysis. For more details, discussion and applications see [8].

For a set $A \subseteq \mathbb{R}^n$, we shall denote the convex hull and the convex cone (containing the origin) of A by $\operatorname{conv}(A)$ and $\operatorname{cone}(A)$, respectively. We use the symbol 0_n for the zero vector of $\mathbb{R}^n$, and $\langle \cdot, \cdot \rangle$ for the standard inner product in Euclidian space.

Having the generally infinite index set T, denote by $\mathbb{R}^{(T)}$ the collection of multipliers $\tau := (\tau_t \big| t \in T\ )$ with $\tau_t \in \mathbb{R}$ and $\tau_t \neq 0$ for finitely many $t \in T$. Let $\mathbb{R}_+^T$ is defined by

$$\mathbb{R}_+^T := \left\{ \tau \in \mathbb{R}^{(T)} \big| \tau_t \geq 0 \ \text{ for all } t \in T \right\}.$$

For a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$, the subdifferential of φ at $x_0 \in \mathbb{R}^n$ is defined by

$$\partial \varphi\left(x_0\right) := \left\{ \rho \in \mathbb{R}^n \big| \ \langle \rho, x - x_0 \rangle \leq \varphi(x) - \varphi(x_0) \ \text{ for all } \ x \in \mathbb{R}^n \right\}.$$

As usual, the symbols $\partial_x \varphi(x_0, y_0)$ and $\partial_y \varphi(x_0, y_0)$ stand for the corresponding partial subdifferential of $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ at (x_0, y_0). The epigraph of φ, the strict epigraph of φ, and the conjugate function $\varphi^* : \mathbb{R}^n \to \mathbb{R}$ to φ, are respectively defined as

$$\begin{aligned}
\operatorname{epi} \varphi &:= \left\{ (x, r) \in \mathbb{R}^n \times \mathbb{R} \ \big| \varphi\left(x\right) \leq r \right\}, \\
\operatorname{Sepi} \varphi &:= \left\{ (x, r) \in \mathbb{R}^n \times \mathbb{R} \ \big| \ \varphi\left(x\right) < r \right\}, \\
\varphi^*\left(u\right) &:= \sup_{x \in \ \mathbb{R}^n} \left\{ \langle u, x \rangle - \varphi\left(x\right) \right\}.
\end{aligned}$$

Recall also that the normal cone of a closed convex subset $A \subseteq \mathbb{R}^n$ at $x_0 \in A$ is defined by $N_A\left(x_0\right) := \partial \mathfrak{I}_A(x_0)$, where $\mathfrak{I}_A(x)$ denote the indicator function of A at x_0, i.e.,

$$\mathfrak{I}_A\left(x\right) := \begin{cases} 0 & \text{if} \quad x \in A \\ +\infty & \text{if} \quad x \notin A. \end{cases}$$

3 Main Results

As a starting point of this section, we define the lower level problem of EGSIP at $\hat{x} \in \mathcal{F}$:

$$\min \psi\left(\hat{x}, y\right) \quad \text{subject to} \quad y \in \Sigma\left(\hat{x}\right) \tag{2}$$

and we denote the value function of this problem with $\mu(\cdot)$, i.e., (with the convention $\inf \emptyset = +\infty$)

$$\mu(\hat{x}) := \inf\left\{\psi(\hat{x}, y) \;\middle|\; y \in \Sigma(\hat{x})\right\}.$$

We shall need to the following simple result.

Lemma 1 (Convexity of lower level value function in EGSIP)**.** Under the standing assumptions, the value function $\mu(\cdot)$ is convex.

Proof. We show that $\mathrm{Sepi}\,\mu$ is a convex set. To this end, choose (x_1, r_1) and (x_2, r_2) in $\mathrm{Sepi}\,\mu$. Hence, $\mu(x_1) < r_1$ and $\mu(x_2) < r_2$. The case when $\Sigma(x_1) = \emptyset$ or $\Sigma(x_2) = \emptyset$ is trivial. Thus, we may choose $y_1 \in \Sigma(x_1)$ and $y_2 \in \Sigma(x_2)$ such that

$$\psi(x_1, y_1) < r_1 \quad \text{and} \quad \psi(x_1, y_1) < r_1\ , \tag{3}$$

Otherwise, if $\psi(x_1, y) \geq r_1$ for each $y \in \Sigma(x_1)$, then by definition of infimum we have

$$\mu(x_1) = \inf_{y \in \Sigma(x_1)} \psi(x_1, y) \geq r_1;$$

i.e., a contradiction occur. The fact that convexity of $\nu_i(\cdot,\cdot)$, $\forall i \in I$, and $\nu_i(x_l, y_l)$ for all $l \in \{1, 2\}$ yield for each $\lambda \in [0, 1]$ that

$$\begin{aligned} &\nu_i(\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)) \leq 0 \\ &\quad \Rightarrow \lambda y_1 + (1-\lambda) y_2 \in \Sigma(\lambda x_1 + (1-\lambda) x_2). \end{aligned}$$

Hence, owning to (3) we obtain

$$\begin{aligned} \mu(\lambda x_1 + (1-\lambda) x_2) &\leq \psi(\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)) \\ &\leq \lambda\ \psi(x_1, y_1) + (1-\lambda)\ \ \psi(x_2, y_2) \\ &< \lambda r_1 + (1-\lambda) r_2. \end{aligned}$$

Consequently $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in \mathrm{Sepi}\,\mu$, and thus $\mathrm{Sepi}\,\mu$ is a convex set. It is known $\mathrm{Sepi}\,\mu$ is convex if and only if μ is convex (see e.g., [8]) . □

Concerning the EGSIP, we consider the system

$$\pi := \left\{(x, y) \in \mathbb{R}^{n+m} \;\middle|\nu_i(x, y) \leq 0 \quad \forall\ i \in I\right\}.$$

Throughout this paper, $\triangle(x, y)$ and K are defined as

$$\begin{aligned} \triangle(x, y) &:= \left\{i \in I \middle| \nu_i(x, y) = 0\right\}, \\ K &:= \mathrm{cone}\left(\bigcup_{i \in I} \mathrm{epi}\ \ {\nu_i}^*\right). \end{aligned}$$

Also, we define the index set of active constraints at $\hat{x} \in \mathcal{F}$ by

$$\Sigma_0(\hat{x}) := \left\{y \in \Sigma(\hat{x}) \;\middle|\; \psi(\hat{x}, y) = 0\right\}.$$

In the remainder of this section, we shall need the following definition:

Definition 1. We say that the EGSIP satisfies the *Closedness Condition* (*CC* in brief), if the set

$$\operatorname{epi}\psi^* + K$$

is closed in the space $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$.

The *CC* was proposed for the first time in [1] and it has been used after in several papers (see, for instance, [3, 4, 6, 7]) to establish optimality conditions of Karush-Kahn-Tucker form, duality and stability results for convex cone-constrained programs or convex infinite programs. Recently, in [10], the author proved some version of the Fritz-John type necessary conditions for DC (difference of convex functions) generalized semi-infinite problems under the *CC*.

We associate with EGSIP the *extended Lagrangian* function as follows:

$$\mathcal{L}: \ \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}_+^{(I)} \to \mathbb{R}$$
$$\mathcal{L}(x, y, a, \beta) := a\psi(x, y) + \sum_{i \in \mathcal{S}(\beta)} \beta_i \nu_i(x, y),$$

where $\mathcal{S}(\beta)$ denotes the support set of $\beta \in \mathbb{R}_+^{(I)}$, defined as

$$\mathcal{S}(\beta) := \left\{ i \in I \ \middle| \beta_i \neq 0 \right\}.$$

We observe that $\mathcal{L}$ is naturally extension of Lagrangian in GSIP theory. Denote the set (maybe empty) of Karush-Kahn-Tucker (KKT) multiplier of the problem (2) at $\hat{y} \in \Sigma_0(\hat{x})$ by

$$\mathcal{K}(\hat{x}, \hat{y}) := \{\beta \in \mathbb{R}_+^{(I)} \big| \ 0_m \in \partial_y \mathcal{L}(\hat{x}, \widehat{y}, 1, \beta), \ \beta_i \nu_i(\hat{x}, \hat{y}) = 0, \ \forall i \in \mathcal{S}(\beta)\}.$$

The next theorem is certainly of its own interest while playing a crucial rule in establishing the main result of this paper on necessary optimality conditions for the EGSIP.

Theorem 1 (Upper estimate for the subdifferential of the lower level value function in EGSIP under CC condition)**.** In addition to the standing assumptions, we suppose that $\hat{y} \in \Sigma_0(\hat{x})$, and the *CC* holds. Then

$$\partial\mu(\hat{x}) \subseteq \bigcup_{\beta \in \mathcal{K}(\hat{x}, \hat{y})} \partial_x \mathcal{L}(\hat{x}, \widehat{y}, 1, \beta).$$

Proof. It is known from Lemma 1 that $\mu(\cdot)$ is a convex function. Let $\xi \in \partial\mu(\hat{x})$. By the definition of subdifferential we have

$$\mu(x) - \mu(\hat{x}) \geq \langle \xi, x - \hat{x} \rangle, \quad \forall x \in \mathbb{R}^n. \tag{4}$$

Since $\mu(\hat{x}) = \psi(\hat{x}, \hat{y})$ by the choice of $\hat{y} \in \Sigma_0(\hat{x})$, and $\mu(x) \leq \psi(x, y)$ for all $(x, y) \in \mathcal{F} \times \Sigma(x)$, from (4) the following inequality holds

$$\psi(x, y) - \psi(\hat{x}, \hat{y}) \geq \langle \xi, x \rangle - \langle \xi, \hat{x} \rangle, \quad \forall (x, y) \in \mathcal{F} \times \Sigma(x).$$

The last relation implies that $(\hat{x}, \hat{y})$ is a solution to the following convex semi-infinite programming problem:

$$\begin{aligned} \min \quad & \psi(x, y) - \langle \xi, x \rangle \\ s.t. \quad & \nu_i(x, y) \leq 0, \quad i \in I. \end{aligned}$$

This implies that

$$0_{n+m} \in \partial \left(\psi - \langle \xi, x \rangle + \mathfrak{J}_\pi\right)(\hat{x}, \hat{y}). \tag{5}$$

Hence

$$(\xi, 0_m) \in \partial \left(\psi + \mathfrak{J}_\pi\right)(\hat{x}, \hat{y}).$$

In the other hand, since the CC holds, we get from ([6], Corollary 3·2) the following upper estimate for the subdifferential of $\psi + \mathfrak{J}_\pi$ at $(\hat{x}, \hat{y})$:

$$\partial \left(\psi + \mathfrak{J}_\pi\right)(\hat{x}, \hat{y}) \subseteq \partial \psi (\hat{x}, \hat{y}) + \bigcup_{\beta \in \mathcal{A}(\hat{x}, \hat{y})} \left[\sum_{i \in \mathcal{S}(\beta)} \beta_i \partial \nu_i (x, y)\right], \tag{6}$$

where

$$\mathcal{A}(\hat{x}, \hat{y}) := \left\{\beta \in \mathbb{R}_+^{(I)} \mid \beta_i \nu_i (\hat{x}, \hat{y}) = 0, \ \forall i \in \mathcal{S}(\beta)\right\}.$$

Then we use the following important relationship between the full and partial subdifferentials of convex functions $h(x, y)$ that holds by e.g., ([2], Prop. 2·3·15)

$$\partial h (\hat{x}, \hat{y}) \subseteq \partial_x h (\hat{x}, \hat{y}) \times \partial_y h (\hat{x}, \hat{y}). \tag{7}$$

With regard to the (5), (6), and (7), there is a $\beta \in \mathcal{A}(\hat{x}, \hat{y})$ such that

$$\xi \in \partial_x \mathcal{L}(\hat{x}, \widehat{y}, 1, \beta) \quad \text{ and } \quad 0_m \in \partial_y \mathcal{L}(\hat{x}, \widehat{y}, 1, \beta). \tag{8}$$

In turn, this implies that

$$\xi \in \bigcup_{\beta \in \mathcal{K}(\hat{x}, \hat{y})} \partial_x \mathcal{L}(\hat{x}, \widehat{y}, 1, \beta).$$

Since ξ was an arbitrary element of $\partial \mu(\hat{x})$, the conclusion holds. □

Constraint qualifications involving epigraphs for convex optimization first introduced by Jeyakumar et al. in [9]. Inspired by this paper as well as [10], we define the following concept.

Definition 2. The EGSIP is said to satisfy the Farkas-Minkowski (FM in brief), if K is a closed set.

Let us introduce another constraint qualification.

Definition 3. We say that the EGSIP is satisfies the *Basic Constraint Qualification* (BCQ, briefly) at $(x_0, y_0) \in \pi$, if

$$N_\pi (x_0, y_0) \subseteq \text{cone}\left(\bigcup_{i \in \triangle(x_0, y_0)} \partial \nu_i (x_0, y_0)\right)$$

EGSIP is said to be BCQ if it is BCQ at every $(x_0, y_0) \in \pi$.

The BCQ, firstly was introduced in [8] in relation to convex optimization problems. It was extended in [13] to the framework of convex semi-infinite programming problems (SIP), and deeply studied in [11] for non-convex SIPs.

Remark 1 (Relationship between qualification conditions)**.** There is no relation of implication between the BCQ and the CC. The relationship between the notions BCQ, FM and CC is shown in the following diagram (see [1, 3, 4, 6, 7], for comprehensive discussion and various examples of these CQs.)

$$BCQ \Longleftrightarrow FM \Longleftrightarrow CC.$$

Remark 2 (Comparison with another CQ)**.** In [13], the following indices subset is considered, instead of $\triangle(x, y)$,

$$\triangle^* (x, y) := \left\{ i \in I \;\middle|\; \nu_i (x, y) = V(x, y) \right\},$$

where

$$V (x, y) := \sup_{i \in I} \; \nu_i (x, y) .$$

Proof. In [13] the continuity of $V(\cdot, \cdot$ is assumed and they formulate the BCQ^* condition at (x_0, y_0) as follows:

$$N_\pi (x_0, y_0) \subseteq \text{cone} \left(\bigcup_{i \in \triangle^*(x_0, y_0)} \partial \nu_i (x_0, y_0) \right).$$

We show that BCQ is equivalent to BCQ^*.

If $V (x_0, y_0)$, the continuity of V entails that (x_0, y_0) is an interior point of π. Then, $N_\pi (x_0, y_0) = \{0\}$ and BCQ and BCQ^* are both trivially satisfied. Thus, we suppose that $V (x_0, y_0) = 0$.

In one hand, $\triangle (x_0, y_0) \subseteq \triangle^* (x_0, y_0)$ because

$$i \in \triangle (x_0, y_0) \Longrightarrow \nu_i (x_0, y_0) = V (x_0, y_0) = 0 \Longrightarrow i \in \triangle (x_0, y_0) .$$

On the other hand, since $V (x_0, y_0) = 0$ and $i \in \triangle^* (x_0, y_0)$, then

$$i \in \triangle^* (x_0, y_0) \Longrightarrow 0 = \nu_i (x_0, y_0) \leq V (x_0, y_0) \leq 0 \Longrightarrow i \in \triangle^* (x_0, y_0) .$$

Hence, $\triangle (x_0, y_0) = \triangle^* (x_0, y_0)$, and BCQ is equivalent to BCQ^*. □

Owning to the Remark 2, it is enough to prove our results under the BCQ and CC conditions. The following theorem is BCQ counterpart of Theorem 1.

Theorem 2 (Upper estimate for the subdifferential of the lower level value function in EGSIP under BCQ)**.** In addition to the standing assumptions, we suppose that $\hat{y} \in \Sigma_0 (\hat{x})$, and the EGSIP satisfies the BCQ at $(\hat{x}, \hat{y})$. Then

$$\partial \mu(\hat{x}) \subseteq \bigcup_{\beta \in \mathcal{K}(\hat{x}, \hat{y})} \partial_x \mathcal{L} (\hat{x}, \widehat{y}, 1, \beta) .$$

Proof. Let $\xi \in \partial \mu(\hat{x})$ is arbitrary. We arrive to (5) from the same argument as in the proof of Theorem 1. Therefore, owning to BCQ, it comes that

$$\begin{aligned}
(\xi, 0_m) &\in \partial \psi (\hat{x}, \hat{y}) + N_\pi (\hat{x}, \hat{y}) \\
&\subseteq \partial \psi (\hat{x}, \hat{y}) + \text{cone} \left(\bigcup_{i \in \triangle(\hat{x}, \hat{y})} \partial \nu_i (\hat{x}, \hat{y}) \right).
\end{aligned}$$

Thus, we can find a finite set $J \subseteq \triangle(\hat{x}, \hat{y})$ and positive numbers $\gamma_j \geq 0$ (for $j \in J$), such that

$$(\xi, 0_m) \in \partial\psi(\hat{x}, \hat{y}) + \sum_{j \in J} \gamma_j \partial\nu_i(\hat{x}, \hat{y}). \tag{9}$$

Now, we define $\beta \in \mathbb{R}_+^{(I)}$ as follows:

$$\beta_i := \begin{cases} \gamma_i & \text{if} \quad i \in J \\ 0 & \text{if} \quad i \notin J. \end{cases}$$

Since the equalities $J = \mathcal{S}(\beta)$ and $\beta_j \nu_j(\hat{x}, \hat{y}) = 0$ (for $j \in J$) hold trivially (thanks to $J \subseteq \triangle(\hat{x}, \hat{y})$), then $\beta \in \mathcal{A}(\hat{x}, \hat{y})$ and the following inclusion is equivalent to (9):

$$(\xi, 0_m) \in \partial\psi(\hat{x}, \hat{y}) + \sum_{i \in \mathcal{S}(\beta)} \beta_i \partial\nu_i(\hat{x}, \hat{y}).$$

Then, owning to (7), the virtues of (8) holds. □

As an immediate consequence of Theorem 2 and Remark 1 we get the following corollary.

Corollary 1 (Upper estimate for the subdifferential of the lower level value function in EGSIP under FM condition)**.** If in the Theorem 2, "CC" is replaced by "FM", then its conclusions hold also true.

Now we are ready to establish the main result of this paper providing Fritz-John type necessary optimality condition for the EGSIP.

Theorem 3 (FJ Necessary optimality Condition for EGSIP)**.** In addition to the standing assumptions, we suppose that $\hat{x}$ is an optimal solution for EGSIP, and $\hat{y} \in \Sigma_0(\hat{x})$. Furthermore, suppose that one of the following conditions holds:

I. EGSIP satisfies the CC,

II. EGSIP satisfies the FM,

III. EGSIP satisfies the BCQ at $(\hat{x}, \hat{y})$.

Then, there are scalars $\lambda_0, \lambda_1 \in [0, 1]$ and vector $\beta \in \mathcal{K}(\hat{x}, \hat{y})$ satisfying the relationships

$$\begin{aligned} &0_n \in \lambda_0 \partial\varphi(\hat{x}) - \lambda_1 \partial_x \mathcal{L}(\hat{x}, \widehat{y}, 1, \beta), \\ &\lambda_0 + \lambda_1 = 1. \end{aligned}$$

Proof. Since $\hat{x}$ is a local minimum of EGSIP and $\mu(\hat{x}) = 0$, it follows that the convex function

$$\omega(x) := \max\{\varphi(x) - \varphi(\hat{x}), -\mu(x)\}$$

attains its local minimum at $\hat{x}$, and yields

$$0 \in \partial\omega(\hat{x}) \subseteq \operatorname{conv}(\partial\varphi(\hat{x}) \cup (-\partial\mu(\hat{x}))).$$

Thus, there are $\lambda_0, \lambda_1 \in [0, 1]$ such that

$$0_n \in \lambda_0 \partial\varphi(\hat{x}) - \lambda_1 \partial\mu(\hat{x}) \quad \text{and} \quad \lambda_0 + \lambda_1 = 1.$$

By Theorems 1 and 2 (and Corollary 1), the proof is now completed. □

Remark 3 (Comparison with known results on optimality conditions for GSIP)**.** To the best of our knowledge, Theorem 2 is the first in the literature on necessary optimality conditions for *generalized semi-infinite programs with semi-infinite lower level problem*. It turns out furthermore that the specifications of Theorem 2 for finite index set I provide significant improvements over previously known necessary optimality conditions for convex GSIPs. The most advanced results for GSIP have been obtained in [14] in differentiable setting; see also the references and commentaries in [14]. In comparison with our Theorem 2 from [12] establishes necessary optimality conditions of FJ type for such GSIPs with some element $\hat{y} \in \Sigma_0(\hat{x})$ therein assuming in addition that $\Sigma(\cdot)$ is uniformly bounded around $\hat{x}$ and imposing a more restrictive Slater type constraint qualification in the lower level problem. Moreover, Theorem 2 is relax to the inner semi-continuity condition of set valued mapping $x \to \Sigma_0(\cdot)$, which is assumed in [10, 12, 14].

To appreciate the above discussion we present an example.

Example 1. Consider the following problem:

$$\begin{aligned}
&\inf \varphi(x) := |x| \quad \text{ s.t. } \quad x \in \mathcal{F}, \\
&\mathcal{F} := \{x \in \mathbb{R} \mid \psi(x,y) := x - 2y \geq 0 \ \text{ for all } \ y \in \Sigma(x)\}. \\
&\Sigma(x) := \{y \in \mathbb{R} | \ \nu_i(x,y) \leq 0 \text{ for all } i \in I\},
\end{aligned}$$

where $I := [0,1] \cup \{2\}$ and

$$\begin{aligned}
&\nu_0(x,y) := |x| + \max\{0,y\}, \\
&\nu_i(x,y) := x^2 + iy - i \quad \forall\, i \in (0,1], \\
&\nu_2(x,y) := x.
\end{aligned}$$

Since I is not finite, the existing results in other literals are not useful for this problem.

It is not difficult to see that $\hat{x} := 0$ is an optimal solution of problem and

$$\begin{aligned}
&\pi = \{0\} \times (-\infty, 0], \\
&\Sigma_0(\hat{x}) = \left\{y \in \mathbb{R}_- \middle| 0 - 2y = 0\right\} = \{0\} \quad (\text{hence, } \hat{y} = 0), \\
&\triangle(0,0) = \{0,2\}, \\
&\partial\nu_0(0,0) = \left\{(u_1,u_2) \in \mathbb{R}^2 \,\middle|\, -1 \leq u_1 \leq 1, \ u_2 = 0\right\} \\
&\qquad\qquad + \left\{(u_1,u_2) \in \mathbb{R}^2 \ 0 \leq u_2 \leq 1, \quad u_1 = 0\right\} \\
&\qquad\quad = [-1,1] \times [0,1], \\
&\partial\nu_2(0,0) = \{(1,0)\}.
\end{aligned}$$

Previous relations imply that

$$N_\pi(0,0) = \mathbb{R} \times [0,+\infty) = \operatorname{cone}(\partial\nu_0(0,0) \cup \partial\nu_2(0,0)).$$

Thus, π satisfies in the BCQ at $(0,0)$.

On the one hand, for each $(x,y,\beta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^{(I)}$, we have

$$\mathcal{L}\left(\hat{x},\widehat{y},1,\beta\right) = x - 2y + \sum_{i\in\mathcal{S}(\beta)} \beta_i \partial\nu_i\left(x,y\right).$$

Take $\beta_0 := 2$ and $\beta_i := 0 \ forall \ i \in (0,1] \cup \{2\}$. Thanks to $\mathcal{S}(\beta) = \{0\}$ and

$$0 \in -2 + \beta_0\,[0,1] = \partial_y \mathcal{L}\left(0,0,1,\beta\right),$$

we give $\beta \in \mathcal{K}\left(0,0\right)$. Finally, there are obviously scalars λ_0, λ_1 (for example: $\lambda_0 = \lambda_1 := \frac{1}{2}$) as in Theorem 3 satisfying the following relationships:

$$\begin{cases} 0 \in \lambda_0\,[-1,1] - \lambda_1\left(1 + 2\,[-1,1]\right) = \partial\varphi\left(0\right) + \partial_x\mathcal{L}\left(0,0,1,\beta\right), \\ \lambda_0 + \lambda_1 = 1. \end{cases}$$

Example 2. Consider the following problem:

$$\inf \varphi\left(x_1,x_2\right) := -x_1 - x_2 \quad \text{ s.t. } \quad x \in \mathcal{F},$$

where

$$\mathcal{F} := \left\{ (x_1,x_2) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} \psi(x_1,x_2,y_1,y_2) := \min\left\{2-y_1,\ 2+y_1,\ 2-y_2,\ 2+y_2\right\} \geq 0 \\ \quad \forall\left(y_1,y_2\right) \in \left(x_1,x_2\right) \end{array} \right\}$$

$$\Sigma\left(x_1,x_2\right) := \left\{ (y_1,y_2) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} \nu_i\left(x_1,x_2,y_1,y_2\right) := \left(x_1-y_1\right)^2 + \left(x_2-y_2\right)^2 - 1 - i \leq 0 \\ \\ \quad \forall\ i \in I := \mathbb{N}\cup\{0\} \end{array} \right\}.$$

It follows that $\mathcal{F} = [-1,1]\times[-1,1]$ and the uniqe optimal point is $(\widehat{x_1},\widehat{x_2}) := (1,1)$. A short calculation shows that $\Sigma\left(\widehat{x_1},\widehat{x_2}\right) = \{(\ddot{y_1},\ddot{y_2}),(\dddot{y_1},\dddot{y_2})\}$, where $(\ddot{y_1},\ddot{y_2}) := (1,2)$ and $(\dddot{y_1},\dddot{y_2}) := (2,1)$,

$$\begin{aligned}
\partial\varphi\left(\widehat{x_1},\widehat{x_2}\right) &= \left\{(-1,-1)\right\}, \\
\partial\psi\left(\widehat{x_1},\widehat{x_2},\ddot{y_1},\ddot{y_2}\right) &= \left\{(0,0,0,-1)\right\}, \\
\partial\psi\left(\widehat{x_1},\widehat{x_2},\dddot{y_1},\dddot{y_2}\right) &= \left\{(0,0,-1,0)\right\}, \\
\triangle\left(\widehat{x_1},\widehat{x_2},\ddot{y_1},\ddot{y_2}\right) &= \left\{0\right\}, \\
\triangle\left(\widehat{x_1},\widehat{x_2},\ddot{y_1},\ddot{y_2}\right) &= \left\{0\right\}, \\
\partial\nu_0\left(\widehat{x_1},\widehat{x_2},\ddot{y_1},\ddot{y_2}\right) &= \left\{2(0,-1,0,1)\right\}, \\
\partial\nu_0\left(\widehat{x_1},\widehat{x_2},\dddot{y_1},\dddot{y_2}\right) &= \left\{2(-1,0,1,0)\right\}, \\
\beta = \frac{1}{2} &\quad \Longrightarrow \quad \partial\ \mathcal{L}\left(\widehat{x_1},\widehat{x_2},\ddot{y_1},\ddot{y_2},1,\beta\right) \\
&= \left\{(0,-1+2\beta,0,-2\beta)\right\} \\
&= \left\{(0,0,0,-1)\right\} \\
\partial\ \mathcal{L}_x\left(\widehat{x_1},\widehat{x_2},\ddot{y_1},\ddot{y_2},1,\beta\right) &= \left\{(0,0)\right\}
\end{aligned}$$

There are scalars λ_0, λ_1 (for example: $\lambda_0 := 0$ and $\lambda_1 := 1$) as in Theorem 3 satisfying the following relationships:

$$\begin{cases} 0_2 \in \lambda_0\left(-1,-1\right) - \lambda_1\left(0,0\right) = \partial\varphi\left(\widehat{x_1},\widehat{x_2}\right) + \partial_x\mathcal{L}\left(\widehat{x_1},\widehat{x_2},\ddot{y_1},\ddot{y_2},1,\beta\right), \\ \lambda_0 + \lambda_1 = 1. \end{cases}$$

Similarly, it shows that the above relations are valid at $(\dddot{y}_1,\dddot{y_2})$.

4 Conclusions

We have established some Fritz-John type necessary optimality conditions for a new extension of convex generalized semi-infinite programming problem (EGSIP). In this line, several constraint qualifications for EGSIP are introduced and their interrelations are studied.

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مسائل بهینه‌سازی دو سطحی با تعداد نامتناهی قید محدب سطح پایین

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چکیده

این مقاله، مدل جدیدی از مسائل بهینه‌سازی را معرفی می‌کند که دو سطحی بوده و شامل بینهایت قید سطح پایین است. در ابتدا ما چندین قید تعریفی سطح پایین برای این مسئله معرفی کرده و سپس این قیدهای تعریفی را برای محاسبه‌ی زیر مشتق تابع مقدار مسئله به کار می‌گیریم. در نهایت، به معرفی چندین شرط لازم بهینگی از نوع فریز-جان برای این مسئله خواهیم پرداخت.

کلمات کلیدی

برنامه‌ریزی دوسطحی، قید تعریفی، شرایط بهینگی، مسئله سطح پایین.