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Finding the Optimal Place of Sensors for a 3-D Damped Wave Equation by using Measure Approach

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Abstract. In this paper, we model and solve the problem of optimal shaping and placing to put sensors for a 3-D wave equation with constant damping in a bounded open connected subset of 3-dimensional space. The place of sensor is modeled by subdomain of this region of a given measure. By using an approach based on the embedding process, first, the system is formulated in variational form; then, by defining two positive Radon measures, the problem is represented in a space of measures. In this way, the shape design problem is turned into an infinite linear problem whose solution is guaranteed. In this step, the optimal solution (optimal control, optimal region and optimal energy) is identified by a 2-phase optimization search technique applying two subsequent approximation steps. Moreover, some numerical simulations are given to compare this new method with other methods.

Keywords. 3-D damped wave equation, Dissipation control, Radon measure, Search technique, Shape optimization.

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1 Introduction and Problem Statement

Shape optimization is part of the field of optimal control. Typically, there is a system governed by a partial differential equation whose solution, u_Ω, depends on some geometric variables of the shape. The problem is to minimize a given cost functional $J(u_\Omega)$ over the set S of all admissible shapes with piecewise smooth boundary. These kinds of problems are typically solved numerically.
Unfortunately, very limited number of articles and books are available on 3-D shape optimization; however, many industrial factors cannot be assumed in a 2-D manner and a 3-D design is needed. But, 3-D optimal shape design methods are problematic because of the following reasons:

i) The main challenge of most optimization methods is the description of the performed shapes in terms of design variables ([20]).

ii) Mesh deformation (such as finite element method) is a major problem for 3-D optimal shape design problems since after a few iterations, the mesh may no longer be feasible. It may cause divergence of the optimization algorithm (see [20]).

iii) In contrast with 2-D shape optimization problems, parameterization techniques for 3-D problems describe the shape or the shape modifications with a large set of constraints which cause some problems in the convergence of the optimization process (see [4]).

iv) Iterative methods, such as the level set method, require the objective function to be decreased; but their main drawback is the possibility of falling into a local (and non-global) minima if the initialization is too far from a global minimum ([2]).

On the other hand, in 1986, Rubio introduced an embedding process for solving optimal control problems governed by ordinary differential equations (see [26]), using positive Radon measures. Then, it was employed to obtain the optimal control for systems governed by partial differential equations (like [16] and [27]). Consequently, since 1999 till now, with the help of this method, different cases of the optimal shape design problems have been solved (a brief report of these kinds of work was given in [11] and we can also emphasize on [8], [9], [19], [10] and [12]). The main goal of this paper is to extend the above-mentioned method for designing unknown general three dimensional optimal shapes. We emphasize that this method does not depend on an initial shape or value and can also cover the above mentioned difficulties.

In many technological situations, a given structure whose optimal position is at rest (for instance), starts to vibrate due to uncontrolled disturbances which we would like to stop. One possibility is through damping mechanisms as is described, certainly in an ideal situation, in [22]. So far, several studies have extensively investigated the problem of optimal stabilization for the 2-D wave equation from different perspectives (see, for instance, [5], [6], [7], [14] and [18]). The performed analysis by Hebrard et al. highlights the effect of the over-damping phenomenon characteristic of this damped wave equation [15]. Freitas [14] and Lopez [18] solved the mentioned problem in which the dissipation vanishes for large values of the constant damping coefficient. In 2006, Munch et al. used Young measure to solve a similar problem and presented a solution method (at least for the problem with a constant damping function); for this purpose, first, the problem was transferred to a variational form (called relax problem) by applying a theorem about Young measure properties. In that study, the damping coefficient was fixed and the best unknown internal region was determined by the use of descend gradient method [21]. In sequence, the best damping coefficient and damping set were determined at different times using level set method [22].

Having a bounded domain Ω of $\mathbb{R}^3$, in this paper, the problem of finding an optimal observation domain $\omega \subset \Omega$ or general damping wave equations is modeled and solved. he aim is to optimize not only the placement but also the shape of ω, over all possible measurable subsets of Ω having a certain prescribed measure. Although such questions are frequently posed in engineering applications, they are under-researched in mathematics. In this regard, for the first time, we consider a shape optimization problem to find the optimal place of a sensor, modeled by a three-dimensional wave equation with a fixed damping coefficient. The objective is to find the shape of the damping set that minimizes the energy at some given end time ([23] and [24]).

2 Optimal Wave Damping Problem

Let $\Omega \subset \mathbb{R}^3$ be a domain with piecewise smooth boundary and consider the three-dimensional damping wave equation with Dirichlet boundary conditions. Consider additionally that ω is a subset of Ω of positive Lebesgue measure and independent of the time $t \in (0, T)$. Moreover, the damping potential a is such that $a(x) = a > 0$ $a.e.$ $x \in \omega$. The resulting equation for the displacement of the sensor is then ([22] and [3])

$$\begin{cases} \ddot{u} - \Delta u + a(\mathbf{x})\dot{u} = 0, & (\mathbf{x}, t) \in \Omega \times (0, T), \\ u = 0, & (\mathbf{x}, t) \in \partial\Omega \times (0, T), \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \dot{u}(\mathbf{x}, 0) = u_1(\mathbf{x}), & x \in \Omega. \end{cases} \tag{1}$$

We refer to ω as the damping set which is an unknown region in Ω, $\partial\omega$ is a smooth and simple closed curve boundary, which must be identified, $\ddot{u}$, u_0 and u_1 are also indicated as $\frac{\partial^2 u}{\partial t^2}$, initial position and velocity, respectively. The energy of the system is known to be [31]

$$E(\omega, a, t) = \frac{1}{2}\int_{\Omega}(|\dot{u}(\mathbf{x}, t)|^2 + |\nabla u(\mathbf{x}, t)|^2)dv. \tag{2}$$

The objective is to minimize the total energy of the membrane at some fixed end time T:

$$Min_{\omega\in S} \quad E(\omega, a, T).$$

This is a shape optimization problem the solution of which depends on the chosen constants a and T. In this study, we choose a moderate in such a way to avoid any problems related to the phenomena of overdamping, and T is large enough so that observability problems related to the finite propagation speed of waves do not occur. Not having any kind of constraint on the damping set, we will obtain the trivial solution $\omega^* = \Omega$; furthermore, we introduce the area constraint

$$V_L = \{\omega \subset \Omega : |\omega| = L|\Omega|, 0 < L < 1\}, \tag{3}$$

in which $|\omega|$ indicates the measure of ω. This constraint can be shown by the following integral relation:

$$\int_{\omega} dv = L\int_{\Omega} dv. \tag{4}$$

System (1) is well-posed (see [17]) and its energy is satisfied in the following dissipation law (see [30]):

$$\dot{E}(\omega, a, t) = -\int_{\Omega} a(\mathbf{x})|\dot{u}(\mathbf{x}, t)|^2 dv \leq 0.$$

Here, u denotes the transversal displacement at point $\mathbf{x}$ in time t.

The mentioned optimal shape design (OSD) problem is defined based on the unknown geometrical pair $(\omega, \partial\omega)$; this pair consists of a measurable set that can be regarded as a nonempty region, and a simple closed surface which is its boundary. The unknown shape which is bounded and has a specified volume is placed on the top of the plane (r, θ). Its boundaries include the unknown surface $\partial\omega$ with equation $z = f(r, \theta) : D \to A$ and its image in the plane (r, θ) is region D; that is, a simple smooth closed curve. Also, its height is bounded between z_{min} and z_{max}. If region D is known, we can obtain the unknown $\partial\omega$ according to the method presented in the next sections. Therefore, we intend to find the optimal unknown surface $\partial\omega$ and the optimal unknown region D simultaneously so that a given performance criterion is minimized. Moreover, a curve can be approximated by broken lines so that ∂D (and hence D) can be approximated with a number M of its points (corners of broken lines belonging to

∂D) which is called the M-representation of D. For a fixed number M, the points in the M-representation set can have the fixed θ-components like $\theta_i = \theta_i', \quad i = 1, 2, \ldots, M$ without losing generality. Hence, each admissible M-representation set called D_M can be characterized by M variables $r_1, r_2, \ \ldots, r_M$. Consequently, ∂D defined by a finite set of M real variables $(r_1, \ r_2, \ldots, r_M)$. Therefore, we introduce the set of admissible surfaces as follow:

$$\omega_A = \{(\partial\omega, D) | \omega = \partial\omega \cup D, D \in D_M, \ z_{min} \le z \le z_{max}, \int_\omega dv = L \int_\Omega dv\},$$

where region $D \subset \mathbb{R}^2$ is defined as follow:

$$D = \{(r, \theta) | \ 0 \le r \le h(\theta), \quad 0 \le \ \theta \le 2\pi\},$$

where $h(\theta)$ is an unknown continuous function. We prefer to solve appropriate problems in cylindrical coordinates since where $0 \le \theta \le 2\pi$ and $r > 0$, the curve ∂D is automatically simple. This simple fact is an essential part in our calculations and also in numerical simulations. Then, we have:

$$|\nabla u| = [(\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2 + (\frac{\partial u}{\partial z})^2],$$

and therefore:

$$E(\omega, a, T) = \frac{1}{2}\int_\Omega [|\dot{u}(r, \theta, z, T)|^2 + (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2 + (\frac{\partial u}{\partial z})^2] r dr d\theta dz. \tag{5}$$

It is worth mentioning that the nature of Ω has not changed but its representation has changed; therefore, we use the same symbol so that at the end, the optimal shape is shown in cylindrical coordinates.

In the present study, for the first time, we determined 3-D unknown w by a linearization method based on the properties of Radon measure. We attempt to find the unknown region ω through a two-phase optimization procedure which is based on an embedding technique. To apply this method, first, we represent the problem into a variational form; next, it is transferred into a new measure theoretical problem in which two unknown positive Radon measures in a product space of measures are sought. Then, the solution procedure is explained and finally, by a 2-phase optimization technique, a nearly optimal shape as well as the minimizing value of system energy are constructed.

The paper is organized as follows: the next section is devoted to the basic deformation in variational form. Section 4 is deals with embedding process and approximation schemes. In Section 5, based on the previous discussions, we present the solution algorithm. Then, a numerical simulation is presented in Section 6. Concluding remarks are also presented in Section 7.

3 Basic Deformation

In general, it is difficult to identify a classical solution for problem (1); thus, attempts have usually been made to find a weak (or generalized) solution of the problem, which is more applicable in our work. The main idea in this replacement is to change the problem into the variational form. To this end, by multiplying (1) with a function $\varphi \in H_0^1((\Omega \times (0,T)))$ and using Green theorem ([28]), we have:

$$\int_\Omega \ddot{u}\varphi dv - \int_\Omega u\Delta\varphi dv + \int_\Omega a(r,\theta,z)\dot{u}\varphi dv = 0. \tag{6}$$

Integrating both sides of (6) with respect to t over $[0,T]$ gives:

$$\int_0^T \int_\Omega \ddot{u}\varphi dvdt - \int_0^T \int_\Omega u\Delta\varphi dvdt + \int_0^T \int_\Omega a(r,\theta,z)\dot{u}\varphi dvdt = 0, \tag{7}$$

By part-by-part integrating the first left expression with respect to t twice and integrating the third expression of (7), we can conclude that:

$$\begin{aligned}
&\textstyle\int_0^T \int_\Omega \ddot{u}\varphi dvdt = \int_\Omega [\dot{u}(T)\varphi(T) - \dot{u}(0)\varphi(0) - u(T)\dot{\varphi}(T) + u(0)\dot{\varphi}(0)]dv \\
&\textstyle+ \int_0^T \int_\Omega u\ddot{\varphi}dvdt; \\
&\textstyle\int_0^T \int_\Omega a(r,\theta,z)\dot{u}\varphi dvdt = \int_\Omega a(r,\theta,z)[u(T)\varphi(T) - u(0)\varphi(0) - \int_0^T u\dot{\varphi}dt]dv \\
&\textstyle= \int_\Omega a(r,\theta,z)[u(T)\varphi(T) - u(0)\varphi(0)]dv - \int_0^T \int_\Omega a(r,\theta,z)u\dot{\varphi}dvdt.
\end{aligned} \tag{8}$$

Now, by substituting initial conditions (1.c) in (8), we have:

$$\begin{aligned}
&\textstyle\int_0^T \int_\Omega \ddot{u}\varphi dvdt = \int_\Omega [\dot{u}(T)\varphi(T) - u_1\varphi(0) - u(T)\dot{\varphi}(T) + u_0\dot{\varphi}(0)]dv \\
&\textstyle+ \int_0^T \int_\Omega u\ddot{\varphi}dvdt; \\
&\textstyle\int_0^T \int_\Omega a(r,\theta,z)\dot{u}\varphi dvdt = \int_\Omega a(r,\theta,z)[u(T)\varphi(T) - u_0\varphi(0)]dv \\
&\textstyle- \int_0^T \int_\Omega a(r,\theta,z)u\dot{\varphi}dvdt.
\end{aligned} \tag{9}$$

By applying (9), the equality (7) is changed to:

$$\begin{aligned}
&\textstyle\int_\Omega \dot{u}(T)\varphi(T)dv - \int_\Omega u(T)\dot{\varphi}(T)dv - \int_0^T \int_\Omega u\Delta\varphi dvdt + \int_\Omega a(r,\theta,z)u(T)\varphi(T)dv \\
&\textstyle- \int_\Omega a(r,\theta,z)u_0\varphi(0)dv - \int_0^T \int_\Omega a(r,\theta,z)u\dot{\varphi}dvdt + \int_0^T \int_\Omega u\ddot{\varphi}dvdt \\
&\textstyle= \int_\Omega [u_1\varphi(0) - u_0\dot{\varphi}(0)]dv, \qquad\qquad \varphi \in H_0^1(\Omega \times (0,T))
\end{aligned} \tag{10}$$

Moreover, for all $(r,\theta,z,t) \in \partial\Omega \times [0,T]$ by the initial condition, we have $u(r,\theta,z,t) = 0$; to applying this condition and using Green theorem, we have:

$$\int_{\partial\Omega} u(r,\theta,z,t)\varphi(r,\theta,z,t).nd\sigma = \int_{\Omega} div(u(r,\theta,z,t)\varphi(r,\theta,z,t))dv = 0, \tag{11}$$

where n is outward normal vector on boundary Ω. With the definition of $a(r,\theta,z)$, we have:

$$\begin{aligned}
&\textstyle\int_{\Omega} a(r,\theta,z)u(T)\varphi(T)dv = \int_{\omega} au(T)\varphi(T)dv\\
&\textstyle\int_{\Omega} a(r,\theta,z)u_0\varphi(0)dv = \int_{\omega} au_0\varphi(0)dv\\
&\textstyle\int_0^T \int_{\Omega} a(r,\theta,z)u\dot{\varphi}dvdt = \int_0^T \int_{\omega} au\dot{\varphi}dvdt.
\end{aligned}$$

Due to the nature of surface $\partial\omega$ (smooth and continuous), we can change the volume integrals on ω into surface integrals in order to simplify the calculations as follow:

$$\begin{aligned}
&\textstyle\int_{\omega} dv = \int\int_D \int_{z_{min}}^{z} dv = \int\int_D (z - z_{min})dA = L\int_{\Omega} dv;\\
&\textstyle\int_{\omega} au(T)\varphi(T)dv = a\int\int_D \int_{z_{min}}^{z} u(T)\varphi(T)dv = a\int\int_D z(u(T)\varphi(T))dA,
\end{aligned}$$

without loss of generality, by just moving plane (r,θ) to plane $z = z_{min}$, we can assume $z_{min} = 0$.

Therefore, problem of obtaining the optimal shape for minimizing energy of system (1) in cylindrical coordinates has the following generalized presentation:

$$\begin{aligned}
\min \quad & E(\omega,a,T) = \frac{1}{2}\int_{\Omega}[|\dot{u}(r,\theta,z,T)|^2 + (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2 + (\frac{\partial u}{\partial z})^2]rdrd\theta dz,\\
s.t.\quad &\\
&\int_{\Omega}\dot{u}(T)\varphi(T)rdrd\theta dz - \int_{\Omega}u(T)\dot{\varphi}(T)rdrd\theta dz - \int_0^T\int_{\Omega}u\Delta\varphi rdrd\theta dzdt\\
&\quad + \int_{\Omega}a(r,\theta,z)u(T)\varphi(T)rdrd\theta dz - \int_{\Omega}a(r,\theta,z)u_0\ \varphi(0)rdrd\theta dz\\
&\quad - \int_0^T\int_{\Omega}a(r,\theta,z)u\dot{\varphi}rdrd\theta dzdt + \int_0^T\int_{\Omega}u\ddot{\varphi}rdrd\theta dzdt = \Phi;\\
&\int_{\Omega}div(u(r,\theta,z,t)\varphi(r,\theta,z,t))rdrd\theta dz = 0, \qquad \forall\varphi \in H_0^1(\Omega\times(0,T));\\
&\int_{\omega}rdrd\theta dz = L\int_{\Omega}rdrd\theta dz,
\end{aligned} \tag{12}$$

in which

$$\Phi = \int_{\Omega}[u_1(r,\theta,z)\varphi(0) - u_0(r,\theta,z)\dot{\varphi}(0)]rdrd\theta dz.$$

To solve (12), situations are ready to use an embedding process; therefore, we change the problem and consider a new one with a different formulation. By applying this method, we show how one can obtain optimal region ω and the amount of the minimized energy simultaneously.

4 Embedding the Solution Space: Metamorphosis

The solution method which is based on an embedding process involves several stages to set up a linear programming problem whose solution is converged to the solution of the original problem (see [26]). This is one of the outstanding advantages of this method even for strongly nonlinear problems. In this manner, we present a new version of shape measure method to solve the optimal shape design (12). First, by defining a new variational formulation, for each obtained surface, an optimal control problem equivalent to the original problem is obtained. Then, a measure theoretical approach with two-stage of approximation is used to convert the optimal control problem to a finite dimensional LP. The solution of this LP is used to construct an approximation solution for the original optimal shape problem in which when the approximation is finer and finer, the solution converges to the solution of the original problem. Thus, the proposed approach is practical and accurate enough whose accuracy can be improved as far as desired (see [9]).

4.1 Step 1: Displaying the problem in variational form

The following conditions put on the functions as well as sets will serve two important purposes. First, they are reasonable conditions which are usually met when considering classical problems. Second, they will allow us to modify these classical problems, which has more advantageous.

We consider function $\psi(r,\theta)$ that is infinite differentiable inside region D (say $\Im(D^0)$) which has compact support; consider $\varphi_1(\theta,r,z) = z\psi(r,\theta)$. Hence, we define function Ψ so that the absolute continuous condition of path function can be imposed on the problem:

$$\Psi = \frac{1}{r}\left(2\,(r-1)\,f_r\psi_\theta + \psi f_\theta + f\psi_\theta + (r-1)\,(f_{r\theta}\psi + f_\theta\psi_r + f\psi_{r\theta})\right).$$

Since each differentiable function with finite derivatives satisfies the Lipschitz condition and is absolutely continuous, function $\psi(r,\theta)$ is also absolutely continuous with respect to each of the independent variables r and θ; if z is absolutely continuous, then, function $z\psi(r,\theta)$ is also absolutely continuous (see [25]). Now, we suppose $F = (\varphi_{1r}, \varphi_{1\theta}, \varphi_{1z})$ and $\varphi_1(\theta,r,z) = z\psi(r,\theta)$, then, for all $\psi \in C(D)$, we have:

$$\nabla\times F = \left(\tfrac{1}{r}\tfrac{\partial\varphi_{1_z}}{\partial\theta} - \tfrac{\partial\varphi_{1_\theta}}{\partial z}\right)\hat{r} + \left(\tfrac{\partial\varphi_{1_r}}{\partial z} - \tfrac{\partial\varphi_{1_z}}{\partial r}\right)\hat{\theta} + \tfrac{1}{r}\left(\tfrac{\partial(r\varphi_{1_\theta})}{\partial r} - \tfrac{\partial\varphi_{1_r}}{\partial\theta}\right)\hat{z},$$

and since the surface equation is $z = f(\theta,r)$, one can conclude that $\nabla f = (-f_r, \frac{-1}{r}f_\theta, 1)$, and according to Stoke's theorem, we have:

$$\oint_{\partial D} F dr = \int\int_S \nabla \times F.n \, d\sigma = \int\int_D \nabla \times F.\nabla f \, dA = \int\int_D \Psi(\theta, r, z, u_i) r dr d\theta$$

$$= \int\int_D \frac{1}{r}\left(2\left(r-1\right) f_r \psi_\theta + \psi f_\theta + f\psi_\theta + \left(r-1\right)\left(f_{r\theta}\psi + f_\theta \psi_r + f\psi_{r\theta}\right)\right) r dr d\theta = 0.$$

Since $supp(\psi) \subset D^0$, the boundary of D is outside of its support and the right hand side of the above integral is equal to zero.

We consider sphere B so that $D \times A \subset B$ and show the space of real-valued and continuously differentiable functions with the first and second order bounded continuous derivatives on B by $C^{'}(B)$. Based on similar reasons for the choice $\psi(r, \theta)$, the second class of functions in $C^{'}(B)$ is selected as the functions that only depend on the independent variables θ and r; we show the set of these functions with $C_1(B)$. In this case, we have:

$$\int_\omega \frac{1}{\sqrt{\frac{1}{r^2} f_\theta^2 + f_r^2 + 1}} f\left(\theta, r\right) d\sigma = \int\int_D f\left(\theta, r\right) r dr d\theta \equiv a_f; \quad f \in C_1\left(B\right).$$

Therefore, problem (12) can be displayed in a new variational form as follow:

$$\begin{aligned}
\min \quad & E(\omega, a) = \frac{1}{2T}\int_0^T \int_\Omega [|\dot{u}(r,\theta,z,T)|^2 + (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2 + (\frac{\partial u}{\partial z})^2] r dr d\theta dz, \\
s.t. \quad & \\
& \frac{1}{T}(\int_0^T \int_\Omega (\dot{u}(T)\varphi(T) - u(T)\dot{\varphi}(T)) r dr d\theta dz dt - \int_0^T \int_\Omega u \Delta\varphi r dr d\theta dz dt) \\
& \quad + \int\int_D a u(T)\varphi(T)(z - z_{min}) r dr d\theta - \frac{1}{T}(\int_0^T \int_\Omega a u_0(r,\theta,z)\varphi(0) r dr d\theta dz dt) \\
& \quad - \int_0^T \int_\Omega a u \dot{\varphi} r dr d\theta dz dt + \int_0^T \int_\Omega u \ddot{\varphi} r dr d\theta dz dt = \Phi; \\
& \frac{1}{T}\int_0^T \int_\Omega div(u(r,\theta,z,t)\varphi(r,\theta,z,t)) r dr d\theta dz dt = 0, \qquad \forall \varphi \in H_0^1(\Omega \times (0,T)); \\
& \int\int_D (z - z_{min}) dA = \frac{L}{T}(\int_0^T \int_\Omega r dr d\theta dz dt). \\
& \int\int_D \frac{1}{r}\left(2\left(r-1\right) f_r \psi_\theta + \psi f_\theta + f\psi_\theta + \left(r-1\right)\left(f_{r\theta}\psi + f_\theta\psi_r + f\psi_{r\theta}\right)\right) r dr d\theta = 0; \\
& \int\int_D f\left(\theta, r\right) r dr d\theta = a_f.
\end{aligned} \tag{13}$$

Now, by determining a suitable control function, we rewrite the problem in the form of an optimal control one.

4.2 Step 2: Embedding into measure space

By considering the vector of functions (r, z, u) as the trajectory and the vector $(\dot{u}, u_r, u_\theta, u_z, f_r, f_\theta, f_{r\theta})$ as controls, problem (13) can be considered as an optimal control problem. In this way, the following definitions need to be presented:

Definition 1. $p = (z, u, \dot{u}, u_r, u_\theta, u_z, f_r, f_\theta, f_{r\theta})$ is called admissible when it satisfies the following conditions:

1. The control functions $\dot{u}$, u_r , u_θ , u_z , f_r, f_θ and $f_{r\theta}$ are bounded and continuous and take their values on compact sets $\dot{U}$, U_r , U_θ , U_z , F_r, F_θ and $F_{r\theta}$ which are subsets of $\mathbb{R}$;

2. $z = f(\theta, r)$ is an absolutely continuous function;

3. u is the bounded solution of the linear damped wave system (1);

4. The relations (13) are satisfied.

The set of all admissible vector p is denoted by P. We also suppose that P is nonempty; in other words, we suppose that the system is controllable (this can be seen in [26], for instance).

Let $D' = [0, T] \times D \times U \times \dot{U} \times U_\theta \times U_z \times U_r$ and $D'' = D \times A \times F_r \times F_\theta \times F_{r\theta}$. For any admissible $p \in P$, we define the linear, positive and bounded functionals Λ_P and Γ_P on $C(D')$ and $C(D'')$ in the following way:

$$\begin{aligned} &\Gamma_P(F) = \int_0^T \int_\Omega F r dr d\theta dz dt, \qquad \forall F \in C(D'); \\ &\Lambda_P(G) = \int_D G r d\theta dr, \qquad \forall G \in C(D''). \end{aligned} \tag{14}$$

Since $\mathbb{R}^8$ is a locally compact space, according to the Heine-Borel theorem ([28]), $D' \subseteq \mathbb{R}^8$ is a compact Hausdorff space. Also, for the same reason, D'' is also a Hausdorff compact space. Therefore, for every given p, Riesz's representation theorem ([28]) indicates two positive Radon measures , μ_P and λ_P uniquely, so that:

$$\begin{aligned} &\Gamma_P(F) = \int_{D'} F d\mu_P \equiv \mu_P(F), \qquad \forall F \in C(D'); \\ &\Lambda_P(G) = \int_{D''} G d\lambda_P \equiv \lambda_P(G), \qquad \forall G \in C(D''). \end{aligned} \tag{15}$$

Consequently, any admissible element can be displayed as (15) by a unique pair of measures, say (μ_P, λ_P), in a subset F of $\mathcal{M}^+(D') \times \mathcal{M}^+(D'')$, where $\mathcal{M}^+(X)$ is the set of all positive Radon measures on X. Therefore, one can transfer the problem (13) into a measure space by:

$$(z, u, \dot{u}, u_r, u_\theta, u_z, f_r, f_\theta, f_{r\theta}) \in P \longmapsto (\mu_P, \lambda_P) \in \mathcal{M}^+(D') \times \mathcal{M}^+(D'');$$

It was proved that such a transformation is an injection (see [10]). To obtain something new, we expand the underlying space and take into account the problem of finding a minimizer pair of measures, say (μ^*, λ^*), on the space of all positive related Radon measures which are just satisfied to the conditions of (13)(not only those inferred from Riesz Representation theorem); therefore, our method is somehow global.

Regarding the famous properties of admissible elements of P and the definitions of the pair of measures (μ, λ) in (15), problem (13) can now be displayed as follows in which the measures λ and μ are its unknown variables:

$$\begin{aligned}
\min \quad & E(\mu,\lambda) = \frac{1}{2T}\mu[|\dot{u}(r,\theta,z,T)|^2 + (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2 + (\frac{\partial u}{\partial z})^2],\\
s.t. \quad & \\
& \frac{1}{T}\mu(\dot{u}(T)\varphi(T) - u(T)\dot{\varphi}(T) - \mu(u\Delta\varphi))\\
& \quad + \lambda(au(T)\varphi(T)(z - z_{min})) - \frac{1}{T}\mu(au_0(r,\theta,z)\varphi(0))\\
& \quad - \mu(au\dot{\varphi} + u\ddot{\varphi}) = \Phi, \qquad \forall \varphi \in H_0^1(\Omega \times (0,T));\\
& \frac{1}{T}\mu(div(u(r,\theta,z,t)\varphi(r,\theta,z,t))) = 0, \qquad \forall \varphi \in H_0^1(\Omega \times (0,T));\\
& \lambda(z - z_{min}) = \frac{L}{T}\mu(1);\\
& \lambda(\frac{1}{r}(2(r-1)f_r\psi_\theta + \psi f_\theta + f\psi_\theta + (r-1)(f_{r\theta}\psi + f_\theta\psi_r + f\psi_{r\theta}))) = 0;\\
& \lambda(f(\theta, r)) = a_f.
\end{aligned} \tag{16}$$

We remind that the theoretical measure problem (16) is linear even though the initial problem is highly nonlinear.

Space $M^+(D') \times M^+(D'')$ is a linear space which will become a locally convex topological vector space when it gives the weak*topology. This can be defined by the family of semi-norms $(\mu, \lambda) \mapsto |\mu(F)| + |\lambda(G)|$ for $F \in C(D')$, $G \in C(D'')$ and $\epsilon > 0$, which can be on the basis of a family of neighborhoods of zero for $M^+(D') \times M^+(D'')$; this family is defined by:

$$U_\epsilon = \{(\mu,\lambda) \in M^+(D') \times M^+(D'') : |\mu(F_j)| + |\lambda(G_j)| < \epsilon; j = 1, 2, ..., r\}$$

which makes a basis for a weak*topology on space $M^+(D') \times M^+(D'')$ (many properties of this topology can be found in the literature such as [32]); in this way, $M^+(D') \times M^+(D'')$ under this topology is a Hausdorff space ([28]).

The proof of the following theorems can be found in [9], [11] and [26].

Theorem 1. a) $Q \subseteq M^+(D') \times M^+(D'')$ is compact under the weak* topology on $M^+(D') \times M^+(D'')$

b) Objective function $i(\mu, \lambda)$ in problem (16) is continuous.

c) There exists a pair of measures (μ^*, λ^*) which is optimal for (16) in set $Q \subset M^+(D') \times M^+(D'')$; that is, for every $(\mu, \lambda) \in Q$, we have:

$$i(\mu^*, \lambda^*) \leq i(\mu, \lambda).$$

Even (16) has an optimal solution in Q, it is still very difficult to achieve the exact solution because the underlying spaces are not finite-dimensional: the number of equations is not finite and the unknowns are measures. Therefore, it is totally acceptable to look for a sub-optimal solution. Thus, first, by choosing suitable dense subsets in the appropriate spaces and then, by choosing the finite number of them, we approximate the problem using a semi-finite linear programming one.

4.3 Identifying a Nearly Optimal Solution

It is possible to approximate the solution of (16) by the solution of a finite-dimensional linear one of sufficiently large dimensions. Besides, by increasing the dimension of the problem, the accuracy of the approximation can be increased. First, we consider the minimization of (16) not only over set Q, but also over its subset called $Q(M_1, M_2, M_3)$ and defined by only a finite number of constraints to be satisfied. This will be achieved by choosing countable sets of functions whose linear combinations are dense in appropriate spaces and then by selecting a finite number of constraints. Let $\{\varphi_i : i \in N\}$, $\{\psi_i : i \in N\}$ and $\{f_i : i \in N\}$ be countable dense (in the converge topology sense) sets in spaces $H_0^1(\Omega \times (0, T))$, $H_0^1(D)$ and $C_1(D)$, respectively. By choosing a finite number of functions in each set, the solution of (16) can be approximated by the following solution:

$$\begin{aligned}
\min \quad & E(\mu, \lambda) = \frac{1}{2T}\mu[|\dot{u}(r, \theta, z, T)|^2 + (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2 + (\frac{\partial u}{\partial z})^2], \\
s.t. & \\
& \frac{1}{T}\mu(\dot{u}(T)\varphi_i(T) - u(T)\dot{\varphi}_i(T) - \mu(u\Delta\varphi_i)) \\
& \quad + \lambda(au(T)\varphi_i(T)(z - z_{min})) - \frac{1}{T}\mu(au_0(r, \theta, z)\varphi_i(0)) \\
& \quad - \mu(au\dot{\varphi}_i + u\ddot{\varphi}_i) = \Phi, \qquad \forall \varphi_i \in H_0^1(\Omega \times (0, T));
\end{aligned}$$

$$\frac{1}{T}\mu(div(u(r,\theta,z,t)\varphi_i(r,\theta,z,t))) = 0,\ \forall\varphi_i \in H_0^1(\Omega\times(0,T)),\quad i=1,2,...,M_1;$$
$$\lambda(z-z_{min}) = \frac{L}{T}\mu(1);$$
$$\lambda(\frac{1}{r}(2\,(r-1)\,f_r\psi_{\theta_j} + \psi_j f_\theta + f\psi_{\theta_j} + (r-1)\,(f_{r\theta}\psi_j + f_\theta\psi_{r_j} + f\psi_{r\theta_j}))) = 0,$$
$$\lambda(f_{sk}\,(\theta,r)) = a_{f_{sk}},\qquad j=1,2,...,M_2;\qquad s=1,2,...,M_3,\quad k=1,2,...,M_4 \quad (17)$$

The density property of the selected sets in (17) causes its solutions to tend to the solution of (16) when $M_1, M_2, ..., M_4 \to \infty$; thus, if numbers $M_1, M_2, ..., M_4$ are selected large enough, (17) is a good approximation of our main problem. Now, the number of constrains of the problem is finite but the problem is still infinite since the underlying space is a subspace of measures. It would be more convenient if we could approximate the solution just by a solution of a simple finite LP.

In [9], Fakharzadeh et al. presented that the pair of optimal measures of (17) is in the form of $\lambda^* = \sum_{m=1}^{M}\beta_m^*\delta(q_m^*)$ and $\mu^* = \sum_{n=1}^{N}\alpha_n^*\delta(Q_n^*)$ in which Q_n^* and q_m^* belong to dense subsets of D' and D'', respectively; moreover, $\delta(t)$ is a unitary atomic measure with support at the singleton set t. Substituting these forms in (17), it might seem that the problem has been made even more difficult, since it is transferred into a non-linear one. But, if function $i(\mu,\lambda)$ can be minimized only with respect to the coefficients α_n^* and β_m^*, it will be turned into a linear programming problem. In other words, the solution can be obtained approximately by solving just the simple finite linear programming like below. If one chooses the points Q_n^* and q_m^* from a dense subsets of D' and D'', this fact could be achieved in the second step of our approximation. (see [9] for more details):

$$\min\quad E = \frac{1}{2T}\sum_{n=1}^{N}\alpha_n[|\dot{u}(Q_n)|^2 + (\frac{\partial u(Q_n)}{\partial r_n})^2 + \frac{1}{r_n^2}(\frac{\partial u(Q_n)}{\partial\theta_n})^2 + (\frac{\partial u(Q_n)}{\partial z_n})^2],$$

$s.t.$

$$\frac{1}{T}\sum_{n=1}^{N}\alpha_n(\dot{u}(Q_n,T)\varphi_i(Q_n,T) - u(Q_n,T)\dot{\varphi}_i(Q_n,T))$$
$$-\sum_{n=1}^{N}\alpha_n(u(Q_n)\Delta\varphi_i(Q_n)) + \sum_{m=1}^{M}\beta_m(au(q_m,T)\varphi_i(q_m,T)(z-z_{min}))$$
$$-\frac{1}{T}\sum_{n=1}^{N}\alpha_n(au_0(Q_n)\varphi_i(0)) - \sum_{n=1}^{N}\alpha_n(au(Q_n)\dot{\varphi}_i(Q_n) + u(Q_n)\ddot{\varphi}_i(Q_n) = \Phi,$$
$$\frac{1}{T}\sum_{n=1}^{N}\alpha_n(div(u(Q_n)\varphi_i(Q_n))) = 0,\qquad i=1,2,\ldots,M_1;$$

$$
\begin{aligned}
&\sum_{m=1}^{M} \beta_m (z_m - z_{min}) = \frac{L}{T} \sum_{n=1}^{N} \alpha_n(1);\\
&\sum_{m=1}^{M} \beta_m (\frac{1}{r_m} (2\,(r_m - 1)\, f_{r_m} \psi_{\theta_j(q_m)} + \psi_j(q_m) f_{\theta_m} + f_m \psi_{\theta_j}(q_m)\\
&\quad + (r_m - 1)\,(f_{r\theta_m} \psi_j(q_m) + f_{\theta_m} \psi_{r_j}(q_m) + f_m \psi_{r\theta_j}(q_m)))) = 0,\\
&\quad j = 1, 2, \ldots, M_2;\\
&\sum_{m=1}^{M} \beta_m (f_{sk}(q_m)) \equiv a_{f_{skm}}, \qquad s = 1, 2, \ldots, M_3, \quad k = 1, 2, \ldots, M_4.
\end{aligned} \tag{18}
$$

Problem (18) is still non-linear because $q_m = (\theta_m, r_m, z_m, f_{r_m}, f_{\theta_m}, f_{r\theta_m})$ and $Q_n = (t_n, \theta_n, r_n, u_n, \dot{u_n}, u_{\theta_n}, u_{z_n}, u_{r_n})$ and r_m are unknowns. Now, by using simultaneous two-phase search techniques for (18), the optimal vector $(r_1, r_2, \ldots, r_M)$ (and hence the optimal domain D) and the optimal coefficients $\alpha_1^*, \ldots, \alpha_N^*, \beta_1^*, \ldots, \beta_M^*$ would be found; one is able to construct the pair of optimal shape and optimal domain in the manner which will be explained in the next section.

5 Algorithm

To apply the mentioned method for solving problem (18) practically, here we present an algorithmic path for the solution procedure. Regarding previous statements, we are able to identify the optimal control and optimal damping set by using the following four-step algorithm:

Step 1: The given sets $[0, T], D, U, \dot{U}, U_\theta, U_r$ and U_z are divided into $n_1, n_2, ..., n_8$ equal parts, and also the sets D, A, F_r, F_θ and $F_{r\theta}$ into n_2, n_3, m_1, m_2, m_3 and m_4 equal parts, respectively; so that, the $N = n_1.n_2.n_3.n_4.n_5.n_6.n_7.n_8$, number of 8-dimensional cells and the $M = n_2.n_3.m_1.m_2.m_3$, number of 5-dimensional cells in the related spaces are obtained. Then, in each of these 8-dimensional and 5-dimensional cells arbitrary points $Q_i = (t_i, \theta_i, r_i, u_i, \dot{u}_i, u_{\theta_i}, u_{z_i}, u_{r_i})$ and $q_j = (\theta_j, r_j, z_j, f_{r_j}, f_{\theta_j}, f_{r\theta_j})$ are selected, respectively.

Step 2: For fixed numbers M_1, M_2 and M_3 and M_4, we select M_1 number of $\varphi_k(Q)$, M_2 of $\psi_l(q)$ and $M_3 \times M_4$ of $f_s(q)$ functions, respectively. Now, one is able to set up the finite linear programming (18) with $N + M$ variables and $M_1 + M_2 + (M_3 \times M_4) + 1$ constrains, which is dependent on variables $r_1, ..., r_M$.

Step 3: To solve problem (18), we use an iterative method with two loops (one in another) and apply two phases of optimization approaches. In this section, a procedure is developed for finding the optimal value of the same functional over the set of all admissible domains D_M; in fact, we aim to solve the above-mentioned problem in (18) by determining the optimal surface and its related optimal image to achieve the minimum value of the objective function $I(S, D)$ on $Q(M_1, ..., M_4)$. Each domain $D \in D_M$, as explained, is determined by a set of finite points (θ_m, r_m) , $m = 1, 2, \ldots, M$. Thus, for a given $D \in D_M$, by solving (18), the nearly optimal value for $I(\alpha^*, \beta^*, D)$ is found as a function of variables $r_1,\ r_2, \ldots, r_M$. Consequently, one can define the following vector function:

$$J{:}(r_1,\ r_2, \ldots, r_M) \in \mathbb{R}^M \to I(\alpha^*, \beta^*, D) \in \mathbb{R}$$

The global minimizer of vector function J, say $(r_1,\ r_2, \ldots, r_M)$, can be identified by using a suitable search technique (like Honey-Bee-Method [1]). Such a method normally needs an initial value (initial domain) for starting the process of minimization. Each time the algorithm calculates a value for J, finite linear programming problem (18) should be solved; thus, the optimal coefficients α_i^*, β_j^* are characterized. Whenever it reaches the minimum value, the minimizer $(r_1^*,\ r_2^*, \ldots, r_M^*)$ (optimal domain D^*) and therefore its associated optimal surface have been obtained. So, the optimal domain and the optimal surface are determined at the same time.

Remark: In each stage where the alternative optimal case is happened, it is enough to select one of them arbitrarily.

Step 4: We summarize the procedure of constructing optimal control functions and path function z derived from a solution of linear programming problem (18): after solving problem (18), we identify the indices n such that the components β_n^* of the extreme point are positive and the corresponding value $\theta_n$and r_n associated with them make $\theta = \theta_n$,$r = r_n$, $u\,(r, \theta, z) = u_n$ and $z(r,\ \theta)\ =\ z_n$. Then, we have optimal points (θ_n, r_n, z_n) and by using curve fitting tool box of MATLAB software, we fit the surface on these points in Cartesian coordinates.

5.1 Total sets

To restrict the number of constraints (17), we considered countable sets of functions whose linear combinations are dense in the specific space. In this section, we attempt to introduce some suitable cases for such sets. In this manner, we explain how one can choose total sets for the constraints of (18). Infinitely differentiable functions consist of functions such as exponential and trigonometric functions. However, exponential functions can never be zero. Therefore, we make use of trigonometric functions whose linear combinations can make Fourier series for each periodic arbitrary function. We choose these functions in the following way and we consider $\psi'_{ji}s$ as:

$$\psi_{1i} = (r - h(\theta\ ))(sin(i\pi\theta\ ));$$

$$\psi_{2i}\ =\ (r - h(\theta\ ))(cos(i\pi\theta\ ));$$

$$\psi_{3i} =\ (r - h(\theta\ ))(cos(i\pi\theta\ )sin(i\pi\theta\ )).$$

Obviously, the linear combinations of these functions are uniformly dense in space $C_1(\Omega)$, infinitely differentiable inside region D and has compact support (see [9]).

To be able to characterize the optimal coefficients of (18), an arbitrary domain will be divided into finite parts and then, an attempt will be made to determine the optimal surface in each part. In this manner, a finite number of angles $\theta = \theta_i,\ i = 1, 2,\ \dots, l$ from the uniform dense subset of $[0, 2\pi]$ would be considered. Then, domain D can be divided into l parts by half-lines $\theta = \theta_i$. Also, the i-th part of D $(i = 1, 2,\ \dots, l)$ can be approximated by sector $R_i = \frac{r_i+r_{i+1}}{2}$, when $\theta_i \le \theta \le \theta_{i+1}$ and $(r_1,\ r_2, \dots, r_M)$ is the optimal value in $(\theta_1, \theta_2,\ \dots \theta_M)$, say D_j. Then, if the number of angles is sufficiently large, the union of D_j's can approximate D arbitrarily. So, we consider f_{sk} as follow:

$$f_{sk}\,(\theta, r) = \begin{cases} 1, & if\ \theta \in J_{1s},\ r \in J_{2k}, \\ 0, & otherwise, \end{cases}$$

where J_{1s} and J_{2k} are determined as follow:

$$J_{1s} = [\frac{(s-1)\,\theta_i}{M_3},\ \frac{(s)\,\theta_i}{M_3})\quad \text{and}\quad J_{2k} = [\frac{(k-1)\,R_i}{M_4},\ \frac{(k)\,R_i}{M_4})\ .$$

Hence:

$$\int\int_D f_{sk}\,(\theta, r)\,r\ drd\theta = \int_{r_{k-1}}^{r_k}\int_{\theta_{s-1}}^{\theta_s} r\ drd\theta = \frac{1}{2}(\theta_s - \theta_{s-1})(r_k^2 - r_{k-1}^2) \equiv a_{f_{sk}}.$$

6 Examples

In this section, by giving a numerical example, we examine the efficiency of the method explained in the previous sections.

As mentioned in the introduction, this problem is resolved in many papers in the one- or two-dimensional spaces. But, so far, it has not been solved in a three-dimensional space. Therefore, we have taken the initial and final conditions from reference [22] and if necessary, we extended these conditions to a three-dimensional situation.

By defining Ω as an cylinder with radius $\sqrt{2}$ and selecting $a = 1.5$ (constant), for chosen initial conditions:

$$\begin{cases} u_0(\theta, r, z) = \sin(\pi r \cos\theta)\sin(\pi r \sin\theta)\sin(\pi z), \\ u_1(\theta, r, z) = 0, \quad (\theta, r, z) \in \Omega, \end{cases}$$

and final conditions as follows ([3] and [29]):

$$u(T = 1) = \dot{u}(T = 1) = 1.$$

We supposed $L = (1/3)$, the volume of the unknown region ω was equal to $\frac{2\pi}{3}$. Then, we chose: $z_{min} = 0, \quad z_{max} = 2$, $0 \leq \theta \leq 2\pi$,$0 \leq t \leq 1$.

To discretize $D' = [0, T] \times D \times U \times \dot{U} \times U_\theta \times U_z \times U_r$, we chose $N = 6^2 \times 5^6$ point in these sets by selecting

1. 6 points in $[0, T]$ for t : $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$;
2. 6 points in $[0, 2\pi]$ for θ: $0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, 2\pi$;
3. 5 points in $[0, \sqrt{2}]$ for r: $0, \frac{\sqrt{2}}{5}, \frac{2\sqrt{2}}{5}, \frac{3\sqrt{2}}{5}, \frac{4\sqrt{2}}{5}, \sqrt{2}$;
4. 5 points in $\dot{U}$ for $\dot{u}$: $0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$;
5. 5 points in U_θ for u_θ: $-1, \frac{-1}{2}, 0, \frac{1}{2}, 1$;
6. 5 points in U_z for u_z: $-1, \frac{-1}{2}, 0, \frac{1}{2}, 1$;
7. 5 points in U_r for u_r: $0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$;

Also, to discretize $D^{"} = D \times A \times F_r \times F_\theta \times F_{r\theta}$, we chose $M = 14 \times 5^5$ point in these sets by selecting

1. 5 points in F_r for f_r : $0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$;
2. 5 points in F_θ for f_θ: $-1, \frac{-1}{2}, 0, \frac{1}{2}, 1$;
3. 5 in $F_{r\theta}$ for $f_{r\theta}$: $-1, \frac{-1}{2}, 0, \frac{1}{2}, 1$;
4. 14 angles in $[0, 2\pi]$ for θ:

$$0, \frac{2\pi}{13}, \frac{4\pi}{13}, \frac{6\pi}{13}, \frac{8\pi}{13}, \frac{10\pi}{13}, \frac{12\pi}{13}, \frac{14\pi}{13}, \frac{16\pi}{13}, \frac{18\pi}{13}, \frac{20\pi}{13}, \frac{22\pi}{13}, \frac{24\pi}{13}, 2\pi;$$

5. 5 value in $A=[0,2]$ for r: $0,\frac{1}{2},1,\frac{3}{2},2$.

Then, by selecting the following functions and setting them in (18) for $M_1=2, M_2=9, M_3=8, M_4=3$ as:

$$\psi_{l_1}=(r-h(\theta\,))(\sin(l_1\pi\theta\,));$$
$$\psi_{l_2}\;=\;(r-h(\theta\,))(\cos(l_2\pi\theta\,));$$
$$\psi_{l_3}=\;(r-h(\theta\,))(\cos(l_3\pi\theta\,)\sin(l_3\pi\theta\,)).$$

$$f_{sk}\left(\theta,r\right)=\begin{cases}1, & if\ \theta\in J_{1s},\ r\in J_{2k}\\ 0, & otherwise\end{cases}$$

where J_{1s} and J_{2k} are determined as follow:

$J_{1s}=[\frac{(s-1)\theta_i}{M_3},\ \frac{(s)\theta_i}{M_3})$ and $J_{2k}=[\frac{(k-1)R_i}{M_4},\ \frac{(k)R_i}{M_4})$ for unknown region D. Where $R_i=\frac{r_i+r_{i+1}}{2}$, the unknowns r_i are optimal solution for $J(r_1,r_2,...,r_M)$ and obtained in Step 3. Hence:

$$\int\int_D f_{sk}\left(\theta,r\right)r\ drd\theta=\int_{r_{k-1}}^{r_k}\int_{\theta_{s-1}}^{\theta_s}r\ drd\theta=\frac{1}{2}(\theta_s-\theta_{s-1})(r_k^2-r_{k-1}^2)\equiv a_{f_{sk}}.$$

Also, for $i=2,3$ we selected $\varphi_i=(r-\sqrt{2})^i\sin(i\theta)t$; therefore:

$$\begin{cases}\varphi_i(T)=(r-\sqrt{2})^i\sin(i\theta)T, \dot{\varphi}_i=(r-\sqrt{2})^i\sin(i\theta),\\ \ddot{\varphi}_i=0,\\ \Delta\varphi_i=(r-\sqrt{2})^{i-2}\sin(i\theta)t(i(i-1)+\frac{i}{r}(r-\sqrt{2})-\frac{i^2}{r}(r-\sqrt{2})^2).\end{cases}\tag{19}$$

we setup the corresponding LP with $M+N$ variables and 38 constrains.

It is noteworthy that, the number of basic functions is chosen arbitrarily and by increasing their number, we will have a better approximate solution.

In 2012, Fakharzadeh et al. dealt with the best standard algorithm to identify the optimal solution for an OSD sample problem governed by an elliptic boundary control problem. Their goal in that paper was to examine and evaluate six different methods according to their ability to find optimality of function. In the mentioned paper, some references (references related to applications or discussions) were given for each method [13]. They conducted a computational examination of several existing derivative free optimization methods to apply solution procedure of OSD problems by the shape-measure technique. These methods consist of Random search, Nelder-Mead algorithm, Hook and Jeeves algorithm, Simulated annealing algorithm, Genetic and Honey bee mating optimization algorithm. The results showed that Random search and Honey bee mating optimization algorithm are more appropriate for use in shape measure method than other algorithms [13]. In this manner, we use Honey Bee Mating optimization algorithm (HBM) to obtain the optimal value of $J(r_1, r_2, \ldots, r_M)$ and the modified Simplex method from MATLAB 7.13 to obtain the optimal value of $I(\beta, D)$.

After solving this problem by 30 iterations, we obtained optimal points (θ_n, r_n, z_n) corresponding to optimal coefficients $\beta_n^* > 0$ in the manner explained in Section 5. Then, we specify the optimal points (x_n, y_n, z_n) with respect to the relationships $x_n = r_n cos(\theta_n)$ and $y_n = r_n sin(\theta_n)$. For a fixed $a = 1.5$, we obtained the nearly optimal region D (see Figure 1), nearly optimal domain (Figure 2), the curve of the objective value in terms of iterations in the LP solver (Figure 3) and the energy value as 6.7495×10^{-20}.

In places with damping of waves, installing sensors is not recommended. Therefore, searching for a place with the least damping of waves for installing sensors is important. In the Figure 2, the spherical area defines ω_A. This area is the damping set and installing sensors in this place is not suitable. In this area, the waves have the highest damping rate and the least energy.

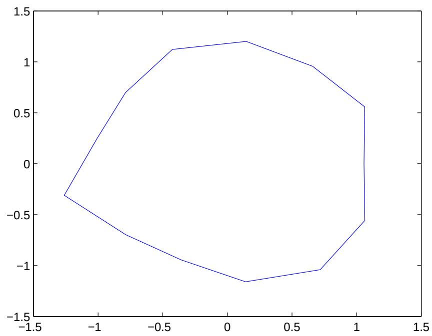

Figure 1: Nearly optimal region D in Example 1.

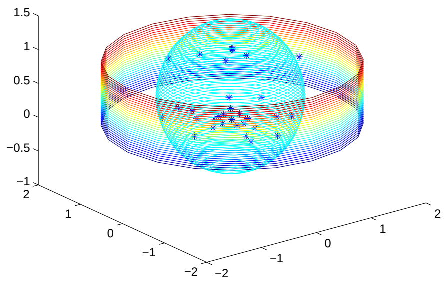

Figure 2: Nearly optimal domain with constant damping coefficient $a = 1.5$.

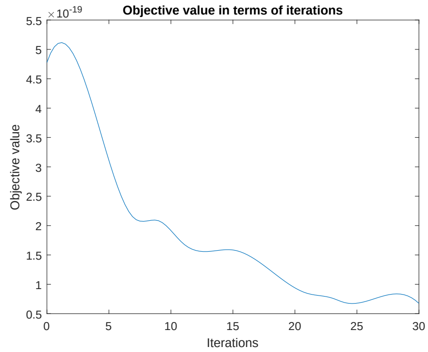

Figure 3: Objective value in terms of iterations of HBM algorithm.

7 Conclusion

By doing an embedding process and using the property of positive Radon measures, we presented a new and very useful technique for solving the problem of minimizing energy of a 3-D damped wave system in an unknown region. In this method, the problem was solved by a 2-phase optimization search technique where the unknown region and unknown damping set were found optimally. The most important characteristic of our method is its simplicity and its independence of the solution of the initial shape. To obtain the optimal domain, we just need to use two-search techniques while solving linear programming problems.

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یافتن مکان بهینه حسگرها برای معادله موج میرای سه بعدی با استفاده از تقریب اندازه‌ها

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چکیده

در این مقاله، مساله تعیین شکل و محل بهینه نصب حسگرها برای معادلات موج سه بعدی با میرایی ثابت را در یک زیر مجموعه پیوسته از فضای سه بعدی، مدلسازی و حل می‌کنیم. محل قرارگرفتن حسگرها به وسیله زیردامنه‌ای از یک ناحیه با اندازه مشخص مدلسازی شده است. با استفاده از یک تقریب بر مبنای روش نشاندن، ابتدا، دستگاه معادلات به صورت تغییراتی فرمولبندی شده؛ سپس، با تعریف دو اندازه رادن مثبت، مساله در فضای اندازه‌ها بازنویسی خواهد شد. در این روش، مساله طراحی شکل به یک مساله برنامه‌ریزی خطی نامتناهی تبدیل شده که وجود جواب آن تضمین شده است. در این مرحله، با استفاده از دو گام تقریب، جواب بهینه (کنترل بهینه، ناحیه و انرژی بهینه) با روش جستجوی دو مرحله‌ای تعیین می‌گردد. علاوه بر این، شبیه سازی عددی برای مقایسه روش جدید با دیگر روش‌ها آورده شده است.

کلمات کلیدی

معادله موج سه بعدی، کنترل پراکندگی، اندازه رادن، روش جستجو، بهینه‌سازی شکل.