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A Numerical Solution of Fractional Optimal Control Problems Using Spectral Method and Hybrid Functions

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Abstract. In this paper, a modern method is presented to solve a class of fractional optimal control problems (FOCPs) indirectly. First, the necessary optimality conditions for the FOCP are obtained in the form of two fractional differential equations (FDEs). Then, the unknown functions are approximated by the hybrid functions, including Bernoulli polynomials and Block-pulse functions based on the spectral Ritz method. Also, two new methods are proposed for calculating the left Caputo fractional derivative and right Riemann-Liouville fractional derivative operators of the hybrid functions that are proportional to the Ritz method. The FOCP is converted into a system of the algebraic equations by applying the fractional derivative operators and collocation method, which determines the solution of the problem. Error estimates for the hybrid function approximation, fractional operators and, the proposed method are provided. Finally, the efficiency of the proposed method and its accuracy in obtaining optimal solutions are shown by some test problems .

Keywords. Fractional optimal control, Hybrid functions, Bernoulli polynomials, Ritz method, Error bound.

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1 Introduction

In recent years, many mathematicians have paid attention to the fractional calculus in the sense of generalized derivatives and integrals of the integer order to fractional one. The FDEs have been considered for describing and modeling many scientific phenomena. Based on the books mentioned in [1, 2, 3] and their references, the fractional systems are widely used in many branches of science such as mathematics, physics, chemistry, biology, economics, and engineering. There are different definitions for fractional derivatives, among which the Riemann-Liouville and Caputo are mostly used.

The optimal control problem (OCP) is to determine the control and state variables so that a process complies with some constraints and at the same time optimizes an objective function. A FOCP is a special case of an OCP in which either objective function, or constraints of the problem, or both contain at least one FDE [4, 5]. Many methods have been used to solve FOCPs numerically. These methods are divided into two general categories, including direct and indirect methods. The indirect methods are based on finding a solution from the necessary optimality conditions, resulted from the calculus of variation and the Pontryagin's minimum principle [6]. These methods lead to a two-point boundary value problem (TPBVP), which can be solved by the well-known numerical methods. Direct methods are according to the discretization of control or state variables and transforming the given problem into a nonlinear optimization problem. Formulation of the FOCPs and some numerical methods used for solving different classes of FOCPs can be found in references [7, 8, 9, 10].

Spectral methods are the special classes of global numerical methods based on the discretization of variables in differential equations, which have recently been extended to use combined methods like the Ritz method [11, 12]. These methods depend on the family of weighted residual methods (WRMs). WRM indicates a specific set of approximation techniques, in which the residuals (or errors) are minimized in a certain way. In these methods, the unknown function is extended by a linear combination of basis functions with unknown coefficients. The other functions known as weighted functions are applied to find the unknown coefficients. Also, the approximation error is reduced as much as possible. With respect to the basis and weights functions, spectral methods are divided into three classes including Galerkin methods, collocation methods and Tau methods. Recently, these methods have been used to solve a variety of FDE and FOCPs. Doha et al. developed an efficient Chebyshev spectral method to solve multi-term FDE [13]. Esmaeili and Shamsi introduced a pseudo-spectral method for solving the fractional initial value problems [14]. Sweilam also used the Legendre spectral-collocation method to solve some types of FOCPs [15]. Nemati introduced a spectral method along with the Bernstein operational matrix for solving the 2D FOCPs [16]. Ejlali et al. proposed the B-spline spectral method for constrained FOCPs [17].

The orthogonal functions can be classified into three families, including the piecewise constant basis function (Block pulse, Harr, Walsh), the orthogonal polynomials (Laguerre, Chebyshev, Legendre), and the set of sine-cosine (Fourier) functions. Orthogonal functions are applied to reduce the dynamical system problems to a system of algebraic equation. The hybrid function is constructed by the combination of Block-Pulse functions with Lagrange polynomials, Bernoulli polynomials, Chebyshev polynomials, and Legendre polynomials [18, 19, 20, 21]. The hybrid functions have been proven to be a mathematical vigorous instrument for discretization a FDE or OCP [22, 23]. Among these hybrid functions, the Bernoulli hybrid functions have been proven to be computationally more effective [24, 25]. Also, the properties of Bernoulli hybrid functions are very effective in computing fractional derivative operators. According to our information, none of these hybrid functions have been used for solving FOCPs by indirect methods. In this paper, at first, the necessary optimality conditions of the FOCP are obtained. Then, the resulted FDEs are converted to a system of algebraic equations, using a numerical method including hybrid functions approximation fitted to the Ritz method along with fractional operators and collocation method. The solution of this system determines the approximate solution of FOCP.

The structure of this paper is as follows: First, in Section 2, we provide some fundamental definitions and properties of the fractional calculus and Bernoulli hybrid functions. Next, Section 3 describes the calculation of fractional derivative operators applicable in the Ritz method. In Section 4, we introduce the problem statement and numerical method for solving the desired problem. Then, in Section 5, the error bounds are presented. After that, in Section 6, illustrative test examples are used to verify the accuracy of the suggested method. And finally, Section 7 is the conclusion of this study.

2 The Required Definitions and Properties

In this section, some definitions of fractional derivatives and integral, hybrid functions and their properties are presented.

2.1 Fractional calculus

The most important definitions of the fractional derivative and integral and their properties are provided in this section. Let $f : [a, b] \to \mathbb{R}$ and $\alpha > 0$ be the order of fractional derivative or integral and $i = [\alpha] + 1$ [1, 2, 3, 26, 27].

Definition 1. The left and right fractional Riemann-Liouville integral of order $\alpha > 0$ are defined, respectively, as follows:

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, $$

$$ {}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau) d\tau, $$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 2. The left and right Riemann-Liouville fractional derivative operators of order α are given by

$$ {}_a^{RL}D_t^\alpha f(t) = \frac{1}{\Gamma(i-\alpha)} \frac{d^i}{dt^i} \int_a^t (t-\tau)^{i-\alpha-1} f(\tau) d\tau, $$

$$ {}_t^{RL}D_b^\alpha f(t) = \frac{(-1)^i}{\Gamma(i-\alpha)} \frac{d^i}{dt^i} \int_t^b (\tau-t)^{i-\alpha-1} f(\tau) d\tau. $$

Definition 3. The left and right Caputo fractional derivative operators are defined by

$$ {}_a^{C}D_t^\alpha f(t) = \frac{1}{\Gamma(i-\alpha)} \int_a^t (t-\tau)^{i-\alpha-1} f^{(i)}(\tau) d\tau, $$

$$ {}_t^{C}D_b^\alpha f(t) = \frac{(-1)^i}{\Gamma(i-\alpha)} \int_t^b (\tau-t)^{i-\alpha-1} f^{(i)}(\tau) d\tau. $$

For the mentioned fractional derivatives and integral, the following properties are established. When $\alpha = z$ is an integer number, the fractional Riemann-liouville and Caputo derivatives reduce to ordinary derivative as follows:

$$ {}_a^{RL}D_t^z x(t) = {}_a^{C}D_t^z x(t) = x^{(z)}(t), \tag{1} $$

$$ {}_t^{RL}D_b^z x(t) = {}_t^{C}D_b^z x(t) = (-1)^z x^{(z)}(t). \tag{2} $$

The relation between Riemann-Liouville fractional derivative and Caputo fractional derivative is as follows:

$$ {}_t^{C}D_b^\alpha f(t) = {}_t^{RL}D_b^\alpha f(t) - \sum_{k=0}^{i-1} \frac{f^{(k)}(b)}{\Gamma(k+1-\alpha)} (b-t)^{k-\alpha}. \tag{3} $$

These operators have the following properties

$$ {}_{t}^{RL}D_{b}^{\alpha}(b-t)^{B-1} = \frac{\Gamma(B)}{\Gamma(B-\alpha)}(b-t)^{B-\alpha-1}, \qquad \alpha, B > 0, \tag{4} $$

$$ \|{}_{t}I_{b}^{\alpha}f(t)\|_{L^2(a,b)} \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|f(t)\|_{L^2(a,b)}, \tag{5} $$

$$ {}_{t}^{C}D_{b}^{\alpha}f(t) = (-1)^{i}{}_{t}I_{b}^{i-\alpha}D^{i}f(t). \tag{6} $$

2.2 The Bernoulli hybrid functions

The Bernoulli hybrid functions $b_{nm}(t)$ for $n = 1, 2, \ldots, N$ and $m = 0, 1, 2, \ldots, M$ are defined on the interval $[0, 1)$ by the following relation [24]

$$ b_{nm}(t) = \begin{cases} \beta_m(Nt-n+1), & t \in [\frac{n-1}{N}, \frac{n}{N}), \\ 0 & \text{otherwise}, \end{cases} \tag{7} $$

where n and m are the orders of the Block-pulse functions and Bernoulli polynomials, respectively. The Bernoulli polynomials of order m can be defined as: [28]

$$ \beta_m(t) = \sum_{k=0}^{m} \binom{m}{k} \alpha_{m-k} t^k, \tag{8} $$

where $\alpha_k = \beta_k(0), k = 0, 1, 2, \ldots, m$, are Bernoulli numbers. The first few Bernoulli polynomials are given as:

$$ \beta_0(t) = 1, \quad \beta_1(t) = t - \frac{1}{2}, \quad \beta_2(t) = t^2 - t + \frac{1}{6}, \quad \beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t. $$

The Bernoulli polynomials satisfy the following property

$$ \beta_m(1-t) = (-1)^m \beta_m(t). \tag{9} $$

2.3 Function approximation

It is obvious that

$$ Y = \text{span}\{b_{10}(t), b_{20}(t), \ldots, b_{N0}(t), b_{11}(t), \ldots, b_{N1}(t), \ldots, b_{1M}(t), \ldots, b_{NM}(t)\}, \tag{10} $$

is a finite dimensional and closed subspace of the Hilbert space $H = L^2[0, 1]$. Therefore, Y is a complete subspace and for each $f \in H$, there is a unique best approximation out of Y such as $f_{NM} \in Y$, that is $\forall y \in Y, \|f - f_{NM}\| \leq \|f - y\|$. Since $f_{NM} \in Y$, there are unique coefficients $a_{10}, a_{20}, \ldots, a_{NM}$, so that [29]

$$f \simeq f_{NM} = \sum_{n=1}^{N} \sum_{m=0}^{M} a_{nm} b_{nm}(t) = A^T \Psi(t), \tag{11}$$

where A and $\Psi(t)$ are the following vectors:

$$A^T = \left[a_{10}, a_{20}, \ldots, a_{N0}, a_{11}, \ldots, a_{N1}, \ldots, a_{1M}, \ldots, a_{NM}\right], \tag{12}$$

$$\Psi^T(t) = \left[b_{10}(t), b_{20}(t), \ldots, b_{N0}(t), b_{11}(t), \ldots, b_{N1}(t), \ldots, b_{1M}(t) \ldots, b_{NM}(t)\right]. \tag{13}$$

3 Fractional derivative operators for hybrid functions

The first step in developing our numerical methods is to compute the left Caputo fractional derivative and right Riemann-Liouville fractional derivative operators of the Bernoulli hybrid functions to be applicable in the Ritz method. We display these operators by D_L^α and D_R^α, respectively. Without the loss of generality, we consider the interval be $[0, 1]$ and $0 < \alpha \leq 1$.

3.1 The left Caputo fractional derivative for Bernoulli hybrid functions

From the Bernoulli hybrid functions (7), $tb_{nm}(t)$ can be equivalently defined as:

$$tb_{nm}(t) = u_{\frac{n-1}{N}}(t) t \beta_m(Nt - n + 1) - u_{\frac{n}{N}}(t) t \beta_m(Nt - n + 1), \tag{14}$$

where $u_r(t)$ is a unit step function defined as:

$$u_r(t) = \begin{cases} 1 & t \geq r, \\ 0 & t < r \end{cases}$$

By using the definition of the Laplace transform and Bernoulli hybrid functions (7) and equations (8), (9) and (14), we have

$$\begin{aligned}
\mathcal{L}\{tb_{nm}(t)\} &= \int_0^{\infty} e^{-st}u_{\frac{n-1}{N}}(t)t\beta_m(Nt-n+1)dt - \int_0^{\infty} e^{-st}u_{\frac{n}{N}}(t)t\beta_m(Nt-n+1)dt \\
&= \int_{\frac{n-1}{N}}^{\infty} e^{-st}t\beta_m(Nt-n+1)dt - \int_{\frac{n}{N}}^{\infty} e^{-st}t\beta_m(Nt-n+1)dt \\
&= e^{-\frac{n-1}{N}s}\int_{\frac{n-1}{N}}^{\infty} e^{-s\left(t-\frac{n-1}{N}\right)}\left(t-\frac{n-1}{N}+\frac{n-1}{N}\right)\beta_m\left(N\left(t-\frac{n-1}{N}\right)\right)dt \\
&\quad -e^{-\frac{n}{N}s}\int_{\frac{n}{N}}^{\infty} e^{-s\left(t-\frac{n}{N}\right)}\left(t-\frac{n}{N}+\frac{n}{N}\right)\beta_m\left(1-(-N)\left(t-\frac{n}{N}\right)\right)dt \\
&= e^{-\frac{n-1}{N}s}\left(\int_{\frac{n-1}{N}}^{\infty}\left(\sum_{k=0}^{m} e^{-s\left(t-\frac{n-1}{N}\right)}\binom{m}{k}\alpha_{m-k}N^k\left(t-\frac{n-1}{N}+\frac{n-1}{N}\right)\left(t-\frac{n-1}{N}\right)^k\right)dt\right) \\
&\quad -e^{-\frac{n}{N}s}\left(\int_{\frac{n}{N}}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k}\alpha_{m-k}(-1)^m(-N)^k e^{-s\left(t-\frac{n}{N}\right)}\left(t-\frac{n}{N}+\frac{n}{N}\right)\left(t-\frac{n}{N}\right)^k\right)dt\right),
\end{aligned}$$

using integration by the change of variable, the Laplace transform is obtained as follows:

$$\begin{aligned}
\mathcal{L}\{tb_{nm}(t)\} &= e^{-\frac{n-1}{N}s}\left(\sum_{k=0}^{m}\binom{m}{k}\alpha_{m-k}N^k\frac{\Gamma(k+2)}{s^{k+2}}+\left(\frac{n-1}{N}\right)\sum_{k=0}^{m}\binom{m}{k}\alpha_{m-k}N^k\frac{\Gamma(k+1)}{s^{k+1}}\right) \\
&\quad -e^{-\frac{n}{N}s}\left(\sum_{k=0}^{m}(-1)^m\binom{m}{k}\alpha_{m-k}(-N)^k\frac{\Gamma(k+2)}{s^{k+2}}\right. \\
&\quad \left.+\left(\frac{n}{N}\right)\sum_{k=0}^{m}(-1)^m\binom{m}{k}\alpha_{m-k}(-N)^k\frac{\Gamma(k+1)}{s^{k+1}}\right). \qquad (15)
\end{aligned}$$

By applying the derivative Laplace transform properties and equation (15), we get

$$\begin{aligned}
\mathcal{L}\{(tb_{nm}(t))'\} &= s\mathcal{L}\{tb_{nm}(t)\} \\
&= e^{-\frac{n-1}{N}s}\left(\sum_{k=0}^{m}\binom{m}{k}\alpha_{m-k}N^k\frac{\Gamma(k+2)}{s^{k+1}}+\left(\frac{n-1}{N}\right)\sum_{k=0}^{m}\binom{m}{k}\alpha_{m-k}N^k\frac{\Gamma(k+1)}{s^{k}}\right) \\
&\quad -e^{-\frac{n}{N}s}\left(\sum_{k=0}^{m}(-1)^m\binom{m}{k}\alpha_{m-k}(-N)^k\frac{\Gamma(k+2)}{s^{k+1}}\right. \\
&\quad \left.+\left(\frac{n}{N}\right)\sum_{k=0}^{m}(-1)^m\binom{m}{k}\alpha_{m-k}(-N)^k\frac{\Gamma(k+1)}{s^{k}}\right). \qquad (16)
\end{aligned}$$

By taking laplace transform of the Convolution $\frac{1}{\Gamma(1-\alpha)t^{\alpha}}$ and $(tb_{nm}(t))'$, we obtain

$$\begin{aligned}
\mathcal{L}\left[{}_0^C D_t^{\alpha}(tb_{nm}(t))\right] &= \mathcal{L}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}(\tau b_{nm}(\tau))' d\tau\right] \\
&= \mathcal{L}\left[\frac{1}{\Gamma(1-\alpha)t^{\alpha}} * (tb_{nm}(t))'\right] = \frac{1}{s^{1-\alpha}} \times \mathcal{L}\{(tb_{nm}(t))'\} \\
&= e^{-\frac{n-1}{N}s}\left(\sum_{k=0}^{m}\binom{m}{k}\alpha_{m-k}N^k\frac{\Gamma(k+2)}{s^{k+2-\alpha}}+\left(\frac{n-1}{N}\right)\sum_{k=0}^{m}\binom{m}{k}\alpha_{m-k}N^k\frac{\Gamma(k+1)}{s^{k+1-\alpha}}\right) \\
&\quad -e^{-\frac{n}{N}s}\left(\sum_{k=0}^{m}(-1)^m\binom{m}{k}\alpha_{m-k}(-N)^k\frac{\Gamma(k+2)}{s^{k+2-\alpha}}\right. \\
&\quad \left.+\left(\frac{n}{N}\right)\sum_{k=0}^{m}(-1)^m\binom{m}{k}\alpha_{m-k}(-N)^k\frac{\Gamma(k+1)}{s^{k+1-\alpha}}\right). \qquad (17)
\end{aligned}$$

By taking inverse Laplace transform of relation (17), the left Caputo fractional derivative is derived as follows:

$$\begin{aligned}
{}_0^C D_t^\alpha (tb_{nm}(t)) &= u_{\frac{n-1}{N}}(t)\sum_{k=0}^{m}\binom{m}{k}\alpha_{m-k}N^k\frac{\Gamma(k+2)}{\Gamma(k+2-\alpha)}\left(t-\frac{n-1}{N}\right)^{k+1-\alpha}\\
&+u_{\frac{n-1}{N}}(t)\left(\frac{n-1}{N}\right)\sum_{k=0}^{m}\binom{m}{k}\alpha_{m-k}N^k\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\left(t-\frac{n-1}{N}\right)^{k-\alpha}\\
&-u_{\frac{n}{N}}(t)\sum_{k=0}^{m}(-1)^m\binom{m}{k}\alpha_{m-k}(-N)^k\frac{\Gamma(k+2)}{\Gamma(k+2-\alpha)}\left(t-\frac{n}{N}\right)^{k+1-\alpha}\\
&-u_{\frac{n}{N}}(t)\left(\frac{n}{N}\right)\sum_{k=0}^{m}(-1)^m\binom{m}{k}\alpha_{m-k}(-N)^k\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\left(t-\frac{n}{N}\right)^{k-\alpha}.
\end{aligned} \tag{18}$$

The relation (18) can be rewritten as follows:

$$D_L^\alpha (tb_{nm}(t)) = {}_0^C D_t^\alpha (tb_{nm}(t)) = \begin{cases} 0, & t\in[0,\frac{n-1}{N}),\\ D_{nm}(t), & t\in[\frac{n-1}{N},\frac{n}{N}),\\ D_{nm}(t)-E_{nm}(t), & t\in[\frac{n}{N},1),\end{cases} \tag{19}$$

where

$$D_{nm}(t)=\sum_{k=0}^{m}\left[\binom{m}{k}\alpha_{m-k}N^k\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\left(t-\frac{n-1}{N}\right)^{k-\alpha}\left(\frac{k+1}{k+1-\alpha}\left(t-\frac{n-1}{N}\right)+\frac{n-1}{N}\right)\right],$$

$$E_{nm}(t)=\sum_{k=0}^{m}\left[(-1)^m\binom{m}{k}\alpha_{m-k}(-N)^k\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\left(t-\frac{n}{N}\right)^{k-\alpha}\left(\frac{k+1}{k+1-\alpha}\left(t-\frac{n}{N}\right)+\frac{n}{N}\right)\right].$$

3.2 The right Riemann-Liouville fractional derivative for Bernoulli hybrid functions

We divide the interval $[0,1)$ into subintervals including $[0,\frac{n-1}{N})$, $[\frac{n-1}{N},\frac{n}{N})$ and $[\frac{n}{N},1)$, $n=1,2,\dots,N$, coordinating with the definition of hybrid functions. Suppose $t\in[\frac{n-1}{N},\frac{n}{N})$, from the definition 2 and Bernoulli hybrid functions (7) and equation (9), we have

$$\begin{aligned}
{}_t^{RL} D_1^\alpha (tb_{nm}(t)) &= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^1(\tau-t)^{-\alpha}\tau b_{nm}(\tau)d\tau\\
&= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\int_t^{\frac{n}{N}}(\tau-t)^{-\alpha}\tau b_{nm}(\tau)d\tau+\int_{\frac{n}{N}}^1(\tau-t)^{-\alpha}\tau b_{nm}(\tau)d\tau\right)\\
&= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^{\frac{n}{N}}(\tau-t)^{-\alpha}\tau\beta_m(N\tau-n+1)d\tau\\
&= {}_t^{RL} D_{\frac{n}{N}}^\alpha (t\beta_m(Nt-n+1)) = {}_t^{RL} D_{\frac{n}{N}}^\alpha \left(t\beta_m\left(1-N\left(\frac{n}{N}-t\right)\right)\right)\\
&= {}_t^{RL} D_{\frac{n}{N}}^\alpha \left(-\left(\left(\frac{n}{N}-t-\frac{n}{N}\right)\right)\left((-1)^m\beta_m\left(N\left(\frac{n}{N}-t\right)\right)\right)\right),
\end{aligned}$$

from equations (4) and (8), we get

$$\begin{aligned}{}_t^{RL}D^{\alpha}_{\frac{n}{N}}\left(t\beta_m(Nt-n+1)\right) &= -{}_t^{RL}D^{\alpha}_{\frac{n}{N}}\left(\left(\left(\frac{n}{N}-t-\frac{n}{N}\right)\right)\left(\sum_{k=0}^{m}(-1)^m\binom{m}{k}N^k\alpha_{m-k}\left(\frac{n}{N}-t\right)^k\right)\right)\\ &= -\left(\sum_{k=0}^{m}\left[(-1)^m\binom{m}{k}N^k\alpha_{m-k}\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\left(\frac{n}{N}-t\right)^{k-\alpha}\right.\right.\\ &\quad\left.\left.\times\left(\frac{(k+1)}{(k+1-\alpha)}\left(\frac{n}{N}-t\right)-\frac{n}{N}\right)\right]\right). \qquad (20)\end{aligned}$$

If $t \in [0, \frac{n-1}{N})$, then from the definition 2 and Bernoulli hybrid functions definition (7), we have

$$\begin{aligned}{}_t^{RL}D^{\alpha}_{1}(tb_{nm}(t)) &= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^1(\tau-t)^{-\alpha}\tau b_{nm}(\tau)d\tau\\ &= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\int_t^{\frac{n-1}{N}}(\tau-t)^{-\alpha}\tau b_{nm}(\tau)d\tau\right.\\ &\left.+\int_{\frac{n-1}{N}}^{\frac{n}{N}}(\tau-t)^{-\alpha}\tau b_{nm}(\tau)d\tau+\int_{\frac{n}{N}}^{1}(\tau-t)^{-\alpha}\tau b_{nm}(\tau)d\tau\right)\\ &= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{\frac{n-1}{N}}^{\frac{n}{N}}(\tau-t)^{-\alpha}\tau\beta_m(N\tau-n+1)d\tau\\ &= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\int_t^{\frac{n}{N}}(\tau-t)^{-\alpha}\tau\beta_m(N\tau-n+1)d\tau-\int_t^{\frac{n-1}{N}}(\tau-t)^{-\alpha}\tau\beta_m(N\tau-n+1)d\tau\right)\\ &= {}_t^{RL}D^{\alpha}_{\frac{n}{N}}\left(t\beta_m(Nt-n+1)\right)-{}_t^{RL}D^{\alpha}_{\frac{n-1}{N}}\left(t\beta_m(Nt-n+1)\right),\end{aligned}$$

${}_t^{RL}D^{\alpha}_{\frac{n}{N}}(t\beta_m(Nt-n+1))$ is equal to equation (20), and ${}_t^{RL}D^{\alpha}_{\frac{n-1}{N}}(t\beta_m(Nt-n+1))$ is obtained as follows:

$$\begin{aligned}&{}_t^{RL}D^{\alpha}_{\frac{n-1}{N}}(t\beta_m(Nt-n+1))\\ &= -{}_t^{RL}D^{\alpha}_{\frac{n-1}{N}}\left(\left(\frac{n-1}{N}-t-\frac{n-1}{N}\right)\beta_m\left(-N\left(\frac{n-1}{N}-t\right)\right)\right)\\ &= -{}_t^{RL}D^{\alpha}_{\frac{n-1}{N}}\left(\left(\frac{n-1}{N}-t-\frac{n-1}{N}\right)\sum_{k=0}^{m}\binom{m}{k}(-N)^k\alpha_{m-k}\left(\frac{n-1}{N}-t\right)^k\right)\\ &= -{}_t^{RL}D^{\alpha}_{\frac{n-1}{N}}\left(\sum_{k=0}^{m}\binom{m}{k}(-N)^k\alpha_{m-k}\left(\left(\frac{n-1}{N}-t\right)^{k+1}-\frac{n-1}{N}\left(\frac{n-1}{N}-t\right)^k\right)\right),\end{aligned}$$

By using equation (4), we have

$$\begin{aligned}{}_t^{RL}D^{\alpha}_{\frac{n-1}{N}}(t\beta_m(Nt-n+1)) &= -\sum_{k=0}^{m}\left[\binom{m}{k}(-N)^k\alpha_{m-k}\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\left(\frac{n-1}{N}-t\right)^{k-\alpha}\right.\\ &\quad\left.\times\left(\frac{(k+1)}{(k+1-\alpha)}\left(\frac{n-1}{N}-t\right)-\frac{n-1}{N}\right)\right]. \qquad (21)\end{aligned}$$

From the equations (20) and (21), we have:

$$D^{\alpha}_{R}(tb_{nm}(t)) = {}_t^{RL}D^{\alpha}_{1}(tb_{nm}(t)) = \begin{cases} R_{nm}(t)-S_{nm}(t), & t\in[0,\frac{n-1}{N}),\\ R_{nm}(t), & t\in[\frac{n-1}{N},\frac{n}{N}),\\ 0, & t\in[\frac{n}{N},1),\end{cases} \qquad (22)$$

where

$$\begin{aligned} R_{nm}(t) &= (-1)^{m+1}\sum_{k=0}^{m}\left[\binom{m}{k}N^k\alpha_{m-k}\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\left(\frac{n}{N}-t\right)^{k-\alpha}\right. \\ &\quad \left.\times\left(\frac{(k+1)}{(k+1-\alpha)}\left(\frac{n}{N}-t\right)-\frac{n}{N}\right)\right], \\ S_{nm}(t) &= -\sum_{k=0}^{m}\left[\binom{m}{k}(-N)^k\alpha_{m-k}\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\left(\frac{n-1}{N}-t\right)^{k-\alpha}\right. \\ &\quad \left.\times\left(\frac{(k+1)}{(k+1-\alpha)}\left(\frac{n-1}{N}-t\right)-\frac{n-1}{N}\right)\right]. \end{aligned}$$

4 Description of the numerical method

In this section, an indirect method based on hybrid functions and fractional derivative operators fitted with the Ritz method is presented for solving following FOCP: [30, 31]

$$\min J[u] = \int_{t_0}^{t_f} f(t, x(t), u(t))dt, \tag{23}$$

subject to the dynamical system

$${}_{t_0}^{C}D_t^{\alpha}x(t) = g(t, x(t)) + b(t)u(t), \tag{24}$$

with the initial condition

$$x(t_0) = x_0, \tag{25}$$

where $0 < \alpha \leq 1, b(t) \neq 0$.

The Hamiltonian function for the problem (23) to (25) is defined as follows:

$$H(x(t), u(t), \lambda(t)) = f(t, x(t), u(t)) + \lambda^T(t)(g(t, x(t)) + b(t)u(t)), \tag{26}$$

the necessary optimality conditions are obtained in the following theorem.

Theorem 1. Let $(x(t), u(t))$ be the optimal solution of the problem (23) to (25), then there exists a costate function $\lambda(t)$ such that

$$\begin{aligned} &{}_{t}^{RL}D_{t_f}^{\alpha}\lambda(t) = \frac{\partial H}{\partial x} = \frac{\partial f}{\partial x}(t, x(t), u(t)) + \lambda^T(t)\frac{\partial g}{\partial x}(t, x(t)), \\ &\frac{\partial H}{\partial u} = \frac{\partial f}{\partial u}(t, x(t), u(t)) + \lambda^T(t)b(t) = 0, \\ &{}_{t_0}^{C}D_t^{\alpha}x(t) = \frac{\partial H}{\partial \lambda} = g(t, x(t)) + b(t)u(t), \quad x(t_0) = x_0, \lambda(t_f) = 0. \end{aligned} \tag{27}$$

Proof. See [4]. $\square$

From equation $\dfrac{\partial H}{\partial u} = 0$, we obtain $u(t)$ in the terms of $\lambda(t)$ and $x(t)$. By substituting the obtained $u(t)$ in other equations (27), the necessary optimality conditions of the problem are determined in the form of

$$\begin{cases} {}_t^{RL}D_{t_f}^{\alpha}\lambda(t) = \theta(t, x(t), \lambda(t)), \\ {}_{t_0}^{C}D_t^{\alpha}x(t) = \vartheta(t, x(t), \lambda(t)), \\ x(t_0) = x_0, \quad \lambda(t_f) = 0, \end{cases} \tag{28}$$

where the functions θ and ϑ are known. We apply the spectral Ritz method to solve the produced TPBVP from the necessary optimality conditions (27). Considering the given initial-boundary conditions, the state and costate functions are estimated as follows:

$$\begin{aligned} x_{NM}(t) &= \phi_1(t)C_{NM}\Psi_{NM}(t) + \Psi_1(t), \\ \lambda_{NM}(t) &= \phi_2(t)\widetilde{C}_{NM}\Psi_{NM}(t) + \Psi_2(t), \end{aligned} \tag{29}$$

where $\Psi_{NM}(t)$ is the Bernoulli hybrid functions vector (13), C_{NM} and $\widetilde{C}_{NM}$ are the following unknown coefficients vectors with appropriate dimensions and will be determined after the numerical approach.

$$C_{NM}^T = [c_{10}, c_{20}, \ldots, c_{N0}, c_{11}, \ldots, c_{N1}, \ldots, c_{1M}, \ldots, c_{NM}],$$

$$\widetilde{C}_{NM}^T = [\widetilde{c}_{10}, \widetilde{c}_{20}, \ldots, \widetilde{c}_{N0}, \widetilde{c}_{11}, \ldots, \widetilde{c}_{N1}, \ldots, \widetilde{c}_{1M}, \ldots, \widetilde{c}_{NM}].$$

Remark 1. It is worth to note that, in the case that the state and costate functions are vectors instead of scalar functions, these coefficients must be regulated to have a matrix form instead of a vector to approximate all the components of the unknown state and costate parameters.

The trial functions of $\phi_i(t)$ and $\psi_i(t)$ for $i = 1, 2$ in (29) are utilized to satisfy the approximate functions in the given initial and boundary conditions. These functions can be simply taken as:

$$\phi_1(t) = t - t_0, \quad \phi_2(t) = t - t_f, \quad \psi_1(t) = x_0, \quad \psi_2(t) = \lambda(t_f). \tag{30}$$

Putting the trial functions (30) into the approximate functions (29) and replacing them into the optimality conditions (28) implies

$$\begin{cases} {}_t^{RL}D_{t_f}^{\alpha}\lambda_{NM}(t) = \theta(t, x_{NM}(t), \lambda_{NM}(t)), \\ {}_{t_0}^{C}D_t^{\alpha}x_{NM}(t) = \vartheta(t, x_{NM}(t), \lambda_{NM}(t)), \end{cases} \tag{31}$$

by applying the left Caputo fractional derivative operator (19) and right Riemann-Liouville fractional derivative operator (22) in equations (31), a system of algebraic equation is obtained. We collocate these equations at equidistant collocation points $t_i, i = 0, 1, \dots, N(M+1)-1$, on the interval $[t_0, t_f]$. we obtain the algebraic system including $2N(M+1)$ equations as follows:

$$\begin{cases} D_R^\alpha \lambda_{NM}(t_i) - \theta(t_i, x_{NM}(t_i), \lambda_{NM}(t_i)) = 0, \\ D_L^\alpha x_{NM}(t_i) - \vartheta(t_i, x_{NM}(t_i), \lambda_{NM}(t_i)) = 0. \end{cases}$$

Therefore, the recent system of algebraic equations is solved by a standard numerical method to find the unknown coefficients. As a result, all the unknown functions of state and costate will be determined.

5 Error Bound for the numerical method

In this section, the error bounds of the Bernoulli hybrid functions, fractional derivative operators, and the proposed method are presented. Let $H^r(a,b)$ be the vector space of the functions $f \in L^2(a,b)$ of order up to r differentiable as

$$H^r(a,b) = \{f \in L^2(a,b): \quad for \quad 0 \le k \le r, \quad f^{(k)} \in L^2(a,b)\}. \tag{32}$$

It can be shown that, the space $H^r(a,b)$ is a Hilbert space associated with the following norm [32]

$$\|f\|_{H^r(a,b)} = \left(\sum_{k=0}^{r} \left\| \frac{d^k f}{dt^k} \right\|^2_{L^2(a,b)} \right)^{\frac{1}{2}}. \tag{33}$$

Theorem 2. Let $f \in H^r(0,1)$ with $r \ge 0$, and $M \ge r-1$. If the function f is approximated by f_{NM} as in equation (11), then

$$\|f - f_{NM}\|_{L^2(0,1)} \le cM^{-r}N^{-r}\|f^{(r)}\|_{L^2(0,1)}, \tag{34}$$

and for $k \ge 1$,

$$\|f - f_{NM}\|_{H^k(0,1)} \le cM^{2k-\frac{1}{2}-r}N^{k-r}\|f^{(r)}\|_{L^2(0,1)}. \tag{35}$$

Proof. See [19]. □

Theorem 3. We assume $f \in H^r(0,1)$ with $r \ge 0$ and $i-1 < \alpha \le i$, then, the error bound for the left Caputo fractional derivative operator is obtained as:

$$\left\| {}_0^C D_t^\alpha f(t) - {}_0^C D_t^\alpha f_{NM}(t) \right\|_{L^2(0,1)} \le \frac{cM^{2k-\frac{1}{2}-r}N^{k-r}\|f^{(r)}\|_{L^2(0,1)}}{\Gamma(i-\alpha+1)}, \tag{36}$$

where $1 < k < r$.

Proof. By using definition 3 and Convolution product, we have

$$\begin{aligned} {}_0^C D_t^\alpha f(t) - {}_0^C D_t^\alpha f_{NM}(t) &= \frac{1}{\Gamma(i-\alpha)} \int_0^t (t-\tau)^{i-\alpha-1} \left(f^{(i)}(\tau) - f_{NM}^{(i)}(\tau) \right) d\tau \\ &= \frac{1}{\Gamma(i-\alpha)} (t)^{i-\alpha-1} * \left(f(t) - f_{NM}(t) \right)^{(i)} . \end{aligned}$$

Young's inequality, $\|f * g\|_p \leq \|f\|_1 \|g\|_p$, yield

$$\begin{aligned} & \left\| {}_0^C D_t^\alpha f(t) - {}_0^C D_t^\alpha f_{NM}(t) \right\|_{L^2(0,1)}^2 \\ = & \left\| \frac{1}{\Gamma(i-\alpha)} (t)^{i-\alpha-1} * \left(f(t) - f_{NM}(t) \right)^{(i)} \right\|_{L^2(0,1)}^2 \\ \leq & \left(\int_0^1 \frac{1}{\Gamma(i-\alpha)} (t)^{i-\alpha-1} dt \right)^2 \left\| \left(f(t)^{(i)} - f_{NM}(t)^{(i)} \right) \right\|_{L^2(0,1)}^2 \\ \leq & \frac{1}{(\Gamma(i-\alpha+1))^2} \left\| f(t) - f_{NM}(t) \right\|_{H^k(0,1)}^2 . \end{aligned}$$

By using equation (35), the relation (36) is obtained. □

Theorem 4. Suppose $f \in H^r(0,1)$ with $r \geq 0$ and $i-1 < \alpha \leq i$, and $f^{(j)}(1) = 0, j = 0, \dots, i-1$, then the error bound for the right Riemann-Liouville fractional derivative operator is obtained as:

$$\left\| {}_t^{RL} D_1^\alpha f(t) - {}_t^{RL} D_1^\alpha f_{NM}(t) \right\|_{L^2(0,1)} \leq \frac{c M^{2k-\frac{1}{2}-r} N^{k-r} \|f^{(r)}\|_{L^2(0,1)}}{\Gamma(i-\alpha+1)}, \tag{37}$$

where $1 < k < r$.

Proof. From equation (3), we have

$${}_t^{RL} D_1^\alpha f(t) = {}_t^C D_1^\alpha f(t),$$

by using the equations (5) and (6), we get

$$\begin{aligned} & \left\| {}_t^{RL} D_1^\alpha f(t) - {}_t^{RL} D_1^\alpha f_{NM}(t) \right\|_{L^2(0,1)}^2 \\ = & \left\| {}_t^C D_1^\alpha f(t) - {}_t^C D_1^\alpha f_{NM}(t) \right\|_{L^2(0,1)}^2 \\ = & \left\| {}_t I_1^{i-\alpha} D^{(i)} \left(f(t) - f_{NM}(t) \right) \right\|_{L^2(0,1)}^2 \\ \leq & \left(\frac{1}{\Gamma(i-\alpha+1)} \right)^2 \left\| D^{(i)} \left(f(t) - f_{NM}(t) \right) \right\|_{L^2(0,1)}^2 \\ \leq & \left(\frac{1}{\Gamma(i-\alpha+1)} \right)^2 \left\| f(t) - f_{NM}(t) \right\|_{H^k(0,1)}^2 \end{aligned}$$

from the inequality (35), the theorem is proved. □

In the following theorem, we estimate the error of the proposed method with respect to the hybrid functions order N, M.

Theorem 5. Consider $x(t)$ and $\lambda(t) \in H^r(0,1)$ are the accurate optimal solutions of the FOCP (23) to (25), with the approximate solutions $x_{NM}(t)$ and $\lambda_{NM}(t)$, which are achieved in the numerical from (29). Also $\theta(t, x(t), \lambda(t))$ and $\vartheta(t, x(t), \lambda(t))$ are Lipschitz functions, with the Lipschitz constants θ_i, ϑ_i, for $i = 1, 2$, respectively.The error bounds of the equations (31) which are shown with ϵ_1 and ϵ_2, are achieved for the mentioned method as follows:

$$\begin{aligned}\|\epsilon_1\|_{L^2(0,1)} &\leq \left(\frac{cM^{2k-\frac{1}{2}-r}N^{k-r}}{\Gamma(2-\alpha)} + \theta_2 cM^{-r}N^{-r}\right) \\ &\times\|\lambda^{(r)}\|_{L^2(0,1)} + \theta_1 cM^{-r}N^{-r}\|x^{(r)}\|_{L^2(0,1)},\end{aligned}$$

$$\begin{aligned}\|\epsilon_2\|_{L^2(0,1)} &\leq \left(\frac{cM^{2k-\frac{1}{2}-r}N^{k-r}}{\Gamma(2-\alpha)} + \vartheta_1 cM^{-r}N^{-r}\right) \\ &\times\|x^{(r)}\|_{L^2(0,1)} + \vartheta_2 cM^{-r}N^{-r}\|\lambda^{(r)}\|_{L^2(0,1)}.\end{aligned} \tag{38}$$

Proof. From first equation (31) and error bounds (34) to (37), we have

$$\begin{aligned}&\|\epsilon_1\|_{L^2(0,1)} \\ =& \left\|{}_t^{RL}D_1^\alpha \lambda(t) - \theta(t, x(t), \lambda(t)) - {}_t^{RL}D_1^\alpha \lambda_{NM}(t) + \theta(t, x_{NM}(t), \lambda_{NM}(t))\right\|_{L^2(0,1)} \\ \leq& \left\|{}_t^{RL}D_1^\alpha \lambda(t) - {}_t^{RL}D_1^\alpha \lambda_{NM}(t)\right\|_{L^2(0,1)} \\ &+ \|\theta(t, x(t), \lambda(t)) - \theta(t, x_{NM}(t), \lambda_{NM}(t))\|_{L^2(0,1)} \\ \leq& \left\|{}_t^{RL}D_1^\alpha \lambda(t) - {}_t^{RL}D_1^\alpha \lambda_{NM}(t)\right\|_{L^2(0,1)} + \theta_1\|x(t) - x_{NM}(t)\|_{L^2(0,1)} \\ &+\theta_2\|\lambda(t) - \lambda_{NM}(t)\|_{L^2(0,1)} \\ \leq& \frac{cM^{2k-\frac{1}{2}-r}N^{k-r}\|\lambda^{(r)}\|_{L^2(0,1)}}{\Gamma(2-\alpha)} + \theta_1 cM^{-r}N^{-r}\|x^{(r)}\|_{L^2(0,1)} \\ &+\theta_2 cM^{-r}N^{-r}\|\lambda^{(r)}\|_{L^2(0,1)} \\ =& \left(\frac{cM^{2k-\frac{1}{2}-r}N^{k-r}}{\Gamma(2-\alpha)} + \theta_2 cM^{-r}N^{-r}\right)\|\lambda^{(r)}\|_{L^2(0,1)} \\ &+\theta_1 cM^{-r}N^{-r}\|x^{(r)}\|_{L^2(0,1)}.\end{aligned}$$

In the same way, we get

$$\|\epsilon_2\|_{L^2(0,1)} = \left\|{}_0^C D_t^\alpha x(t) - \vartheta(t, x(t), \lambda(t)) - {}_{t_0}^C D_t^\alpha x_{NM}(t) + \vartheta(t, x_{NM}(t), \lambda_{NM}(t))\right\|_{L^2(0,1)},$$

where

$$\|\epsilon_2\|_{L^2(0,1)} \leq \left(\frac{cM^{2k-\frac{1}{2}-r}N^{k-r}}{\Gamma(2-\alpha)} + \vartheta_1 cM^{-r}N^{-r}\right)\|x^{(r)}\|_{L^2(0,1)} + \vartheta_2 cM^{-r}N^{-r}\|\lambda^{(r)}\|_{L^2(0,1)}.$$

Relations (38) show that the errors ϵ_1 and ϵ_2 tend to zero, when M and N increase and the approximate solution converges to the exact solution to the problem. □

6 Illustrative Examples

In this section, the proposed numerical scheme is applied to solve three FOCPs. Moreover, to show the superiority of the method, the numerical solutions are compared with the results reported in the previous works [15, 33, 34, 35]. Example 1 was studied in [15] by the Legendre spectral-collocation method. Also, this example was solved by the Ritz method and Legendre operational matrix directly [33]. In [34], Example 2 was solved by using a direct Epsilon-Ritz method. Example 3, including multiple components of the state function, is solved to show the convergence as well as the applicability of the suggested technique for more complicated problems.

Example 1. Consider the following time-invariant FOCP with quadratic performance index [15, 33]

$$\min J[u] = \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right] dt$$

subject to the given constraint

$${}_0^C D_t^\alpha x(t) = -x(t) + u(t),$$

with the given initial conditions as $x(0) = 1$. For $\alpha = 1$, the exact solution is given by $x(t) = \cosh(\sqrt{2}t) + \beta\sinh(\sqrt{2}t)$ and $u(t) = (1+\sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2}+\beta)\sinh(\sqrt{2}t)$, where

$$\beta = -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})},$$

and minimum value of cost function is $J = 0.192909$. To apply the proposed numerical scheme, first the following necessary optimality conditions are obtained:

$$\begin{cases} {}_0^C D_t^\alpha x(t) = -x(t) + u(t), \\ u(t) + \lambda(t) = 0, \\ {}_t^{RL} D_1^\alpha \lambda(t) = x(t) - \lambda(t), \\ x(0) = 1, \quad \lambda(1) = 0. \end{cases} \tag{39}$$

By replacing $u(t) = -\lambda(t)$ in first equation of (39) and applying relation (29), we have

$$\begin{cases} {}_0^C D_t^\alpha x_{NM}(t) = -x_{NM}(t) - \lambda_{NM}(t), \\ {}_t^{RL} D_1^\alpha \lambda_{NM}(t) = x_{NM}(t) - \lambda_{NM}(t). \end{cases} \tag{40}$$

Then the state and costate functions are approximated using the suggested trial functions. By applying fractional operators (19) and (22), and collocating equations (40) at the given nodes, a system of algebraic equations is achieved. Figures 1 and 2 show the exact and approximate state and control functions for $\alpha = 1$ and approximate solutions for different values of α. Table 1 demonstrates the cost function for different values of the basis function and fractional order variable. As shown in Table 1, the present method is superior to other methods utilized in the literature with the same conditions.

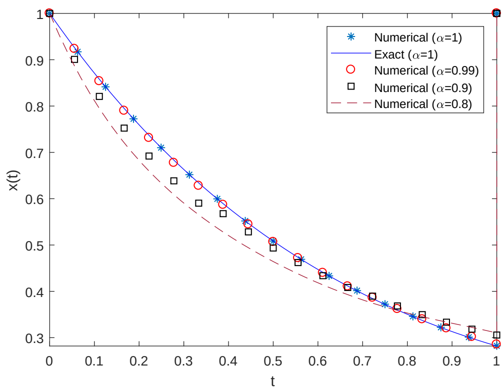

Figure 1: Exact and numerical solutions of state variable with N=1, M=8 for Example 1.

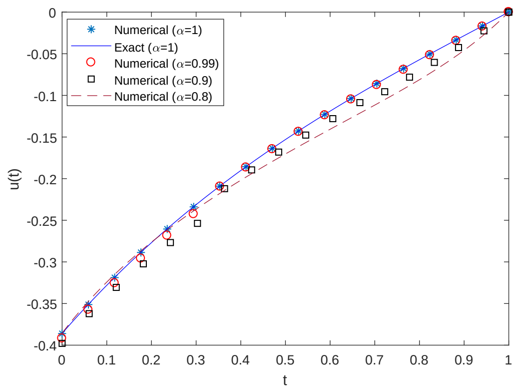

Figure 2: Exact and numerical solutions of control variable with N=1, M=8 for Example 1.

Example 2. Consider the following fractional optimal control problem as [34]:

$$\min J[u] = \int_0^1 \left[\left(x_1(t) - t^{\frac{3}{2}} - 1 \right)^2 + \left(x_2(t) - t^{\frac{5}{2}} \right)^2 + \left(u(t) - \frac{3\sqrt{\pi}}{4} t + t^{\frac{5}{2}} \right)^2 \right] dt$$

subject to the given constraint

$$\begin{cases} {}_0^C D_t^{\alpha} x_1(t) = x_2(t) + u(t), \\ {}_0^C D_t^{\alpha} x_2(t) = x_1(t) + \dfrac{15\sqrt{\pi}}{16} t^2 - t^{1.5} - 1, \end{cases}$$

with the given initial conditions as $x_1(0) = 1, x_2(0) = 0$. For this problem in $\alpha = 0.5, x_1(t) = 1 + t^{1.5}, x_2(t) = t^{2.5}$, and $u(t) = \dfrac{3\sqrt{\pi}}{4} t - t^{2.5}$ minimize the cost function and minimum value is $J = 0$. The necessary optimality conditions are as follows:

$$\begin{cases} {}_0^C D_t^\alpha x_1(t) = x_2(t) + u(t), \\ {}_0^C D_t^\alpha x_2(t) = x_1(t) + \dfrac{15\sqrt{\pi}}{16} t^2 - t^{1.5} - 1, \\ {}_t^{RL} D_1^\alpha \lambda_1(t) = \lambda_2(t) - 2t^{1.5} + 2x_1(t) - 2, \\ {}_t^{RL} D_1^\alpha \lambda_2(t) = \lambda_1(t) - 2t^{2.5} + 2x_2(t), \\ u(t) + \dfrac{1}{2}\lambda_1(t) - \dfrac{3\sqrt{\pi}}{4} t + t^{2.5} = 0, \\ x_1(0) = 1, \quad x_2(0) = 0, \quad \lambda_1(1) = 0, \quad \lambda_2(1) = 0. \end{cases} \tag{41}$$

The system of fractional differential equation is obtained using the proposed method. Table 2 demonstrates the approximate cost function with different choices of number basis hybrid functions. As the order of the basis increased, the performance index is improved. Moreover, our results are compared with the numerical method in [34] to show the efficiency of the proposed scheme. In Figures 3-5, the exact solution ($\alpha = 0.5$) as well as numerical solution are plotted for different choices of fractional order α.

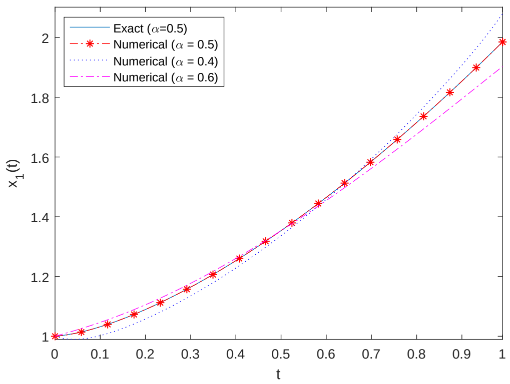

Figure 3: Comparison between exact and numerical solutions of $x_1(t)$ with N=2, M=5 for Example 2.

Example 3. Consider the following fractional optimal control problem as [35]

$$\min J[u] = \frac{1}{2}\int_0^1 \left[3x^2(t) + u^2(t)\right] dt$$

subject to the given constraint

$$\dot{x}(t) + 3{}_0^C D_t^\alpha x(t) = 4x(t) - 4u(t),$$

with the given initial condition as $x(0) = 1$. The exact solution of this problem for $\alpha = 1$ is given as follows:

$$x^*(t) = \frac{3}{3+e^4}e^{2t} + \frac{e^4}{3+e^4}e^{-2t}, \quad u^*(t) = \frac{3e^4}{3+e^4}e^{-2t} - \frac{3}{3+e^4}e^{2t}.$$

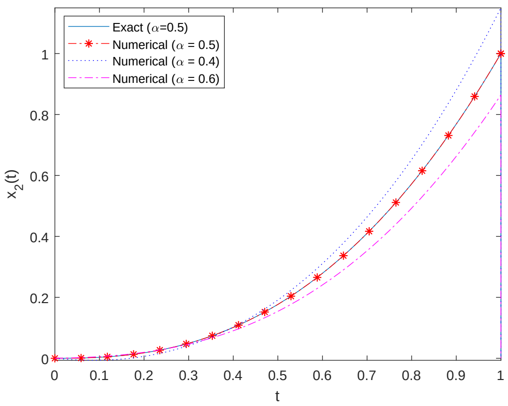

Figure 4: Comparison between exact and numerical solutions of $x_2(t)$ with N=2, M=5 for Example 2.

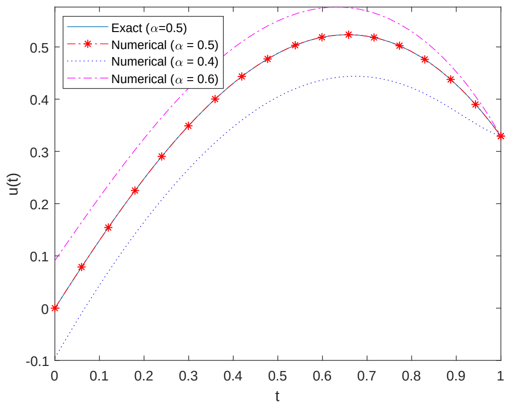

Figure 5: Comparison between exact and numerical solutions of $u(t)$ with N=2, M=5 for Example 2.

To find the solution, the proposed numerical scheme is applied. Applying the Pontryagin's minimum principle yields the following necessary conditions of optimality as: [15]

$$\begin{cases} \dot{x}(t) + 3{}_0^C D_t^\alpha x(t) = 4x(t) - 4u(t), \\ u(t) - 4\lambda(t) = 0, \\ 3{}_t^{RL} D_1^\alpha \lambda(t) - \dot{\lambda}(t) = 3x(t) + 4\lambda(t), \\ x(0) = 1, \quad \lambda(1) = 0. \end{cases} \tag{42}$$

For this problem, we take the state and costate functions as follows:

$$\begin{aligned} x_{NM}(t) &= tC_{NM}\Psi_{NM}(t) + 1, \\ \lambda_{NM}(t) &= (t-1)\widetilde{C}_{NM}\Psi_{NM}(t), \end{aligned} \tag{43}$$

For $\alpha = 1$, from equations (1) and (2), we obtain:

$$ {}_0^C D_t^1 x(t) = \dot{x}(t), \quad {}_t^{RL} D_1^1 \lambda(t) = -\dot{\lambda}(t). \tag{44} $$

By substituting relations (43) and (44) into system (42) and using the proposed method the unknown state and costate functions are approximated. Tables 3 and 4 demonstrate the absolute error of the exact and approximate state and control functions with respect to different values of N and M for $\alpha = 1$. Absolute error of the cost function for $\alpha = 1$ is presented in Table 5. In Figures 6 and 7, the exact and approximate $x(t)$ and $u(t)$ for different values of α are plotted. The absolute errors of state and control variables for $\alpha = 1$ are depicted in Figures 8 and 9. These results show that as the approximation orders of N and M increase, the error decrease. Also, the approximate solution convergences to the exact solution.

Table 1: Comparing the approximate cost function for different values of M, N and fractional order α with the existing works for Example 1.

N	M	α	LSCM(Alg.I) [15]	Ritz method [33]	Present method
1	3	0.99	0.195687	-	0.1919454
1	3	0.9	0.193929	-	0.1831375
1	3	0.8	0.193035	-	0.1737882
2	5	0.99	-	-	0.1917515
2	5	0.9	0.187676	-	0.1816494
2	5	0.8	-	-	0.1711172
1	8	1	-	0.1929094 ($m = n = 3$)	0.1929092

Table 2: The comparison of the optimal cost function J obtained for Example 2.

Method	Epsilon-Ritz Method [34]	Present Method	Exact Solution
Optimal Cost J^*	8.0027×10^{-6} $k = 8, \epsilon = 0.0001$	1.3665244×10^{-6} ($M = 2, N = 2$) 8.6126414×10^{-6} ($M = 3, N = 2$) $4.78442792 \times 10^{-8}$ ($M = 6, N = 1$)	0

Table 3: Absolute error of the state function for $\alpha = 1$ of Example 3.

t	$N = 1, M = 3$	$N = 2, M = 5$	$N = 1, M = 10$
0.1	9.614×10^{-4}	1.125×10^{-5}	4.418×10^{-11}
0.2	7.643×10^{-4}	5.801×10^{-6}	3.812×10^{-11}
0.3	3.299×10^{-4}	3.523×10^{-6}	3.417×10^{-11}
0.4	8.734×10^{-5}	2.827×10^{-6}	3.200×10^{-11}
0.5	8.225×10^{-5}	8.667×10^{-7}	3.077×10^{-11}
0.6	1.262×10^{-4}	1.243×10^{-6}	3.108×10^{-11}
0.7	1.945×10^{-5}	1.922×10^{-6}	3.229×10^{-11}
0.8	4.280×10^{-4}	3.592×10^{-6}	3.526×10^{-11}
0.9	7.623×10^{-4}	9.113×10^{-6}	3.987×10^{-11}

Table 4: Absolute error of the control function for $\alpha = 1$ of Example 3.

t	$N = 1, M = 3$	$N = 2, M = 5$	$N = 1, M = 10$
0.1	1.739×10^{-3}	2.411×10^{-5}	1.0345×10^{-10}
0.2	1.582×10^{-3}	2.634×10^{-5}	8.264×10^{-11}
0.3	1.884×10^{-3}	2.291×10^{-5}	6.461×10^{-11}
0.4	1.788×10^{-3}	1.801×10^{-5}	4.895×10^{-11}
0.5	1.322×10^{-3}	1.657×10^{-5}	3.544×10^{-11}
0.6	9.026×10^{-4}	1.585×10^{-5}	2.318×10^{-11}
0.7	9.210×10^{-4}	1.315×10^{-5}	1.206×10^{-11}
0.8	1.391×10^{-3}	1.295×10^{-5}	1.143×10^{-12}
0.9	1.634×10^{-3}	1.913×10^{-5}	9.664×10^{-12}

Table 5: Absolute error of the cost function for $\alpha = 1$ of Example 3.

Approximation Order	$N = 1, M = 3$	$N = 2, M = 5$	$N = 1, M = 10$
Optimal cost function	2.332×10^{-3}	2.890×10^{-5}	1.404×10^{-10}

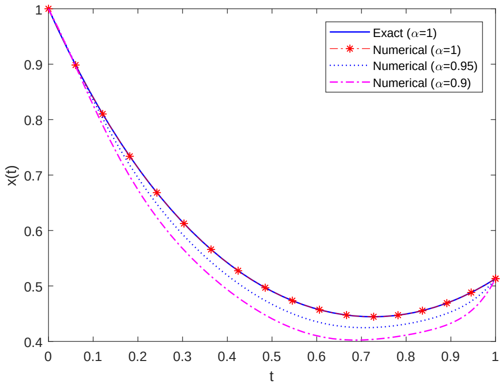

Figure 6: The exact and numerical state functions for different values of fractional order and N=1, M=10 for Example 3.

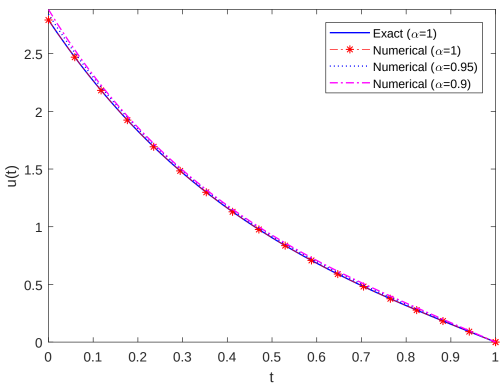

Figure 7: The exact and numerical control functions for different values of fractional order and N=1, M=10 for Example 3.

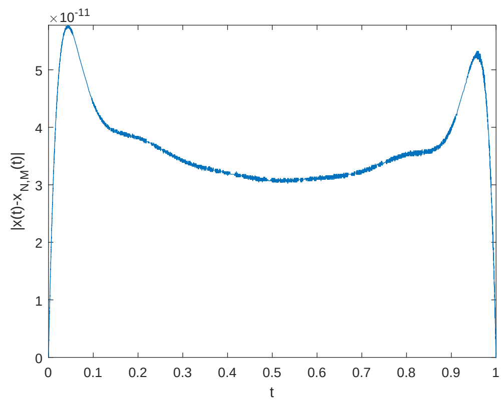

Figure 8: Error of the numerical state function for $\alpha = 1$ and $N = 1$, $M = 10$ for Example 3.

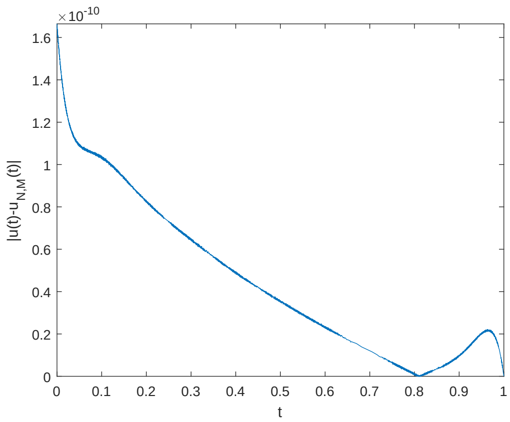

Figure 9: Error of the numerical control function for $\alpha = 1$ and $N = 1$, $M = 10$ for Example 3.

7 Conclusion

In this paper, a new method was presented for solving FOCPs. At first, the left Caputo fractional derivative and right Riemann-Liouville fractional derivative operators for Bernoulli hybrid functions were computed directly and without any approximation. Next, the necessary optimality conditions were used to transform the solution of FOCP to that of a set of FDEs. Also, the unknown state and costate functions were approximated by the hybrid functions and Ritz method. Then, using the fractional operators and collocation method, the resulted equations were reduced to a system of the algebraic equations. Afterward, the optimal results were determined by the solution to the algebraic system. Furthermore, the error bounds and convergence of the proposed method were discussed. And finally, the applicability and effectiveness of the proposed method were verified by solving some examples.

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یک روش عددی برای حل مسائل کنترل بهینه کسری با استفاده از روش طیفی و توابع ترکیبی

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چکیده

در این مقاله، یک روش جدید برای حل یک کلاس از مسائل کنترل بهینه کسری (FOCP) بر مبنای روش غیر مستقیم ارائه شده است. در ابتدا، شرایط لازم بهینگی برای مساله کنترل بهینه کسری به صورت دستگاهی شامل دو معادله دیفرانسیل کسری (FDEs) به دست می آید. سپس توابع مجهول توسط توابع ترکیبی شامل چند جمله ایهای برنولی و توابع بلاک پالس بر اساس روش طیفی ریتز، تقریب می شوند. همچنین دو روش جدید برای محاسبه عملگرهای مشتق چپ کسری کاپوتو و مشتق راست کسری ریمان-لیوویل توابع ترکیبی که متناسب با روش ریتز هستند، پیشنهاد می شود. مساله کنترل بهینه کسری با استفاده ازعملگرهای مشتق کسری و روش هم مکانی به یک سیستم معادلات جبری تبدیل می شود که با حل آن جواب مساله کنترل بهینه به دست می آید. برآورد خطای تقریب توابع ترکیبی، عملگرهای کسری و روش پیشنهادی، ارائه شده است. در نهایت، کارایی روش پیشنهادی و دقت آن در به دست آوردن جواب بهینه با حل چند مساله نمونه نشان داده شده است.

کلمات کلیدی

مساله کنترل بهینه کسری، توابع ترکیبی، چند جمله ایهای برنولی، روش ریتز، کران خطا.