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A General Scalar-Valued Gap Function for Nonsmooth Multiobjective Semi-Infinite Programming

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Abstract. For a nonsmooth multiobjective mathematical programming problem governed by infinitely many constraints, we define a new gap function that generalizes the definitions of this concept in other articles. Then, we characterize the efficient, weakly efficient, and properly efficient solutions of the problem utilizing this new gap function. Our results are based on $(\Phi, \rho)-$invexity, defined by Clarke subdifferential.

Keywords. Semi-infinite programming, Multiobjective optimization, Constraint qualification, Optimality conditions, Gap function

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1 Introduction

In this paper, we consider the following multiobjective semi-infinite programming problem (MSIP):

$$\begin{array}{lll}(\text{P}) & \inf & \big(f_1(x), f_2(x), \ldots, f_p(x)\big) \\ \text{s.t.} & g_t(x) \leq 0 \quad t \in T, & x \in \mathbb{R}^n,\end{array}$$

where $f_i,\ i \in I := \{1, 2, \ldots, p\}$ and $g_t,\ t \in T$ are locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$, and the index set $T \neq \emptyset$ is arbitrary, not necessarily finite. When T is finite, (P) is a multiobjective optimization problem, and when $p = 1$ and T is infinite, (P) is a semi-infinite optimization problem.

Necessary and sufficient optimality conditions for efficient, weakly efficient, and isolated efficient solutions of MSIP have been studied by many authors; see for instance [13, 18] in linear case, [12, 14] in convex case, [5] in smooth case, and [7, 11, 19, 20, 21, 23] in locally Lipshitz case. In almost all of the mentioned articles, the Karush-Kuhn-Tucker (KKT) type necessary conditions are justified for MSIPs under some constraint qualifications, and sufficient conditions are proved under several kinds of generalized convexity and generalized invexity. We know that the most general generalization of concept of invexity is $(\Phi, \rho)-$invexity, has been introduced by Caristi *et al.* in [5, 6] for smooth functions. Antczak and his coauthor presented the concept of $(\Phi, \rho)-$invexity for nonsmooth functions [1, 2], and Kanzi [19] extended this definition to a wider range of nonsmooth functions. In the present paper, we will use this most general form of $(\Phi, \rho)-$invexity.

On the other hand, the gap function for mathematical programming problems has been studied in various publications in recent years. Hearn [17] introduced a gap function for scalar convex optimization problems. Chen *et al.* [9] investigated a gap function for differentiable multiobjective optimization problems. The weak point of the gap function introduced in [9] is set-valued, i.e., brings a set to any point. Recently, Caristi *et al.* [4] can present some scalar-valued gap functions to nonsmooth multiobjective problems. Given the complexity of set-valued maps, these new single-valued gap functions are very useful. The defect gap functions introduced in [4] is that they work only for problems with convex\quasiconvex data. In the present article, this weakness will be resolved. For this end, we will define a gap function for nonsmooth MSIP, using $(\Phi, \rho)-$invexity. Of course, it should be mentioned that, in this study, if we replace "$(\Phi, \rho)-$invex" by "invex", the results will still be original which are the extensions of the existing theorems in mentioned articles.

We organize the paper as follows. In the next section, we provide the preliminary results to be used in the rest of the paper. In Section 3, we first overview some necessary optimality conditions for weakly efficient and efficient solutions, that are presented in literatures. Then, we state a similar result for properly efficien solutions. In Section 4, we introduce a new gap function involving $(\Phi, \rho)-$invexity, and present some characterizations for efficient, weakly efficient and properly efficient solutions of MSIP respect to considered gap function, unlike of other papers that consider separate gap functions for each kind of efficiency.

2 Preliminaries

In this section, we briefly overview some notions of nonsmooth analysis widely used in formulations and proofs of main results of the paper. For more details, discussion, and applications see [8].

As usual, $\langle x, y\rangle$ stands for the standard inner product $x, y \in \mathbb{R}^n$. Given $x, y \in \mathbb{R}^n$, we write $x \leq y$ (resp. $x < y$) when $x \neq y$ and $x_i \leq y_i$ (resp. $x_i < y_i$) for all $i \in \{1, \ldots, n\}$. The zero vector of $\mathbb{R}^n$ is denoted by 0_n.

Given a nonempty set $A \subseteq \mathbb{R}^n$, we denote by A^0 and A^-, the polar and strictly polar cones of A, defined respectively by

$$A^0 := \{x \in \mathbb{R}^n \mid \langle x, a\rangle \leq 0, \quad \forall a \in A\},$$
$$A^- := \{x \in \mathbb{R}^n \mid \langle x, a\rangle < 0, \quad \forall a \in A\}.$$

Also, we denote the cotingent tangent cone of A at $\hat{x} \in A$ by $T(A, \hat{x})$, i.e.,

$$T(A, \hat{x}) := \big\{v \in \mathbb{R}^n \mid \exists t_r \downarrow 0,\ \exists v_r \to v \text{ such that } \hat{x} + t_r v_r \in A \ \ \forall r \in \mathbb{N}\big\}.$$

Let $\hat{x} \in \mathbb{R}^n$ and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Clarke directional derivative of φ at $\hat{x}$ in the direction $v \in \mathbb{R}^n$, and the Clarke subdifferential of φ at $\hat{x}$ are respectively given by

$$\varphi^0(\hat{x}; v) := \limsup_{y \to \hat{x},\ t \downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t}$$

and

$$\partial_c \varphi(\hat{x}) := \big\{\xi \in \mathbb{R}^n \mid \langle \xi, v\rangle \leq \varphi^0(\hat{x}; v) \qquad \text{for all } v \in \mathbb{R}^n\big\}.$$

The Clarke subdifferential is a natural generalization of the classical derivative since it is known that when function φ is continuously differentiable at $\hat{x}$, $\partial_c \varphi(\hat{x}) = \{\nabla \varphi(\hat{x})\}$. Moreover when a function φ is convex, the Clarke subdifferential coincides with $\partial \varphi(\hat{x})$, the subdifferential in the sense of convex analysis, i.e.

$$\partial\varphi(\hat{x}) := \big\{\xi \in \mathbb{R}^n \mid \varphi(x) \geq \varphi(\hat{x}) + \big\langle \xi, x - \hat{x} \big\rangle \qquad \forall\, x \in \mathbb{R}^n \big\}.$$

It is worth to observe that $\partial_c\varphi(\hat{x})$ is a nonempty, convex, and compact subset of $\mathbb{R}^n$.

Theorem 1. Let ϑ_1 and ϑ_2 be locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $\hat{x} \in \mathbb{R}^n$. Then,

$$\partial_c(\alpha\vartheta_1 + \beta\vartheta_2)(\hat{x}) \subseteq \alpha\partial_c\vartheta_1(\hat{x}) + \beta\partial_c\vartheta_2(\hat{x}), \qquad \forall \alpha, \beta \in \mathbb{R}.$$

3 KKT Type Necessary Conditions

At starting point of this section, we observe that the feasible set of (P) is denoted by M, i.e.,

$$M := \{x \in \mathbb{R}^n \mid g_t(x) \leq 0, \quad \forall t \in T\}.$$

For each $\hat{x} \in M$, set

$$F_{\hat{x}} := \bigcup_{i \in I} \partial_c f_i(\hat{x}), \quad \text{and} \quad G_{\hat{x}} := \bigcup_{t \in T(\hat{x})} \partial_c g_t(\hat{x}),$$

where, $T(\hat{x})$ denotes the set of active constraints at $\hat{x}$,

$$T(\hat{x}) := \{t \in T \mid g_t(\hat{x}) = 0\}.$$

There exist different kind of optimality, named efficiency, in multiobjective optimization. A feasible point $\hat{x}$ is said to be efficient solution [resp. weakly efficient solution] for (P) if and only if there is no $x \in M$ satisfying $f(x) \leq f(\hat{x})$ [resp. $f(x) < f(\hat{x})$]. As well as in the classical case, the KKT type optimality conditions hold at efficient and weakly efficient solutions of (P), provided some constraint qualifications (CQ) are satisfied. For example, Kanzi [20] emphasized on weakly efficiency, and introduced the CCQ as,

Definition 1. Let $\hat{x} \in S$. We say that (P) satisfies the Cottle constraint qualification (CCQ, in brief) at $\hat{x}$, if J is a compact subset of $\mathbb{R}^p$, and the function $(x, t) \to g_t(x)$ is upper semicontinuous on $\mathbb{R}^n \times T$, and $\partial^c g_t(x)$ is an upper semicontinuous mapping in t for each x, and $(G_{\hat{x}})^- \neq \emptyset$.

Then, following KKT type theorem is proved in [20, Theorem 3.6].

Theorem 2. *(KKT Necessary Condition)* Let $\hat{x} \in M$ be a weakly efficient solution of (P) and CCQ holds at $\hat{x}$. Then there exist $\alpha_i \geq 0$ (for $i \in I$) with $\sum_{i=1}^m \alpha_i = 1$, and $\beta_t \geq 0$ (for $t \in T(\hat{x})$) with $\beta_t \neq 0$ for at most finitely many indices, such that

$$0 \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(\hat{x}) + \sum_{t \in T(\hat{x})} \beta_t \partial_c g_t(\hat{x}).$$

Caristi and Kanzi [7] considered the efficient solutions of (P), considered a Meda type CQ as,

$$\text{(MCQ)}: \qquad (F_{\hat{x}})^0 \cap (G_{\hat{x}})^0 \subseteq \bigcap_{i=1}^{p} T(Q^i, \hat{x}),$$

where, $Q^i(\hat{x}) := \Big\{x \in M \mid f_l(x) \leq f_l(\hat{x}) \quad \forall l \in I \setminus \{i\}\Big\}$, and in [7, Theorem 3.3] proved the strong KKT type result as follows.

Theorem 3. *(Strong KKT Necessary Condition).* Let $\hat{x}$ be an efficient solution of (P). If in addition, (MCQ) and the condition

$$(F_{\hat{x}})^0 \setminus \{0_n\} \subseteq \bigcup_{i=i}^{p} \big(\partial_c f_i(\hat{x})\big)^-, \tag{1}$$

hold at $\hat{x}$, then there exist scalars $\alpha_i > 0$, $i \in I$, and an integer $k \geq 0$, and a set $\{t_1, t_2, \ldots, t_k\} \subseteq T(\hat{x})$, and scalars $\beta_{t_r} \geq 0$ for $r \in \{1, 2, \ldots, k\}$, such that

$$0 \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(\hat{x}) + \sum_{r=1}^{k} \beta_{t_r} \partial_c g_{t_r}(\hat{x}).$$

Also, Kanzi in [19, Theorem 3] (resp. [19, Theorem 4]) presented the KKT (resp. strong KKT) condition under Zangwill (resp. strong Zangmill) CQ, that introduced there.

Everywhere in the above, we consider the efficiency and weakly efficiency for (P). Proper efficiency is a very important notion used in studying multiobjective optimization problems. There are many definitions of proper efficiency in literature, as those introduced by Geoffrion, Benson, Borwein, and Henig; see [16] for a comparison among the main definitions of this notion. We recall the following definition from [15, pp. 110].

Definition 2. A point $\hat{x} \in M$ is called a properly efficient solution of (P) when there exists a $\lambda > 0_p$ such that

$$\langle \lambda, f(\hat{x}) \rangle \leq \langle \lambda, f(x) \rangle, \qquad \forall x \in M.$$

As proved in [10, Section 3], the above definition of proper efficiency is weaker than its other definitions (under some assumed conditions). The following theorem gives us a strong KKT condition for properly efficient solutions of (P).

Theorem 4. (Strong KKT Necessary Condition) Let $\hat{x}$ be a properly efficient solution of (P), and CCQ holds at $\hat{x}$. Then, there exist $\alpha_i > 0$ (for $i \in I$) with $\sum_{i=1}^{p} \alpha_i = 1$, and $\beta_t \geq 0$, (for $t \in T(\hat{x})$), with $\beta_t \neq 0$ for finitely many indexes, such that

$$0 \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(\hat{x}) + \sum_{t \in T(\hat{x})} \beta_t \partial_c g_t(\hat{x}).$$

Proof. By the definition of proper efficiency, there exist some scalars $\lambda_i > 0$ (for $i \in I$) such that

$$\sum_{i=1}^{p} \lambda_i f_i(\hat{x}) \leq \sum_{i=1}^{p} \lambda_i f_i(x), \qquad \forall x \in M.$$

This means that $\hat{x}$ is a minimizer of the following scalar semi-infinite problem:

$$\min_{x \in M} \sum_{i=1}^{p} \lambda_i f_i(x).$$

Applying Theorem 2, we get

$$0_n \in \tau \partial_c \Big(\sum_{i=1}^{p} \lambda_i f_i(\cdot) \Big)(\hat{x}) + \sum_{t \in T(\hat{x})} \mu_t \partial_c g_t(\hat{x}), \tag{2}$$

for some $\tau > 0$ and $\mu_t \geq 0$, ($t \in T(\hat{x})$), with $\mu_t \neq 0$ for finitely many indexes. Since Theorem 1 guaranties that

$$\partial_c \Big(\sum_{i=1}^{p} \lambda_i f_i(\cdot) \Big)(\hat{x}) \subseteq \sum_{i=1}^{p} \lambda_i \partial_c f_i(\hat{x}),$$

(2) concludes that

$$0_n \in \tau \sum_{i=1}^{p} \lambda_i \partial_c f_i(\hat{x}) + \sum_{t \in T(\hat{x})} \mu_t \partial_c g_t(\hat{x}).$$

Dividing both sides of above inclusion to $\tau \sum_{i=1}^{p} \lambda_i$, we conclude that

$$0_n \in \sum_{i=1}^{p} \frac{\lambda_i}{\sum_{i=1}^{p} \lambda_i} \partial_c f_i(\hat{x}) + \sum_{t \in T(\hat{x})} \frac{\mu_t}{\tau \sum_{i=1}^{p} \lambda_i} \partial_c g_t(\hat{x}). \tag{3}$$

For each $i \in I$ and $t \in T(\hat{x})$ take

$$\alpha_i := \frac{\lambda_i}{\sum_{i=1}^{p} \lambda_i}, \quad \text{and} \quad \beta_t := \frac{\mu_t}{\tau \sum_{i=1}^{p} \lambda_i}.$$

Since $\sum_{i=1}^{p} \alpha_i = 1$, (3) completes the proof. □

We illustrate the application of Theorem 4 by an example.

Example 1. Consider the following problem:

$$\begin{aligned} & \inf \ (x_1, x_2) \\ \text{s.t.} \quad & (\cos t) x_1 + (\sin t) x_2 \leq 0, \quad t \in \Big[\pi, \frac{3\pi}{4}\Big]. \end{aligned}$$

It is easy to check that

$$M = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\} + \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0\}.$$

We consider the feasible point $\hat{x} = (\cos\alpha, \sin\alpha)$ for some $\alpha \in \left(\pi, \frac{3\pi}{4}\right)$.
Since $f_1(x_1, x_2) = x_1$, $f_2(x_1, x_2) = x_2$, $g_t(x_1, x_2) = (\cos t)x_1 + (\sin t)x_2$, and $T = \left[\pi, \frac{3\pi}{4}\right]$, we get

$$T(\hat{x}) = \{\alpha\}, \qquad G_{\hat{x}} = \{(\cos\alpha, \sin\alpha)\}, \qquad F_{\hat{x}} = \{(1, 0), (0, 1)\}.$$

Therefore, according to Theorem 4, we conclude $\hat{x}$ is a properly efficient solution for the problem.

4 Characterization via gap function

This section is started by a definition from [19].

Definition 3. Suppose that the functions $\Phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and the nonempty set $X \subseteq \mathbb{R}^n$ are given. A locally Lipschitz function $\hbar : \mathbb{R}^n \to \mathbb{R}$ is said to be $(\Phi, \rho)-$invex at $x^* \in X$ with respect to X, if for each $x \in X$ one has:

$$\Phi\big(x, x^*, 0_n, r\big) \geq 0 \qquad \text{for all } r \geq 0, \tag{4}$$

$$\Phi(x, x^*, ., .) \text{ is convex on } \mathbb{R}^n \times \mathbb{R}, \tag{5}$$

$$\Phi\Big(x, x^*, \xi, \rho(x, x^*)\Big) \leq \hbar(x) - \hbar(x^*), \qquad \forall \xi \in \partial_c \hbar(x^*). \tag{6}$$

Notice that this definition is more general that [1, Definition 4] and [2, Definition 6], since there considered ρ are real number and here is a function. Everywhere in the following, we will assume X equals to feasible solution of (P), i.e., $X = M$, but for the sake of simplicity we will omit to mention X.

Since 1982, an important function respect to convex optimization problems was defined by Hearn [17]. As mentioned in introduction, all existing literatures the gap function was defined for optimization programming with convex or quasiconvex data. Now, we define the gap function for nonsmooth MSIPs with (Φ, ρ)-invex functions.

Definition 4. Suppose that the f_i functions are (Φ, ρ_i)-invex at $x \in M$. For each

$$\xi := (\xi_1, \ldots, \xi_p) \in \prod_{i=1}^{p} \partial_c f_i(x) \text{ and } \lambda := (\lambda_1, \ldots, \lambda_p) \geq 0_p \text{ with } \sum_{i=1}^{p} \lambda_i = 1,$$

the gap function of problem (P) is defined as

$$\Upsilon(x, \xi, \lambda) := \inf_{y \in M} \left\{ \sum_{i=1}^{p} \lambda_i \Phi(y, x, \xi_i, \rho_i(y, x)) \right\}.$$

It is worth mentioning that all the gap functions considered in [7, 9, 12, 17] are special cases of above gap function. At the rest of this section, we will characterize efficient, weakly efficient, and properly efficient solutions of (P) utilizing $\Upsilon(x, \xi, \lambda)$.

Theorem 5. Let the f_i function be (Φ, ρ_i)-invex at $\hat{x} \in M$ for each $i \in I$.

(a) If $\Upsilon(\hat{x}, \hat{\xi}, \hat{\lambda}) = 0$ for some $\hat{\xi} := (\hat{\xi}_1, \dots, \hat{\xi}_p) \in \prod_{i=1}^{p} \partial_c f_i(\hat{x})$ and $\hat{\lambda} := (\hat{\lambda}_1, \dots, \hat{\lambda}_p) \geq 0_p$ with $\sum_{i=1}^{p} \hat{\lambda}_i = 1$, then $\hat{x}$ is a weak efficient solution for (P).

(b) If $\Upsilon(\hat{x}, \hat{\xi}, \hat{\lambda}) = 0$ for some $\hat{\xi} := (\hat{\xi}_1, \dots, \hat{\xi}_p) \in \prod_{i=1}^{p} \partial_c f_i(\hat{x})$ and $\hat{\lambda} := (\hat{\lambda}_1, \dots, \hat{\lambda}_p) > 0_p$ with $\sum_{i=1}^{p} \hat{\lambda}_i = 1$, then $\hat{x}$ is an efficient solution for (P).

Proof. (a) By contradiction assume that $\Upsilon(\hat{x}, \hat{\xi}, \hat{\lambda}) = 0$ while $\hat{x}$ is not a weak efficient solution for (P). Then, we can find a feasible point $x_0 \in M$ such that $f_i(x_0) < f_i(\hat{x})$ for all $i \in I$. Thus, the (Φ, ρ_i)-invexity of f_i functions implies that

$$\Phi\big(x_0, \hat{x}, \hat{\xi}_i, \rho_i(x_0, \hat{x})\big) \leq f_i(x_0) - f_i(\hat{x}) < 0, \qquad \forall i \in I. \tag{7}$$

On the other hand, since $\hat{\lambda} \geq 0_p$, then there exists an index $k \in I$ such that

$$\hat{\lambda}_k > 0, \quad \text{and} \quad \hat{\lambda}_i \geq 0 \quad \forall i \in I \setminus \{k\}. \tag{8}$$

Clearly, (7) and (8) imply

$$\hat{\lambda}_k \Phi\big(x_0, \hat{x}, \hat{\xi}_k, \rho_k(x_0, \hat{x})\big) < 0, \quad \text{and} \quad \hat{\lambda}_i \Phi\big(x_0, \hat{x}, \hat{\xi}_i, \rho_i(x_0, \hat{x})\big) \leq 0 \quad \forall i \in I \setminus \{k\}.$$

Hence,

$$\sum_{i=1}^{p} \hat{\lambda}_i \Phi\big(x_0, \hat{x}, \hat{\xi}_i, \rho_i(x_0, \hat{x})\big) < 0,$$

which consequences that $\Upsilon(\hat{x}, \hat{\xi}, \hat{\lambda}) < 0$. This contradiction completes the proof. (b) If $\Upsilon(\hat{x}, \hat{\xi}, \hat{\lambda}) = 0$ while $\hat{x}$ is not an efficient solution for (P), there exist some $x_0 \in M$ and some index $k \in I$ such that

$$f_i(x_0) \leq f_i(\hat{x}), \quad \forall i \in I, \quad \text{and} \quad f_k(x_0) < f_k(\hat{x}).$$

According to the above inequalities, the (Φ, ρ_i)-invexity of f_i functions, and the assumption of $\hat{\lambda} > 0_p$, we get

$$\sum_{i=1}^{p} \hat{\lambda}_i \Phi\big(x_0, \hat{x}, \hat{\xi}_i, \rho_i(x_0, \hat{x})\big) \leq \sum_{i=1}^{p} \hat{\lambda}_i \left(f_i(x_0) - f_i(\hat{x})\right) < 0.$$

So, $\Upsilon(\hat{x}, \hat{\xi}, \hat{\lambda}) < 0$, which contradicts the assumption. □

Since properly efficientcy is stronger than weakly efficiency and efficiency, the following sufficient condition needs some assumptions which are stronger than Theorem 4, containing equality of ρ_i functions for each $i \in I$.

Theorem 6. Suppose that for each $i \in I$, the f_i function is (Φ, ρ)-invex at $\hat{x} \in M$. If there exists a $\hat{\xi} := (\hat{\xi}_1, \dots, \hat{\xi}_p) \in \prod_{i=1}^p \partial_c f_i(\hat{x})$ such that $\Upsilon(\hat{x}, \hat{\xi}, \lambda) = 0$ for all $\lambda := (\lambda_1, \dots, \lambda_p) > 0_p$ with $\sum_{i=1}^p \lambda_i = 1$, then $\hat{x}$ is a proper efficient solution for (P).

Proof. If $\hat{x}$ is not a proper efficient solution for (P), we can find some $x_0 \in M$ and $\lambda^* := (\lambda_1^*, \dots, \lambda_p^*) > 0_p$ such that

$$\sum_{i=1}^p \lambda_i^* f_i(x_0) < \sum_{i=1}^p \lambda_i^* f_i(\hat{x}).$$

Taking $\widetilde{\lambda}_i := \frac{\lambda_i^*}{\sum_{i=1}^p \lambda_i^*}$, we conclude that $\sum_{i=1}^p \widetilde{\lambda}_i = 1$, and

$$\sum_{i=1}^p \widetilde{\lambda}_i f_i(x_0) < \sum_{i=1}^p \widetilde{\lambda}_i f_i(\hat{x}). \tag{9}$$

We claim that $\sum_{i=1}^p \widetilde{\lambda}_i f_i$ is a $(\Phi, \rho)-$invex function at $\hat{x}$. Suppose that $\zeta \in \sum_{i=1}^p \widetilde{\lambda}_i \partial_c f_i(\hat{x})$ is given. It is enough to show that

$$\Phi\big(x, \hat{x}, \zeta, \rho(x, \hat{x})\big) \le \sum_{i=1}^p \widetilde{\lambda}_i f_i(x) - \sum_{i=1}^p \widetilde{\lambda}_i f_i(\hat{x}), \qquad \forall x \in M. \tag{10}$$

For this end, we recall from Theorem 1 that $\zeta = \sum_{i=1}^p \widetilde{\lambda}_i \zeta_i$ for some $\zeta_i \in \partial_c f_i(\hat{x})$. The $(\Phi, \rho)-$invexity of f_i functions at $\hat{x}$ and the convexity of $\Phi(x, \hat{x}, ., .)$ imply that

$$\begin{aligned}
\Phi\big(x, \hat{x}, \zeta, \rho(x, \hat{x})\big) &= \Phi\Big(x, \hat{x}, \sum_{i=1}^p \widetilde{\lambda}_i \zeta_i, \sum_{i=1}^p \widetilde{\lambda}_i \rho(x, \hat{x})\Big) \\
&\le \sum_{i=1}^p \widetilde{\lambda}_i \Phi\big(x, \hat{x}, \zeta_i, \rho(x, \hat{x})\big) \\
&\le \sum_{i=1}^p \widetilde{\lambda}_i \big(f(x) - f(\hat{x})\big) = \sum_{i=1}^p \widetilde{\lambda}_i f_i(x) - \sum_{i=1}^p \widetilde{\lambda}_i f_i(\hat{x}).
\end{aligned}$$

Thus, (10) is proved. Now, (9) and the $(\Phi, \rho)-$invexity of $\sum_{i=1}^p \widetilde{\lambda}_i f_i$ at $\hat{x}$ conclude that

$$\sum_{i=1}^p \widetilde{\lambda}_i \Phi\big(x_0, \hat{x}, \hat{\xi}_i, \rho(x_0, \hat{x})\big) \le \sum_{i=1}^p \widetilde{\lambda}_i f_i(x_0) - \sum_{i=1}^p \widetilde{\lambda}_i f_i(\hat{x}) < 0.$$

This means $\Upsilon(\hat{x}, \hat{\xi}, \widetilde{\lambda}) < 0$, which contradicts the assumption. □

The following new definition will be required in the sequel.

Definition 5. A locally Lipschitz function $\hbar : \mathbb{R}^n \to \mathbb{R}$ is said to be "symmetric (Φ, ρ)-invex" at $\tilde{x} \in \mathbb{R}^n$ if

- $\hbar$ is (Φ, ρ)-invex at $\tilde{x}$,
- $\Phi(\tilde{x}, \tilde{x}, \xi, \rho(\tilde{x}, \tilde{x})) = 0$ for all $\xi \in \partial_c \hbar(\tilde{x})$.

$\hbar(.)$ is said to be symmetric (Φ, ρ)-invex, if it is symmetric (Φ, ρ)-invex at each point in its domain.

We recall from [23] that for r-convex ($r \in \mathbb{R}_+$) functions we have $\rho(x, y) := r$ and

$$\Phi\big(x, y, \xi, \rho\big) = \langle \xi, y - x \rangle + r\|x - y\|^2.$$

So, r-convex functions are symmetric (Φ, ρ)-invex. Also, the skew invex functions, which are defined in [22], are examples for nonconvex symmetric (Φ, ρ)-invex functions. The following example shows that a symmetric (Φ, ρ)-invexity function does not need to be invex.

Example 2. Consider a function $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(x, y, u, w) := \begin{cases} -\dfrac{u}{3y^2}|x^3 - y^3| & \text{if} \quad y \neq 0, \\ \\ w|x^3| & \text{if} \quad y = 0. \end{cases}$$

Let x and y be arbitrary elements of $\mathbb{R}$. Since $\Phi(x, y, ., .)$ is a linear function and

$$\Phi(x, y, 0, r) = \begin{cases} 0 & \text{if} \quad y \neq 0, \\ r|x^3| & \text{if} \quad y = 0, \end{cases}$$

the conditions (12) and (26) hold. Take $\rho(x, y) := -1$ for all $x, y \in \mathbb{R}$, and $\hbar(x) := x^3$. Since $\hbar(.)$ is continuously differentiable on $\mathbb{R}$, then $\partial_c \hbar(y) = \{3y^2\}$. Now, owing to

$$\begin{aligned} \Phi(x, y, 3y^2, -1) \quad &= \quad \begin{cases} -|x^3 - y^3| & \text{if} \quad y \neq 0, \\ -|x^3| & \text{if} \quad y = 0, \end{cases} \\ &\leq \quad x^3 - y^3 = \hbar(x) - \hbar(y), \end{aligned}$$

we understand that $\hbar(.)$ is a $(\Phi, \rho)-$invex function at each $y \in \mathbb{R}$ with respect to $\mathbb{R}$. Also, th equality of

$$\Phi(y, y, 3y^2, -1) = 0,$$

shows that $\hbar(.)$ is a symmetric $(\Phi, \rho)-$invex function at each $y \in \mathbb{R}$. Furthermore, as it follows by [3, Theorem 1], $\hbar(.)$ is not an invex function on $\mathbb{R}$.

Theorem 7. Let $\hat{x} \in M$ be a weakly efficient solution of (P) and CCQ holds at $\hat{x}$. Suppose that for each $i \in I$ the f_i function is symmetric (Φ, ρ_i)-invex at $\hat{x}$, and for each $t \in T(\hat{x})$ the g_t function is (Φ, ρ_t)-invex at $\hat{x}$, satisfying

$$\rho_r(y, \hat{x}) \geq 0, \qquad \forall r \in I \cup T(\hat{x}),\ \forall y \in M. \tag{11}$$

Then, there exist $\xi := (\xi_1, \ldots, \xi_p) \in \prod_{i=1}^{p} \partial_c f_i(\hat{x})$ and $\lambda := (\lambda_1, \ldots, \lambda_p) \geq 0_p$ with $\sum_{i=1}^{p} \lambda_i = 1$, such that $\Upsilon(\hat{x}, \xi, \lambda) = 0$.

Proof. According to Theorem 2, we can find some $\lambda_i \geq 0$ and $\xi_i \in \partial_c f_i(\hat{x})$ (for $i \in I$) with $\sum_{i=1}^{p} \lambda_i = 1$, a finite subset T^* for $T(\hat{x})$, some $\mu_t \geq 0$ and $\zeta_t \in \partial_c g_t(\hat{x})$ (for $t \in T^*$), such that

$$\sum_{i \in I} \lambda_i \xi_i + \sum_{t \in T^*} \mu_t \zeta_t = 0_n. \tag{12}$$

For each $(i, t) \in I \times T^*$ set

$$\hat{\lambda}_i := \frac{\lambda_i}{1 + \sum_{t \in T^*} \mu_t}, \quad \text{and} \quad \hat{\mu}_t := \frac{\mu_t}{1 + \sum_{t \in T^*} \mu_t}.$$

Assume that $t \in T^*$ and $y \in M$ are arbitrarily chosen. Since $T^* \subseteq T(\hat{x})$, the (Φ, ρ_t)-invexity of g_t implies that

$$g_t(y) \leq 0 = g_t(\hat{x}) \implies \Phi\big(y, \hat{x}, \zeta_t, \rho_t(y, \hat{x})\big) \leq 0, \qquad \forall y \in M.$$

So, by $\hat{\mu}_t \geq 0$ (for $t \in T^*$), we get

$$\sum_{t \in T^*} \hat{\mu}_t \Phi\big(y, \hat{x}, \zeta_t, \rho_t(y, \hat{x})\big) \leq 0, \qquad \forall y \in M. \tag{13}$$

On the other hand, Definition 3, (11) and (12) conclude that

$$\begin{aligned} 0 \;&\leq\; \Phi\Big(y, \hat{x}, 0_n, \sum_{i \in I} \hat{\lambda}_i \rho_i(y, \hat{x}) + \sum_{t \in T^*} \hat{\mu}_t \rho_t(y, \hat{x})\Big) \\ &=\; \Phi\Big(y, \hat{x}, \sum_{i \in I} \hat{\lambda}_i \xi_i + \sum_{t \in T^*} \hat{\mu}_t \zeta_t, \sum_{i \in I} \hat{\lambda}_i \rho_i(y, \hat{x}) + \sum_{t \in T^*} \hat{\mu}_t \rho_t(y, \hat{x})\Big) nonumber \qquad (14) \\ &\leq\; \sum_{i \in I} \hat{\lambda}_i \Phi\big(y, \hat{x}, \xi_i, \rho_i(y, \hat{x})\big) + \sum_{t \in T^*} \hat{\mu}_t \Phi\big(y, \hat{x}, \zeta_t, \rho_t(y, \hat{x})\big), \qquad (15) \end{aligned}$$

where (15) is implied by $\sum_{i \in I} \hat{\lambda}_i + \sum_{t \in T^*} \hat{\mu}_t = 1$ and convexity of $\Phi(y, \hat{x}, ., .)$. Combining the last inequality and (13), yields

$$\sum_{i \in I} \hat{\lambda}_i \Phi\big(y, \hat{x}, \xi_i, \rho_i(y, \hat{x})\big) \geq 0 \implies \sum_{i \in I} \lambda_i \Phi\big(y, \hat{x}, \xi_i, \rho_i(y, \hat{x})\big) \geq 0, \qquad \forall y \in M. \tag{16}$$

Since the symmetric (Φ, ρ_i)-invexity of f_i functions at $\hat{x}$ concludes

$$\sum_{i\in I}\lambda_i\Phi\big(\hat{x},\hat{x},\xi_i,\rho_i(\hat{x},\hat{x})\big)=0,$$

the inequality (16) deduces that

$$\Upsilon(\hat{x},\xi,\lambda)=\inf_{y\in M}\left\{\sum_{i=1}^{p}\lambda_i\Phi\big(y,\hat{x},\xi_i,\rho_i(y,\hat{x})\big)\right\}=0,$$

as requested. □

Applying Theorems 3 and 4, and repeating the proof of Theorem 7, we can state the following theorem for efficient and properly efficient solutions of (P), respectively.

Theorem 8. Assume that $\hat{x}\in M$is an efficient solution of (P), the (MCQ) is satisfied at $\hat{x}$, and (1) holds. Suppose that for each $i\in I$ the f_i function is symmetric (Φ,ρ_i)-invex at $\hat{x}$, and for each $t\in T(\hat{x})$ the g_t function is (Φ,ρ_t)-invex at $\hat{x}$, satisfying (11). Then, there exist $\xi:=(\xi_1,\ldots,\xi_p)\in\prod_{i=1}^{p}\partial_c f_i(\hat{x})$ and $\lambda:=(\lambda_1,\ldots,\lambda_p)>0_p$ with $\sum_{i=1}^{p}\lambda_i=1$, such that $\Upsilon(\hat{x},\xi,\lambda)=0$.

Theorem 9. Suppose that $\hat{x}$ is a properly efficient solution for (P) and CCQ holds at $\hat{x}$. Suppose that for each $i\in I$ the f_i function is symmetric (Φ,ρ_i)-invex at $\hat{x}$, and for each $t\in T(\hat{x})$ the g_t function is (Φ,ρ_t)-invex at $\hat{x}$, satisfying (11). Then, there exist $\xi:=(\xi_1,\ldots,\xi_p)\in\prod_{i=1}^{p}\partial_c f_i(\hat{x})$ and $\lambda:=(\lambda_1,\ldots,\lambda_p)>0_p$ with $\sum_{i=1}^{p}\lambda_i=1$, such that $\Upsilon(\hat{x},\xi,\lambda)=0$.

We note that the difference between the Theorem7 with Theorems 8 and 9 is that in the first we have $\lambda\geq 0_p$, whereas in the latter ones we have $\lambda>0_p$. Also, it is worth mentioning that the presented results generalize

5 Conclusion

In this paper, we considered the class of nonsmooth multiobjective optimization problems with arbitrary many constraints. We proved a Karush-Kuhn-Tucker type optimality condition for properly efficient solutions of the problems. We introduced a new gap function that can characterizes efficient, weakly efficient, and properly efficient solutions the problem, under (Φ,ρ_i)-invexity and symmetric (Φ,ρ_i)-invexity assumptions.

References

[1] Antczak T. (2015) ”Saddle point criteria and Wolfe duality in nonsmooth $(\Phi, \rho)-$invex vector optimization problems with inequality and equality constraints”, International J. Computer Math. 92, 882-907.

[2] Antczak T., Stasiak A. (2011) ”$(\Phi, \rho)-$Invexity in Nonsmooth Optimization”, Numer. Func. Anal. Optim.32, 1-25.

[3] Ben-Israel A., Mond B. (1986) ”What is invexity?”, J. Australian Math. Sci. 28, 1-9.

[4] Caristi G., Kanzi M. and Soleimani-damaneh M. (2017) ”On gap functions for nonsmooth multiobjective optimization problems”, Optim Letters. DOI: 10.1007/s11590-017-1110-4.

[5] Caristi G., Ferrara M. and Stefanescu A. (2010) ”Semi-infinite multiobjective programming with grneralized invexity”, Mathematical Reports, 62, 217-233.

[6] Caristi G., Ferrara M. and Stefanescu A. (2006) ”Mathematical programming with (ρ, Φ)-invexity”, In Generalized Convexity and Related Topics. Lecture Notes in Economics and Mathematical Systems, Vol. 583. (I.V. Konnor, D.T. Luc, and A.M. Rubinov, eds.). Springer, Berlin-Heidelberg-New York, 167-176.

[7] Caristi G., Kanzi N. (2015) ”Karush-Kuhn-Tuker Type Conditions for Optimality of Non-Smooth Multiobjective Semi-Infinite Programming”, International Journal of Mathematical Analysis 9, 1929-1938.

[8] Clarke F. H. (1983) ”Optimization and Nonsmooth Analysis”, Wiley, Interscience.

[9] Chen C. Y., Goh C. J., Yang, X. Q. (1998) ”The gap function of a convex multicriteria optimization problem”, Eur. J. Oper. Res. 111, 142–151.

[10] Ehrgott M. (2005) ”Multicriteria Optimization”, Springer, Berlin.

[11] Gao X. Y. (2013) ”Optimality and duality for non-smooth multiobjective semi-infinite programming”, J. Netw. 8, 413-420.

[12] Goberna M.A., Kanzi N. (2017) ”Optimality conditions in convex multiobjective” SIP. Math. Programming. DOI 10.1007/s10107-016-1081-8.

[13] Goberna M. A., Guerra-Vazquez F. and Todorov M. I. (2016) ”Constraint qualifications in linear vector semi-infinite optimization”, European J. Oper. Res. 227, 32-40.

[14] Goberna M. A., Guerra-Vazquez F. and Todorov M. I. (2013) ”Constraint qualifications in convex vector semi-infinite optimization”, European J. Oper. Res. 249, 12-21.

[15] Gopfert A., Riahi H., Tammer C. and Zalinescu C. (2003) ”Variational methods in partioal ordered spaces”, Springer, New York.

[16] Guerraggio A., Molho E. and Zaffaroni A. (1994) ”On the notion of proper efficiency in vector optimization”, J. Optim. Theory Appl. 82, 1-21.

[17] Hearn D. W., (1982) ”The gap function of a convex program”, Oper. Res. Lett. 1, 67–71.

[18] Kanzi N., Shaker Ardekani J., Caristi G. (2018) ”Optimality, scalarization and duality in linear vector semi-infinite programming”, Optimization. Doi: 10.1080/02331934.2018.1454921.

[19] Kanzi N. (2017) ”Necessary and sufficient conditions for (weakly) efficient of nondifferentiable multi-objective semi-infinite programming”, Iranian Journal of Science and Technology Transactions A: Science, DOI 10.1007/s40995-017-0156-6.

[20] Kanzi N. (2015) ”Karush-Kuhn-Tucker Types Optimality Conditions For Non-Smooth Semi-Infinite Vector Optimization Problems”, Journal of Mathematical Extension 9, 45-56.

[21] Kanzi N. (2015) ”Regularity Conditions for Non-Differentiable Infinite Programming Problems Using Michel-Penot Subdifferential”, Biquarterly Control and Optimization in applied Mathematics 1, 21-30.

[22] Noor M. A. (2004) ”On generalized preinvex functions and monotonicities”, Journal of Inequalities in Pure and Applied Mathematics. 5, 1–9.

[23] Rezayi A. (2018) ”Characterization of Isolated Efficient Solutions in Nonsmooth Multiobjective Semi-infinite Programming”, Iran J Sci Technol Trans Sci, Doi.org/10.1007/s40995-018-0637-2.

یک تابع شکاف حقیقی مقدار برای مسائل برنامه ریزی چند هدفی نیمه نامتناهی غیر هموار

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چکیده

ما در این مقاله برای یک مسئله برنامه ریزی چند هدفه غیر همواری که توسط تعداد بینهایت قید تعریف می‌شود تابع شکاف جدیدی را معرفی می‌کنیم که تعمیم این مفهوم در مقالات دیگر است. آنگاه ما کارایی، کارایی ضعیف و کارایی سره مسئله فوق را توسط این تابع شکاف جدید مشخص سازی می‌کنیم تمام مفاهیم ما بر مبنای مفهوم توابع $\Phi, \rho-$ اینوکس و زیر مشتق کلارک تنظیم گشته‌اند.

کلمات کلیدی

برنامه ریزی نیمه نامتناهی، بهینه سازی چندهدفه، کیفیت محدود، شرایط بهینگی، تابع شکاف.