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Characterization of Properly Efficient Solutions for Convex Multiobjective Programming with Nondifferentiable vanishing constraints

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Abstract. This paper studies the convex multiobjective optimization problem with vanishing constraints. We introduce a new constraint qualification for these problems, and then a necessary optimality condition for properly efficient solutions is presented. Finally by imposing some assumptions, we show that our necessary condition is also sufficient for proper efficiency. Our results are formulated in terms of convex subdifferential.

Keywords. Multiobjective optimization, Vanishing constraints, Convex optimization, Constraint qualification

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1 Introduction

We consider the following *multiobjective mathematical programming with vanishing constraints* (MMPVC in brief):

$$\text{MMPVC}: \quad \min_{x\in\Omega} \ F(x) := (f_1(x), \ldots, f_p(x)),$$
$$\Omega := \left\{x \in \mathbb{R}^n \mid H_i(x) \geq 0, \ G_i(x)H_i(x) \leq 0, i \in I\right\}, \tag{1}$$

where, the considered functions f_j (for $j \in J := \{1, \ldots, p\}$), H_i (for $i \in I := \{1, \ldots, m\}$), and G_i (for $i \in I$) are convex, not necessarily differentiable, and defined from $\mathbb{R}^n$ to $\mathbb{R}$.

If $p = 1$, then MMPVC reduces to "mathematical programming with vanishing constraints" (MPVC) which were introduced by Kanzow and his coauthors in 2007 [1, 9]. After defining the MPVC, finding the optimality conditions, named stationary conditions, for it become an interesting subject for some researchers; see [7, 8, 9, 13] in smooth case and [10, 11] in nonsmooth case).

If $G_i(x) = 0$ for $i \in I$, the MMPVC coincides to classical multiobjective programming problem which is an important field in optimization theory. Also, the MMPVC is a direct generalization for the following "mathematical problem with equilibrium constraints" (MPEC), considered in a lot of papers (see [14, 16] and their references):

$$\begin{array}{lll} \min & F(x) & \\ s.t. & H_i(x) \geq 0, \ G_i(x) \geq 0, & i \in I, \\ & G_i(x)H_i(x) = 0, & i \in I. \end{array}$$

To the best of our knowledge, there is no work available dealing with MMPVC with nondifferentiable data, and the present paper is the first to consider it. So far under differentiablity assumption, there is only one conference paper that considered MMPVC [12].

As well as classic multiobjective optimization, we can consider different kinds of optimality (efficiency) for MMPVC, including weakly efficient, efficient, strictly efficient, isolated efficient, and properly efficient solutions. Some characterizing of weakly efficient solutions for MMPVCs with smooth data are presented in [12]. In order to obtain optimality in which, given any objective, the trade-off between that objective and some other objective is bounded, Geoffrion [3] suggested restricting attention to efficient solutions that are proper. After Geoffrion, proper efficiency became a very important notion in studying multiobjective optimization, and many definitions for proper efficiency were introduced in literature, such as those introduced by Benson, Borwein, Henig, Kuhn-Tucker; see [2] for a comparison among the main definitions of this notion. Here, we will consider the newest definition of proper efficiency that is introduced in [4], and will characterize it for nonsmooth convex MMPVC. This characterization is made for the first time, even for MMPVCs with smooth data.

Since the product function of two convex functions is not necessarily convex, the feasible set Ω is not necessarily convex. Consequently, to set optimality conditions for properly efficient solutions of MMPVC, we can select different normal cones for S. Here we focus on Mordukhovich normal cone of Ω. This kind of optimality condition has been studied in [7, 8, 9, 14, 16] for

MPVCs and MPECs. We would mention that all mentioned references to MPVC have considered the problems with continuously differentiable functions, and the present paper extends their results to MMPVC with nondifferentiable functions.

The structure of this paper is as follows: Section 2 contains some definitions and theorems from convex analysis and non-smooth analysis. In section 3, we will introduce a new constraint qualification for MMPVC, and will present a necessary condition for properly efficient solutions of MMPVC. Then, we will show our necessary condition is also sufficient under some weak assumptions.

2 Preliminaries

In this section we present some preliminary results on convex analysis and nonsmooth analysis from [6, 15]. Suppose that $g : \mathbb{R}^n \to \mathbb{R}$ is a convex function, and $x_0 \in \mathbb{R}^n$. The subdifferential of g at x_0 is defined as

$$\partial g(x_0) := \{\zeta \in \mathbb{R}^n \mid g(x) - g(x_0) \geq \langle \zeta, x - x_0 \rangle \,, \quad \forall x \in \mathbb{R}^n\}.$$

Notice that if g_1 and g_2 are two convex functions from $\mathbb{R}^n$ to $\mathbb{R}$, and α is a non-negative real number, then $\alpha g_1 + g_2$ is convex and

$$\partial(\alpha g_1 + g_2)(x_0) = \alpha \partial g_1(x_0) + \partial g_2(x_0).$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Mordukhovich subdifferential of φ at x_0 is defined as

$$\partial_M \varphi(x_0) := \limsup_{x \to x_0} \Big\{ \xi \in \mathbb{R}^n \mid \liminf_{y \to x} \frac{\varphi(y) - \varphi(x) - \langle \xi, y - x \rangle}{\|y - x\|} \geq 0 \Big\}.$$

We observe that if g is a convex function, then $\partial_M g(x_0) = \partial g(x_0)$ and $\partial_M(-g)(x_0) = -\partial g(x_0)$. Also, for two locally Lipschitz functions φ_1 and φ_2 from $\mathbb{R}^p$ to $\mathbb{R}$, and for an arbitrary real number α, we have

$$\partial_M \big(\alpha \varphi_1 + \varphi_2\big)(x_0) \subseteq \alpha \partial_M \varphi_1(x_0) + \partial_M \varphi_2(x_0).$$

Notice that if x_0 is a minimizer of φ on $\mathbb{R}^p$, then $0_p \in \partial_M \varphi(x_0)$, where 0_p denotes the zero vector of $\mathbb{R}^p$.

The Mordukhovich normal cone of a closed subset $\Lambda \subseteq \mathbb{R}^p$ at $x_0 \in \Lambda$ is defined by $N_M(\Lambda, x_0) := \partial_M \mathcal{I}_\Lambda(x_0)$, where

$$\mathcal{I}_\Lambda(x) := \begin{cases} 0 & x \in \Lambda, \\ +\infty & x \notin \Lambda. \end{cases}$$

It is not difficult to show that for given $\Lambda_i \subseteq \mathbb{R}^{p_i}$ and $x^{(i)} \in \Lambda_i$, $i = 1, \ldots, s$, we have

$$N_M\left(\Lambda_1 \times \cdots \times \Lambda_s, \big(x^{(1)}, \ldots, x^{(s)}\big)\right) = N_M\big(\Lambda_1, x^{(1)}\big) \times \cdots \times N_M\big(\Lambda_s, x^{(s)}\big). \tag{2}$$

If $h(y) = (h_1(y), \ldots, h_s(y))$, where h_is are locally Lipschitz from $\mathbb{R}^n$ to $\mathbb{R}$, and $x^* = (x_1^*, \ldots, x_s^*)$, then the Mordukhovich coderivative of h is defined as

$$D^*h(\overline{y})(x^*) = \partial_M \left(\sum_{k=1}^{s} x_k^* h_k(y) \right) (\overline{y}).$$

Let $\Pi : \mathbb{R}^r \rightrightarrows \mathbb{R}^s$ be a set-valued function, and $\bar{x} \in \Pi(\bar{y})$. We say that Π is calm at $(\overline{y}, \overline{x})$ if there exist some $L > 0$ and neighborhoods U and V around $\overline{x}$ and $\overline{y}$, respectively, such that $d_{\Pi(\overline{y})}(x) \leq L\|y - \overline{y}\|$,for all $y \in V$ and $x \in U \cap \Pi(y)$, where $d_{\Pi(\overline{y})}(x)$ denotes the distance between x to $\Pi(\overline{y})$.

Theorem 1. [5, Theorem 4.1] Suppose that the set-valued mapping $\mathcal{F} : \mathbb{R}^l \rightrightarrows \mathbb{R}^k$ is defined as

$$\mathcal{F}(y) := \{x \in C \mid g(x) + y \in E\},$$

where the function $g : \mathbb{R}^k \to \mathbb{R}^l$ is locally Lipschitz and $(C, E) \subseteq \mathbb{R}^k \times \mathbb{R}^l$ is closed. If $\mathcal{F}$ is calm at $(0, \overline{x}) \in Gph\mathcal{F}$, then

$$N_M\big(\mathcal{F}(0), \overline{x}\big) \subseteq \bigcup_{y^* \in N_M(E, g(\overline{x}))} D^*g(\overline{x})(y^*) + N_M(C, \overline{x}).$$

Theorem 2. [5, Corollary 3.4] Consider the set-valued function $\mathcal{F} : \mathbb{R}^p \rightrightarrows \mathbb{R}^k$,

$$\mathcal{F}(y) := \{x \in \mathbb{R}^k \mid g(x, y) \in E\},$$

where $g : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^q$ is locally Lipschitz and $E \subseteq \mathbb{R}^q$ is closed. Let $(\bar{y}, \bar{x}) \in Gph\mathcal{F}$. Further, assume the following qualification condition holds,

$$\bigcup_{z^* \in N_M(E, g(\bar{x}, \bar{y})) \setminus \{0\}} [\partial_M \langle z^*, g \rangle (\overline{x}, \overline{y})]_x = \emptyset,$$

where $[\ ]_x$ denotes projection onto the x-component. Then, $\mathcal{F}$ is calm at $(\bar{y}, \bar{x})$.

For two vectors $x, y \in \mathbb{R}^p$, the inequality $x \leqq y$ stands for $x_i \leq y_i$ for all $i \in \{1, 2, \ldots, p\}$. The inequality $x \leq y$ means $x \leqq y$ and $x \neq y$. Furthermore, $x < y$ stands for $x_i < y_i$ for all $i \in \{1, 2, \ldots, p\}$.

3 Main Results

At the start of this section, we recall that the feasible solution set of MMPVC which is defined in (1) is denoted by Ω. Also, we recall the following definition from [4, pp. 110].

Definition 1. A feasible point $x_0 \in \Omega$ is called a properly efficient solution to MMPVC when there exists a vector $\lambda > 0_p$ such that

$$\langle \lambda, F(x_0) \rangle \leq \langle \lambda, F(x) \rangle, \qquad \forall x \in \Omega.$$

Throughout this paper, we fix a feasible point $\hat{x} \in \Omega$, and divide the index set I as

$$I_+ := \{i \in I \mid H_i(\hat{x}) > 0\}, \quad \text{and} \quad I_0 := \{i \in I \mid H_i(\hat{x}) = 0\}.$$

Also, we divide these two index sets as

$$I_+^0 := \{i \in I_+ \mid G_i(\hat{x}) = 0\}, \qquad I_+^- := \{i \in I_+ \mid G_i(\hat{x}) < 0\},$$
$$I_0^+ := \{i \in I_0 \mid G_i(\hat{x}) > 0\}, \qquad I_0^0 := \{i \in I_0 \mid G_i(\hat{x}) = 0\},$$
$$I_0^- := \{i \in I_0 \mid G_i(\hat{x}) < 0\}.$$

Now, we introduce a new constraint qualification for MMPVC that plays a key rule in this section.

Definition 2. The MMPVC is said to be satisfy to $(\mathfrak{CQ})$ at $\hat{x}$ if there are not, non-zero together, scalars α_i and β_i for $i \in I$, satisfying $\alpha_i \geq 0$ for $i \in I_0^0 \cup I_+^0$, $\beta_i \geq 0$ for $i \in I_0^-$, $\alpha_i \beta_i = 0$ for $i \in I_0^0$, and

$$0 \in \sum_{i \in I_0^0 \cup I_+^0} \alpha_i \partial G_i(\hat{x}) - \sum_{i \in I_0} \beta_i \partial H_i(\hat{x}).$$

We should mention that $(\mathfrak{CQ})$ is a generalization of a constraint qualification that is defined by Ye [16] for mathematical programming with equilibrium constraints (MPEC), named "No Nonzero Abnormal Multiplier Constraint Qualification". This constraint qualification was extended to nonsmooth MPECs by Movahedian and Nobakhtian [14], and is considered for MMPVC, for the first time, in the present paper.

Example 1. Let

$$\Omega = \{x \in \mathbb{R}^2 \mid x_1 \geq -x_2, \quad x_2(x_1 + x_2) \leq 0\},$$

and $\hat{x} = 0_2 \in \Omega$. This set can be considered as feasible set of a MMPVC with following data:

$$H_1(x_1, x_2) = x_1 + x_2, \qquad \text{and} \qquad G_1(x_1, x_2) = x_2.$$

Obviously, $I_0 = \{1\}$, $\partial H_1(\hat{x}) = \{(1,1)\}$ and $\partial G_1(\hat{x}) = \{(0,1)\}$. A short calculation shows that

$$0_2 \in \alpha_1 \partial G_1(\hat{x}) - \beta_1 \partial H_1(\hat{x}), \quad \alpha_1 \geq 0, \ \beta_1 \geq 0 \quad \Longrightarrow \quad \alpha_1 = \beta_1 = 0,$$

and so, the $\mathfrak{CQ}$ holds at $\hat{x}$.

The following theorem presents the first main result of this section.

Theorem 3. Let $\hat{x}$ be a properly efficient solution to MMPVC. If $(\mathfrak{CQ})$ holds at $\hat{x}$, then there exist scalars μ_j^F, μ_i^H and μ_i^G, for $j \in J$ and $i \in I$, such that:

$$0_n \in \sum_{j=1}^{p} \mu_j^F \partial f_j(\hat{x}) + \sum_{i=1}^{m} \left[\mu_i^G \partial G_i(\hat{x}) - \mu_i^H \partial H_i(\hat{x})\right], \tag{3}$$

$$\mu_i^G \geq 0, \ i \in I_0^0 \cup I_+^0; \qquad \mu_i^G = 0, \ i \in I_0^+ \cup I_0^- \cup I_+^-, \tag{4}$$

$$\mu_i^H \text{ free}, \ i \in I_0^0 \cup I_0^+; \qquad \mu_i^H \geq 0, \ i \in I_0^-; \qquad \mu_i^H = 0, \ i \in I_+, \tag{5}$$

$$\mu_i^H \mu_i^G = 0, \quad i \in I_0^0, \tag{6}$$

$$(\mu_1^F, \ldots, \mu_p^F) > 0_p. \tag{7}$$

Proof. Since $\hat{x}$ is a properly efficient solution to MMPVC, Definition 1 concludes that there exist some positive scalars $\mu_j^F > 0$, for $j \in J$, such that $\hat{x}$ is a minimizer to the following weighted problem:

$$\min \sum_{j=1}^{p} \mu_j^F f_j(x) \qquad \text{subject to} \qquad x \in \Omega.$$

Therefore, $\sum_{j=1}^{p} \mu_j^F f_j + \mathcal{I}_S$ attains its global minimum at $\hat{x}$. Hence,

$$\begin{aligned} 0_n &\in \partial_M \left(\sum_{j=1}^{p} \mu_j^F f_j + \mathcal{I}_\Omega \right)(\hat{x}) \subseteq \sum_{j=1}^{p} \mu_j^F \partial_M f_j(\hat{x}) + \partial_M \mathcal{I}_\Omega(\hat{x}) \\ &= \sum_{j=1}^{p} \mu_j^F \partial f_j(\hat{x}) + N_M(\Omega, \hat{x}). \end{aligned} \tag{8}$$

For estimating of $N_M(\Omega, \hat{x})$, for all $i \in I$ take $\Theta_i(x) := \big(G_i(x), H_i(x)\big)$, and let $\Theta(x) := \big(\Theta_1(x), \dots, \Theta_m(x)\big)$. Also, set

$$\mathcal{X}_* := \{(v^1, v^2) \in \mathbb{R}^2 \mid v^2 \geq 0 \ \text{ and } v^1 v^2 \leq 0\},$$

and $\mathcal{X} := \big\{(v_1, \dots, v_m) \in \big(\mathbb{R}^2\big)^m \mid v_i := (v_i^1, v_i^2) \in \mathcal{X}_*, \quad \forall i \in I\big\}$. Since $\mathcal{X} = \prod_{i=1}^m \mathcal{X}_*$, then

$$N_M\big(\mathcal{X}, \Theta(\hat{x})\big) = \prod_{i=1}^{m} N_M\big(\mathcal{X}_*, \Theta_i(\hat{x})\big), \tag{9}$$

by (2). On the other hand, the following equality has been proved in [7, Lemma 3.2]:

$$N_M(\mathcal{X}_*, \Theta_i(\hat{x})) = \begin{cases} \mathcal{X}_* & \text{for} \quad i \in I_0^0 \\ \{0\} \times \mathbb{R} & \text{for} \quad i \in I_0^+ \\ \{0\} \times \mathbb{R}_- & \text{for} \quad i \in I_0^- \\ \mathbb{R}_+ \times \{0\} & \text{for} \quad i \in I_+^0 \\ \{0\} \times \{0\} & \text{for} \quad i \in I_+^-. \end{cases} \tag{10}$$

Owing to (9)-(10), the $(\mathfrak{CQ})$ at $\hat{x}$ implies that for each $\rho = \big(\rho_1^G, \rho_1^H, \dots, \rho_m^G, \rho_m^H\big) \in N_M\left(\mathcal{X}, \Theta(\hat{x})\right)$ we have

$$0_n \in \sum_{i \in I} \left[\rho_i^G \partial G_i(\hat{x}) + \rho_i^H \partial H_i(\hat{x})\right] \implies \rho = 0_{2m}.$$

Thus,

$$0_n \notin \bigcup_{0_{2m} \neq \rho \in N_M(\mathcal{X}, \Theta(\hat{x}))} \left[\partial\left(\langle \rho, \Theta(x) + y \rangle\right)(\hat{x}, 0_m)\right]_x.$$

From this and Theorem 2 we conclude that the set-valued function $\widehat{\Omega}(.)$ is calm at $(\hat{x}, 0_m)$, where $\widehat{\Omega}(y) := \{x \in \mathbb{R}^n \mid \Theta(x) + y \in \mathcal{X}\}$ for each $y \in \mathbb{R}^{2m}$. Since $\widehat{\Omega}(0_m) = \Omega$, Theorem 1 deduces that

$$N_M(\Omega, \hat{x}) \subseteq \bigcup_{\lambda \in N_M(\mathcal{X}, \Theta(\hat{x}))} D^*\Theta(\hat{x})(\lambda) + N_M(\mathbb{R}^n, \hat{x}). \tag{11}$$

On the other hand, by(2), for each $\lambda := \big(\lambda_1^H, \lambda_1^G, \dots, \lambda_m^H, \lambda_m^G\big) \in \mathbb{R}^{2m}$ we have

$$D^*\Theta(\hat{x})(\lambda) = \partial_M \langle \lambda, \Theta(.)\rangle(\hat{x}) = \partial \left[\sum_{i=1}^{m} \left(\lambda_i^H H_i + \lambda_i^G G_i\right)\right](\hat{x})$$

$$= \sum_{i=1}^{m} \left[\lambda_i^H \partial H_i(\hat{x}) + \lambda_i^G \partial G_i(\hat{x})\right].$$

According to above equality,(11) and the fact that $N_M(\mathbb{R}^n, \hat{x}) = \{0_n\}$, we get tho following estimate for $N_M(\Omega, \hat{x})$:

$$N_M(\Omega, \hat{x}) \subseteq \bigcup_{\lambda \in N_M(\mathcal{X}, \Theta(\hat{x}))} \left[\sum_{i=1}^{m} \left(\lambda_i^H \partial H_i(\hat{x}) + \lambda_i^G \partial G_i(\hat{x})\right)\right].$$

Hence, the last inclusion and (8) imply that

$$0_n \in \sum_{j=1}^{p} \mu_j^F \partial f_j(\hat{x}) + \bigcup_{\lambda \in N_M(\mathcal{X}, \Theta(\hat{x}))} \left[\sum_{i=1}^{m} \left(\lambda_i^H \partial H_i(\hat{x}) + \lambda_i^G \partial G_i(\hat{x})\right)\right].$$

Therefore, there exists some $\lambda := (\lambda_1^H, \lambda_1^G, \ldots, \lambda_m^H, \lambda_m^G) \in N_M(\mathcal{X}, \Theta(\hat{x}))$ such that

$$0 \in \sum_{j=1}^{p} \mu_j^F \partial f_j(\hat{x}) + \sum_{i=1}^{m} \left[\lambda_i^H \partial H_i(\hat{x}) + \lambda_i^G \partial G_i(\hat{x})\right]. \tag{12}$$

From (10) and $\lambda \in N_M(\mathcal{X}, \Theta(\hat{x}))$, we can conclude that $\lambda_i^G \geq 0$ for $i \in I_0^0 \cup I_+^0$, $\lambda_i^G = 0$ for $i \in I_0^+ \cup I_0^- \cup I_+^0$, λ_i^H is free for $i \in I_0^0 \cup I_0^+$, $\lambda_i^H \leq 0$ for $i \in I_0^-$, $\lambda_i^H = 0$ for $i \in I_+^0 \cup I_+^-$, and $\lambda_i^H \lambda_i^G = 0$ for $i \in I_0^0$. Taking $\mu_i^G := \lambda_i^G$ for $i \in I$, $\mu_i^H := -\lambda_i^H$ for $i \in I_0^0$, $\mu_i^H := \lambda_i^H$ for $i \in I \setminus I_0^0$, and considering (12), the result is justified. □

It is worth mentioning that when $p = 1$, the relations (3)-(7), named M-stationary condition, are proved in [7, 8] for the problems with smooth data, and in [14] for nonsmooth MPECs. The present paper is the first that studies this kind of stationary condition for MMPVCs.

We know from classic nonlinear optimization that necessary optimality conditions are also to be sufficient under convexity assumption. These results cannot be applied for MMPVC since the product function $H_i G_i$ does not convex. The following theorem, which is our second main result in this section, shows the sufficient condition holds for MMPVCs, under some additional weak assumptions.

Theorem 4. Let $\hat{x} \in \Omega$ be a feasible solution that satisfies in (3)-(7) for some scalars μ_j^F, μ_i^H, and μ_i^G, $(i, j) \in I \times J$.

(a): If

$$\mathcal{A} := \{i \in I_0^0 \mid \mu_i^H < 0\} \cup \{i \in I_0^0 \mid \mu_i^H = 0,\ \mu_i^G > 0\} = \emptyset,$$

then $\hat{x}$ is a local properly efficient to MMPVC.

(b): If

$$\mathcal{B} := \mathcal{A} \cup \{i \in I_0^+ \mid \mu_i^H < 0\} \cup \{i \in I_+^0 \mid \mu_i^H = 0,\ \mu_i^G > 0\} = \emptyset,$$

then $\hat{x}$ is a global properly efficient to MMPVC.

Proof. **(a):** Suppose that $\hat{x}$ is not locally properly efficient to MMPVC. Then, for each neighborhood $U \subseteq \mathbb{R}^n$ to $\hat{x}$, and for each vector $\lambda = (\lambda_1, \ldots, \lambda_p) > 0_p$, we can find a point $x_\lambda^U \in \Omega \cap U$ such that

$$\sum_{j=1}^{p} \lambda_j f_j(\hat{x}) > \sum_{j=1}^{p} \lambda_j f_j(x_\lambda^U).$$

Notice that (7) leads us take $\lambda = \mu^F := (\mu_1^F, \ldots, \mu_p^F)$ in above inequality. So, the convexity of $\sum_{j=1}^{p} \mu_j^F f_j$ implies that

$$\left\langle \varsigma, x_\mu^U - \hat{x} \right\rangle \le \sum_{j=1}^{p} \mu_j^F f_j(x_\mu^U) - \sum_{j=1}^{p} \mu_j^F f_j(\hat{x}) < 0, \qquad \forall \varsigma \in \partial \left(\sum_{j=1}^{p} \mu_j^F f_j \right)(\hat{x}).$$

The last inequality and the fact that $\partial \left(\sum_{j=1}^{p} \mu_j^F f_j \right)(\hat{x}) = \sum_{j=1}^{p} \mu_j^F \partial f_j(\hat{x})$ conclude that

$$\sum_{j=1}^{p} \mu_j^F \left\langle \varsigma_j, x_\mu^U - \hat{x} \right\rangle < 0, \quad \exists x_\mu^U \in U \cap \Omega, \ \forall \varsigma_j \in \partial f_j(\hat{x}). \tag{13}$$

On the other hand, (3) implies that

$$\sum_{j=1}^{p} \mu_j^F \xi_j^F + \sum_{i=1}^{m} (\mu_i^G \xi_i^G - \mu_i^H \xi_i^H) = 0, \tag{14}$$

for some $\xi_j^F \in \partial f_j(\hat{x})$, $\xi_i^H \in \partial H_i(\hat{x})$ and $\xi_i^G \in \partial G_i(\hat{x})$, for $(i, j) \in I \times J$.

Let $i \in I_0^+$. The continuity of G_i concludes that there exists a neighborhood U_i for $\hat{x}$ such that $G_i(x) > 0$ for all $x \in U_i$. Thus, $G_i(x) > 0$, $H_i(x) \ge 0$ and $G_i(x)H_i(x) \le 0$, for all $x \in U_i \cap \Omega$, which imply $H_i(x) = 0$. Similarly, for each $i \in I_+^0$ there exists a neighborhood $\widehat{U}_i$ for $\hat{x}$ such that $H_i(x) > 0$ and $G_i(x) \le 0$. Summarizing, for all $x \in \Omega \cap V$ in which $V := \bigcap_{i \in I_0^+} U_i \cap \bigcap_{i \in I_0^+} \widehat{U}_i$, we have $G_i(x) \le 0 = G_i(\hat{x})$, for $i \in I_+^0$, and $H_i(x) = 0 \le H_i(\hat{x})$, for $i \in I_0^+$. Hence

$$\langle \xi_i^G, x - \hat{x} \rangle \le 0, \quad \forall i \in I_+^0, \quad \text{and} \quad \langle \xi_i^H, x - \hat{x} \rangle \le 0, \quad \forall i \in I_0^+.$$

So, owing to (4)-(6), we get

$$\langle \sum_{i \in I_+^0 \cup I_0^+} \left(\mu_i^G \xi_i^G - \mu_i^H \xi_i^H \right), x - \hat{x} \rangle \le 0, \qquad \forall x \in \Omega \cap V.$$

By the above inequality, convexity of functions, assumption that $\mathcal{A} = \emptyset$, (4)-(6), and a short calculation, we deduce that

$$\langle \sum_{i=1}^{m} \left(\mu_i^G \xi_i^G - \mu_i^H \xi_i^H \right), x - \hat{x} \rangle \le 0, \qquad \forall x \in \Omega \cap V. \tag{15}$$

Now, inner-producing two sides of (14) to $x - \hat{x}$ and regarding (15), we conclude that

$$\sum_{j=1}^{p} \mu_j^F \left\langle \xi_j^F, x - \hat{x} \right\rangle \ge 0, \qquad \forall x \in \Omega \cap V,$$

which contradicts (13). Thus, the proof is complete.

(b): Emptiness assumption of $\mathcal{B}$ leads us to repeat the proof of (a) without considering any neighborhood for $\hat{x}$.

□

Example 2. Consider the MMPVC with following data:

$$\begin{aligned} &f_1(x_1,x_2) = x_1^2 + |x_2|, && f_2(x_1,x_2) = 2x_1^4 + 3|x_2|, \\ &H_1(x_1,x_2) = -x_2, && H_2(x_1,x_2) = |x_1| + x_2, \\ &G_1(x_1,x_2) = -1, && G_2(x_1,x_2) = -x_1. \end{aligned}$$

Taking $\hat{x} = 0_2$, we conclude that $I_0^- = \{1\}$ and $I_0^0 = \{2\}$. Since the conditions (3)-(7) hold for $\mu_1^F = \mu_2^F = 1$, $\mu_1^H = \mu_2^H = \frac{1}{4}$ and $\mu_1^G = \mu_2^G = 0$, and also $\mathcal{B} = \emptyset$, Theorem 4 implies that $\hat{x}$ is properly sufficient for the problem.

4 Conclusion

In this paper, we considered a new class of nonsmooth multiobjective optimization problems, denoted by MMPVC, as an extension of the mathematical programs with vanishing constraints from the scalar case and the multiobjective mathematical programming with equilibrium constraints. We introduced a suitable modification of the "No Nonzero Abnormal Multiplier Constraint Qualification". We gave Karush-Kahn-Tucker type necessary optimality condition for proper efficient solutions, and derived that this necessary condition is also sufficient for proper efficiency under some additional assumptions in emptiness kind.

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مشخص‌سازی جواب‌های موثر سره برای مسائل چندهدفه‌ی محدب با قیود غیرمشتق‌پذیر پوچ‌شونده

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چکیده

ما در این مقاله یک مسئله‌ی بهینه‌سازی چندهدفه‌ی محدب را در نظر می‌گیریم که توسط قیدهای پوچ‌شونده تعریف می‌شود. در ابتدا، یک قید تعریفی جدید برای مسئله معرفی کرده و توسط مخروط نرمال مردخویچ، یک شرط لازم برای جواب‌های موثر سره‌ی مسئله ارائه خواهیم داد. آنگاه ثابت خواهیم کرد که شرط لازم بیان شده، شرط کافی نیز برای جواب‌های موثر سره می‌باشد. احکام ما بر حسب زیرمشتق محدب فرمول‌بندی شده‌اند.

کلمات کلیدی

بهینه‌سازی چندهدفه، قیود چندهدفه، بهینه‌سازی محدب، قیدهای تعریفی.