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Solution of Fractional Optimal Control Problems with Noise Function Using the Bernstein Functions

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Abstract. This paper presents a numerical solution of a class of fractional optimal control problems (FOCPs) in a bounded domain having a noise function by the spectral Ritz method. The Bernstein polynomials with the fractional operational matrix are applied to approximate the unknown functions. By substituting these estimated functions into the cost functional, an unconstrained nonlinear optimization problem is achieved. In order to solve this optimization problem, the Matlab software and its optimization toolbox are used. In the considered FOCP, the performance index is expressed as a function of both state and control functions. The method is robust enough because of its computational consistency in the presence of the noise function. Moreover, the proposed scheme has a good pliability satisfying the given initial and boundary conditions. At last, some test problems are investigated to confirm the efficiency and applicability of the new method.

Keywords. Optimization, Spectral method, Noise function, Fractional optimal control, Operational matrix.

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1 Introduction

In recent years, fractional order calculus has an effective role in many areas such as modeling and simulation, optimization, signal processing and control. Therefore the theory of fractional differential equations has attracted the attention of scientists in many fields. Due to continuous order on fractional calculus, it will bring about more accurate and concise behavioral description of many practical processes and plants such as finance and hydrology, bioengineering [1], diffusion and stochastic processes [2] mobile sensor [3], electrochemistry [4], optimal control systems [5] and fractional delayed systems [6]. The authors would like to refer the interested in the context of fractional calculus to the books [7, 8] and several published articles on optimal control [9, 10, 11, 12]. One of the astounding contexts of the fractional calculus is the fractional optimal control problem (FOCP). An FOCP definitely states the problem aims at finding an input function on the fractional differential equation governing the dynamics of the system that minimizes the performance index in terms of the state and control variables [13, 14, 15]. It defines different types of FOCPs depending on the types of fractional derivative. Here, we consider the Caputo fractional derivative. The procedure to find the solution of an FOCPs can be treated similarly like ordinary optimal control problems by two approaches, indirect or direct. In the first approach, we form a Hamiltonian system then the arising two-point boundary value problem is solved. In the other approach, the given FOCP is solved directly by discretizing the unknown functions [16, 17, 18]. Since both of the right-left fractional derivatives appear in the Hamiltonian system [19], therefore, the numerical solution of FOCPs using the direct method has been studied by many authors [20, 21]. The presence of the noise signals in system modeling are inevitable. Recently, a finite horizon linear quadratic (LQ) OCP for a discrete-time linear fractional system (LFS) imposed by multiplicative and independent random perturbations is studied [22, 23]. Moreover, the OCP of continuous-time systems with general noises, according to sampled data, is surveyed in the literature [24]. In this paper, a FOCP with a noise function on the system dynamics is considered. In order to find the solution of the resulted problem, an extended Bernstein fractional operational matrix [25, 26] is formed to estimate the integer-fractional order derivatives of the basis. As instances, the Bernstein polynomials over a finite domain is beneficial for applicable computations, because of their geometrical outlook [27] as well as their intrinsic numerical stability [28]. The Bernstein polynomials also provide us with explicit approximations to a given continuous function in which the approximate sequence converges uniformly to that function. These polynomials have also compensating “shape-preserving” properties such as mapping a convex function to another convex function as well [29]. After approximating the unknown functions using the proposed basis and by enforcing the necessary optimality conditions, a system of nonlinear algebraic is achieved. Meanwhile, the convergence discussion about the operational matrix is provided as well. The rest of the paper is constructed in the following way. In Section 2, some prerequisites and a new operational matrix of the Bernstein polynomials are briefly reminded. In Section 3, the main problem and the proposed numerical scheme are mentioned. Furthermore, we state a new operational matrix with its error upper. In Section 4, the applicability of the new method is illustrated by providing two examples. Finally, the important achievements of the paper are briefed in Section 5.

2 The Fractional Derivative and the Bernstein Basis Functions

In this part, several preliminary definitions, such as fractional derivative as well as the Bernstein polynomials are stated. In the remaining part, an approximation of a function is given using the basis.

Definition 1. Let $\alpha > 0$, be a real number and n be an integer number. The fractional order Riemann-Liouville and Caputo derivative of the function f are explained as follows, respectively [30]

$$
\begin{aligned}
{}_{a}^{RL}D_t^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) \mathrm{d}\tau, \\
{}_{a}^{C}D_t^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n}{\mathrm{d}\tau^n} f(\tau) \mathrm{d}\tau
\end{aligned}
$$

Definition 2. The Bernstein polynomials of degree $m \geq 0$ are stated as [25]

$$
\beta_{i,m}(t) = \binom{m}{i} t^i (1-t)^{m-i}, \quad 0 \leq i \leq m;
$$

The Bernstein polynomials of degree 5 are as follows:

$$
\begin{aligned}
\beta_{0,5}(t) = (1-t)^5, \beta_{1,5}(t) = 5t(1-t)^4, \beta_{2,5}(t) = 10t^2(1-t)^3, \\
\beta_{3,5}(t) = 10t^3(1-t)^2, \beta_{4,5}(t) = 5t^4(1-t), \beta_{5,5}(t) = t^5.
\end{aligned}
$$

In Figure 1, the Bernstein polynomials of order 5 are plotted. These polynomials have some interesting properties that all of them are positive and have unity summation on every degree of them. These properties make them useful for probability purpose and stochastic systems usage [31].

The Bernstein polynomials are dense in the space $L^2[0,1]$ [32]. Moreover, every polynomial of degree m like $P_m(t)$ can be displayed in terms of the Bernstein polynomials, that is, there are real numbers c_j such that

$$
P_m(t) = \sum_{j=0}^{m} c_j \beta_{j,m}(t). \tag{1}
$$

Now we try to approximate a function in $L = L^2[0,1]$ in terms of the Bernstein polynomials. The space spanned by the first m Bernstein polynomials are denoted by $Y_m = \{\beta_{0,m}, \beta_{1,m}, \ldots, \beta_{m,m}\}$. Let f be a given function in the space L. Because Y_m is a finite-dimension, there exists the best estimation element out of Y_m like $f_0(t) = \sum_{j=0}^{m} \lambda_j \beta_{j,m}(t)$ in which

$$
\|f - f_0\|_2 \leq \|f - g\|, \quad \text{for all } \ g \in Y_m. \tag{2}
$$

Here one may write the following theorem for the polynomial estimations:

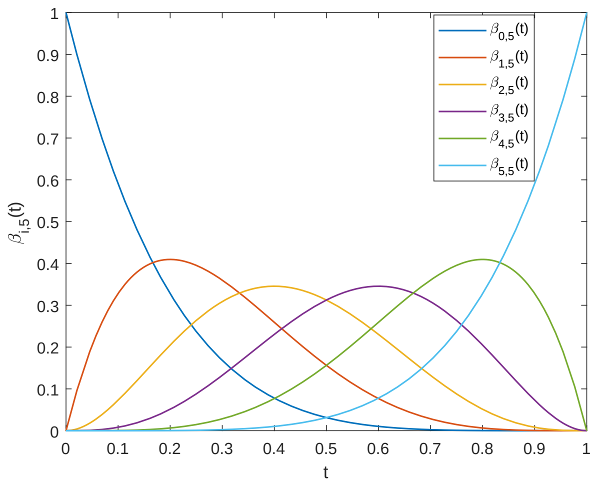

Figure 1: Plots of several Bernstein polynomials ($m = 5$).

Theorem 1. Suppose that $f \in L^2[0,1]$ is estimated by f_m with the Bernstein functions $\{\beta_{i,m}(t)\}_{i=0}^{m}$ like

$$f_m(t) = \sum_{i=0}^{m} \lambda_i \beta_{i,m}(t). \tag{3}$$

where $\lambda_i = \int_0^1 f(t) d_{i,m}(t) dt$ and $d_{i,m}(t)$ are the dual Bernstein polynomials. They are defined with respect to the Bernstein polynomials. The duals of the Bernstein polynomials are orthonormal to them. If $e_m(f) = \int_0^1 (f(t) - f_m(t))^2 dt$ then

$$\lim_{m\to\infty} e_m(t) = 0. \tag{4}$$

Proof. See [26]. $\square$

Theorem 2. Suppose that $m \in \mathbb{N}$ and we have the Bernstein polynomials set as $\{\beta_{0,m}(t), \beta_{1,m}(t), \dots, \beta_{m,m}(t)\}$. The dual basis $\{d_{0,m}(t), d_{1,m}(t), \dots, d_{m,m}(t)\}$ of the Bernstein basis by a proper inner product, has the Bernstein-Bazier presentation as follows

$$d_{p,m}(t) = \sum_{q=0}^{m} c_{p,q} B_{q,m}(t), \quad p = 0, 1, \dots, m, \tag{5}$$

where

$$c_{p,q} = \frac{(-1)^{p+q}}{\binom{m}{p}\binom{m}{q}} \sum_{j=0}^{\min(p,q)} (2j+1) \binom{m+j+1}{m-p} \times \binom{m-j}{m-p} \binom{m+j+1}{m-q} \binom{m-j}{m-q}. \tag{6}$$

Also, the dual polynomials satisfy the following relation

$$\langle \beta_{i,m}(t), d_{j,m}(t) \rangle = \int_0^1 \beta_{i,m}(t) d_{j,m}(t) \mathrm{d}t = \delta_{i,j}, \tag{7}$$

where δ is the Kronecker function.

Proof. See [33]. □

Remark 1. The Bernstein functions and their duals have the following orthonormality property [33]:

$$\int_0^1 \beta_{i,m}(t)d_{j,m}(t)\mathrm{d}t = \begin{cases} 1, & i=j, \\ 0, & i \neq j. \end{cases} \tag{8}$$

Next, a new fractional Bernstein operational matrix congruous with the proposed method is formed to ease the computational complexity. First of all, expanding the Bernstein polynomial $\beta_{i,m}(t)$ yields

$$\beta_{i,m}(t) = \binom{m}{i} t^i (1-t)^{m-i} = \sum_{k=0}^{m-i} \binom{m}{i}\binom{m-i}{k}(-1)^{m-i-k}\, t^{m-k},\ 0 \leq i \leq m. \tag{9}$$

Multiplying both sides of Eqn. (9) to t^s and taking the fractional derivative implies

$$\begin{aligned}
{}_0^C D_t^\alpha(t^s B_{i,m}(t)) &= \sum_{k=0}^{m-i} \binom{m}{i}\binom{m-i}{k}(-1)^{m-i-k}\, {}_0^C D_t^\alpha(t^{s+m-k}) \\
&= \sum_{k=0}^{\min(m-i,m+s-[\alpha]-1)} \binom{m}{i}\binom{m-i}{k}(-1)^{m-i-k} \\
&\quad \times \frac{(s+m-k)!}{\Gamma(s+m-k-\alpha+1)} t^{s+m-k-\alpha}.
\end{aligned} \tag{10}$$

By approximating $t^{s+m-k-\alpha}$ by $m+1$ items of the Bernstein polynomials of degree m, we get

$$t^{s+m-k-\alpha} \simeq \sum_{j=0}^{m} b_j \beta_{j,m}(t), \tag{11}$$

where the coefficients can be computed according to Theorem 1 as

$$\begin{aligned}
b_j &= \int_0^1 t^{s+m-k-\alpha} d_{j,m}(t)\mathrm{d}t \\
&= \sum_{q=0}^{m}\sum_{r=0}^{m-q}\sum_{l=0}^{\min(j,q)} \frac{2l+1}{\binom{m}{j}\binom{m}{q}} \binom{m+l+1}{m-j}\binom{m-l}{m-j} \times \binom{m+l+1}{m-q} \\
&\quad \times \binom{m-l}{m-q}\binom{m}{q}\binom{m-q}{r}(-1)^{m-r+j} \frac{1}{s+2m-\alpha-r-k+1}.
\end{aligned} \tag{12}$$

Hence, we have

$${}_0^C D_t^\alpha(t^s \beta_{i,m}(t)) \simeq \sum_{j=0}^{m} b_{i,j}^{\alpha;s} \beta_{j,m}(t), \tag{13}$$

where $b_{i,j}^{\alpha;s}$ is the(i,j)- entry of the operational matrix computed as

$$\begin{aligned}
b_{i,j}^{\alpha;s} &= < {}_0^C D_t^\alpha(t^s \beta_{i,m}(t)), d_{j,m}(t) > \\
&= \sum_{k=0}^{\min(m+s-[\alpha]-1,\, m-i)} \sum_{q=0}^{m}\sum_{r=0}^{m-q}\sum_{l=0}^{\min(j,q)} (-1)^{2m-i-k-r+j} \frac{(2l+1)}{\binom{m}{j}}
\end{aligned}$$

$$\begin{aligned}
&\times \binom{m}{i}\binom{m-i}{k}\binom{m+l+1}{m-j}\binom{m-l}{m-j}\binom{m+l+1}{m-q} \\
&\times \binom{m-l}{m-q}\binom{m-q}{r}\frac{(s+m-k)!}{(s+2m-\alpha-r-k+1)\Gamma(s+m-k-\alpha+1)}, \\
&0 \le i, j \le m. \qquad (14)
\end{aligned}$$

To calculate the integral appeared in the performance index, the Legendre-Gauss quadrature law is used as

$$\int_0^1 f(t)dt \simeq \frac{1}{2}\sum_{j=1}^{\ell} w_j f(\frac{\tau_j+1}{2}), \qquad (15)$$

where $\{w_j\}_{j=1}^{\ell}$ and $\{\tau_j\}_{j=1}^{\ell}$ are the Legendre-Gauss weights and nodes, respectively,

$$w_j = \frac{2}{(1-\tau_j)^2}[\dot{L}_\ell(\tau_j)]^2 \qquad (16)$$

3 The Main Problem and the Numerical Approach

3.1 The Numerical Spectral Method

Stated by the researchers, the dynamics of many systems can be presented more accurately regarding the fractional order derivative than just the integer ones [7, 34]. As an instance, one can address modeling the system of ”light amplification Erbium-doped fiber amplifier”. This kind of amplifier is one of the best ordinary used types of fiber amplifiers having applications in metro optical networks [35]. Our considered problem is similar to this one. Hence, regarding the application of the proposed problem, the following FOCP is considered:

$$\min \quad J[u] = \int_0^1 F(t, f(t), u(t))\mathrm{d}t, \qquad (17)$$

$$s.t.$$

$$\begin{aligned}
&M\dot{f}(t) + N\,{}_0^C D_t^\alpha f(t) = h(t, f(t)) + n(t) + u(t), \qquad (18)\\
&f(0) = f_0,\ \dot{f}(0) = f_1 \quad or \quad f(0) = f_0,\ f(1) = f_2,
\end{aligned}$$

where $0 < \alpha < 2$ is the fractional order and two kinds of initial or initial-boundary conditions are given. The parameters M and N are real numbers. The function $n(t)$ is an independent random perturbation function applied to the system. Commonly, the appeared perturbation function in the dynamic system is as small as possible. On the other hand, in order not to chatter the system dynamics, the perturbation function appearing in the system is assumed to be $\|n\| \le \min\{\|f\|, \|u\|\}$. Moreover, h is a differentiable function with respect to its parameters and f is a continuously differentiable function with the provided initial-boundary conditions. In order to execute the numerical method, first, the basis vector is constructed using the Bernstein polynomials as

$$B_m(t) = [\beta_{0,m}(t), \beta_{1,m}(t), \ldots, \beta_{m,m}(t)]^T, \qquad (19)$$

Using the basis functions (19), the state function is estimated as

$$f(t) \simeq \omega(t) X_m^T B_m(t) + \nu(t), \tag{20}$$

where $X_m \in \mathbb{R}^{m+1}$ is the real unknown coefficients to be computed, the functions ω and ν are the auxiliary trial functions and must be selected to impose all the given conditions. The function ω and ν shall be selected to impose homogeneous conditions (i.e., $\omega(0) = \nu(0) = 0$ or $\omega(0) = \nu(1) = 0$, depending on the given initial conditions). Depending on the type of just initial or initial-boundary conditions, we put $\omega(t) = t^2$ or $\omega(t) = t^2 - t$, respectively. A right candidate to satisfy the conditions is the Hermite or common interpolating polynomials as $\nu(t) = f_0 + tf_1$ or $\nu(t) = f_0 + (f_2 - f_0)t$, respectively.

To proceed with the numerical procedure, we consider the trial functions for the just first case. The other one can be dealt in a similar manner. By calculating the control parameter from the system dynamics (26), utilizing the fractional operational matrix (13) and then by substituting the consequent approximate functions to the cost functional, we get

$$\begin{aligned} \min \quad J[X_m] = \int_0^1 F\Big(& t, t^2 X_m^T B_m(t) + tf_1(t) + f_0, M(X_m^T D_{m,m}^{1;2} B_m(t) + f_1) \\ &+ N(X_m^T D_{m,m}^{\alpha;2} B_m(t) + \frac{f_1 t^{1-\alpha}}{\Gamma(2-\alpha)}) - h(t, t^2 X_m^T B_m(t) + tf_1 + f_0) - n(t)\Big) \mathrm{d}t. \end{aligned} \tag{21}$$

Using the quadrature law (15) in order to approximate the integral appeared in (21), an unconstrained nonlinear optimization problem is obtained. To find the optimal solution, the optimization toolbox of Matlab 2019 software is used.

Remark 2. It is worth noting that the noise function that appeared in the system description is finally conveyed to the cost functional through the system dynamics. Hence, the optimization process of the proposed algorithm is updated such that the control function neutralizes the perturbation affection on its optimal situation.

3.2 Bound for the Error of the Operational Matrix

In this section, a topmost bound for the error of the operational Bernstein matrix is provided. Let $z_1, z_2, \ldots, z_m$ be some elements in the Hilbert space H, the Gram determinant of them is determined as

$$G(z_1, z_2, \ldots, z_m) = \begin{vmatrix} \langle z_1, z_1 \rangle & \langle z_1, z_2 \rangle & \cdots & \langle z_1, z_m \rangle \\ \langle z_2, z_1 \rangle & \langle z_2, z_2 \rangle & \cdots & \langle z_2, z_m \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z_m, z_1 \rangle & \langle z_m, z_2 \rangle & \cdots & \langle z_m, z_m \rangle \end{vmatrix}.$$

Theorem 3. Let H be a Hilbert space and M is a finite dimensional subspace of H. Suppose that $\{z_1, z_2, \ldots, z_m\}$ is a basis for M. For an arbitrary element z in H and its best unique approximation z_0, we have

$$\|z - z_0\|_2^2 = \frac{G(z, z_1, z_2, \ldots, z_m)}{G(z_1, z_2, \ldots, z_m)}. \tag{22}$$

Proof. See [36]. □

Let us define the error matrix as follows

$$E_m^{\alpha;s} = D_{m,m}^{\alpha;s}.B_m(t) - {}_0^C D_t^{\alpha}(t^s B_m(t)) = \begin{bmatrix} e_{0,m}^{\alpha;s}(t) \\ e_{1,m}^{\alpha;s}(t) \\ \vdots \\ e_{m,m}^{\alpha;s}(t) \end{bmatrix}. \tag{23}$$

Referring to (11) and considering Theorem 1, we get

$$\|t^{s+m-k-\alpha} - \sum_{j=0}^{m} b_j \beta_{j,m}(t)\|_2 = \left(\frac{G\left(t^{s+m-k-\alpha}, \beta_{0,m}(t), \beta_{1,m}(t), \dots, \beta_{m,m}(t)\right)}{G\left(\beta_{0,m}(t), \beta_{1,m}(t), \dots, \beta_{m,m}(t)\right)} \right)^{1/2}.$$

By considering the relations (11) and (12) , we conclude that

$$\begin{aligned} \|e_{i,m}^{\alpha;s}\|_2 &= \|{}_0^C D_t^{\alpha}(t^s \beta_{i,m}(t)) - \sum_{j=0}^{m} \beta_{i,j}^{\alpha;s} \beta_{j,m}(t)\|_2 \\ &\le \sum_{k=0}^{\min(m-i,\, m+s-[\alpha]-1)} \binom{m}{i}\binom{m-i}{k}(-1)^{m-i-k} \frac{(s+m-k)!}{\Gamma(s+m-k-\alpha+1)} \\ &\times \left(\frac{G\left(t^{s+m-k-\alpha}, \beta_{0,m}(t), \beta_{1,m}(t), \dots, \beta_{m,m}(t)\right)}{G\left(\beta_{0,m}(t), \beta_{1,m}(t), \dots, \beta_{m,m}(t)\right)} \right)^{1/2}, 0 \le i \le m. \end{aligned} \tag{24}$$

therefore, an error upper bound is provided. Hence regarding Eq. (24), it is obvious that the error vector (23) tends to zero by raising the number of the Bernstein basis,.

4 Illustrative Test Problems of FOCP

Example 1. Let the following FOCP with variable fractional order adopted from [37] with a dynamical system constraint affected by a noise function and the given initial-boundary conditions as

$$\min \qquad J[u] = \int_0^1 [(\alpha+2)f(t) - tu(t)]^2 \mathrm{d}t, \tag{25}$$

$$s.t.$$

$$\dot{f}(t) + {}_0^C D_t^{\alpha} f(t) = u(t) + t^2 + 0.01\sin(1.5t), \tag{26}$$

$$f(0) = 0, \;\; f(1) = \frac{2}{\Gamma(3+\alpha)},$$

where $0.01\sin(1.5t)$ is the noise function applied to the system.

This problem without considering the noise function has the following exact solution

$$f^*(t) = \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}, \quad u^*(t) = \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{27}$$

in which its optimal value of the cost functional without considering the noise function is $J^* = 0$. Approximating the state function and utilizing the dynamical system (25) to estimate the input function, we obtain

$$f_m(t) = (t^2 - t)X_m^T B_m(t) + \frac{2}{\Gamma(3+\alpha)}t, \tag{28}$$

$$u_m(t) = \dot{f}_m(t) + {}_0^C D_t^\alpha f_m(t) - t^2 - 0.01\sin(1.5t), \tag{29}$$

Table 1 displays the error of the state variable without a noise function along with the approximate solution. Moreover, Table 1 shows the cost functional for variable values orders of the basis by taking $\alpha = 1.5$ in the presence of the noise function. Figure 2 depicts plots of the exact [without any noise function] and numerical [with the noise function] solution for various amounts of the fractional order.

Table 1: Absolute error of the state function (exact [without noise] and estimated [with noise]) for $\alpha = 1.5$ on the selected nodes with different basis orders.

t	$m = 1$	$m = 3$
[0.5ex] 0.1	3.72×10^{-3}	1.38×10^{-4}
0.2	4.33×10^{-3}	2.75×10^{-4}
0.3	3.06×10^{-3}	4.23×10^{-4}
0.4	0.89×10^{-3}	5.55×10^{-4}
0.5	1.42×10^{-3}	6.41×10^{-4}
0.6	3.29×10^{-3}	6.61×10^{-4}
0.7	4.30×10^{-3}	6.07×10^{-4}
0.8	4.15×10^{-3}	4.78×10^{-4}
0.9	2.73×10^{-3}	2.77×10^{-4}

Table 2: Optimal cost functional (J*) for Example 1 in presence of noise function for ($\alpha = 1.5$).

Polynomial Order	$m = 1$	$m = 2$	$m = 7$
Optimal cost functional	8.90×10^{-4}	3.98×10^{-8}	1.48×10^{-13}

Also, in Table 3, a comparison between the current paper results is made to the results of [37]. The estimated optimal performance index of the current paper for $\alpha = 0.5$ is better than the results of [37].

Example 2. Assume the following FOCP with nonlinear dynamical system constraint having a sinusoidal noise function given by

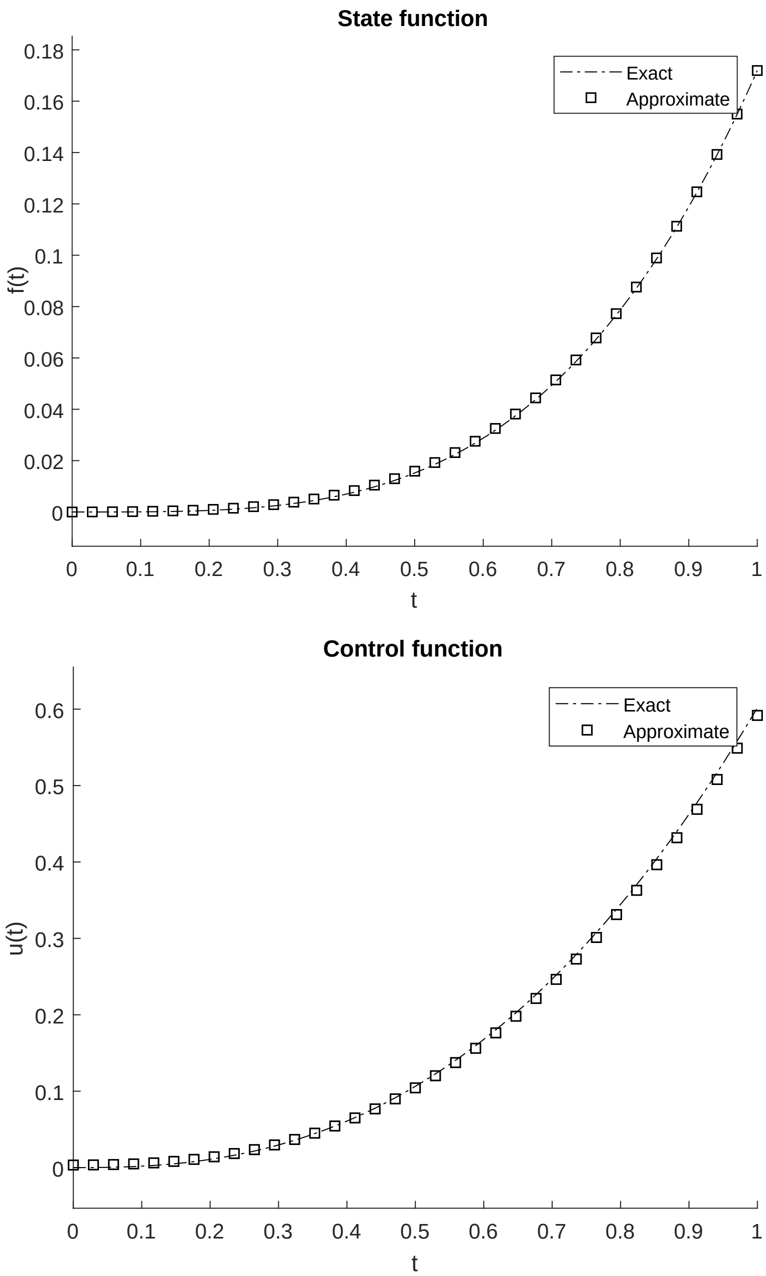

Figure 2: Plots of exact [without noise] and estimated [with noise] state and control input ($\alpha = 1.5$) in the presence of noise function for Example 1.

$$\begin{aligned}
\min\ J = \int_0^1 [&-2e^{1+t^2+f(t)} + e^{2(1+t^2+f(t))} + \frac{8\sqrt{t}}{\sqrt{\pi}}u(t) - (2\sin(1+t^2) - 4t)u(t) \\
&+u^2(t) + \frac{16t}{\pi} + 4t^2 + 4\sin(f(t))(t - \frac{2\sqrt{t}}{\sqrt{\pi}}) + \frac{16t\sqrt{t}}{\sqrt{\pi}} + \sin^2(1+t^2) + 1]\mathrm{d}t, \\
&\dot{f}(t) + {}_0^C D_t^{1.5} f(t) = \sin(f(t)) + u(t) - 0.1\cos(12t)e^{-t}, \\
&f(0) = -1,\ \dot{f}(0) = 0.
\end{aligned} \tag{30}$$

Table 3: Optimal cost function (J*) for Example 1 in the absence of a noise function for ($\alpha = 1/2$).

The Method	The method of [37]($N = 3$)	Current study ($m = 3$)
Optimal cost functional	2.4503×10^{-4}	1.6857×10^{-5}

where the function $n(t) = -0.1\cos(12t)e^{-t}$ is the imposed noise function. The exact solution of the function without considering the noise function is as follows

$$f^*(t) = -t^2 - 1, \quad u^*(t) = \sin(1 + t^2) - 2t - \frac{4\sqrt{t}}{\sqrt{\pi}}.$$

We intend to apply the proposed method to solve this FOCP. Therefore, the unknown functions are estimated as

$$f_m(t) = t^2 X_m^T B_m(t) - 1 \tag{31}$$

$$u_m(t) = \dot{f}_m(t) + {}_0^C D_t^{1.5} f_m(t) - \sin f_m(t) + 0.1\cos(12t)e^{-t}. \tag{32}$$

Applying the Bernstein operational matrix to compute fractional derivatives on the input function implies

$$\begin{aligned}\tilde{u}_m(t) &= 2tX_m^T B_m(t) + X_m^T(D_m^{1;2} + D_m^{1.5;2})B_m(t)\\ &\quad - \sin(t^2 X_m^T B_m(t) - 1) + 0.1\cos(12t)e^{-t}.\end{aligned}$$

By substituting the approximate state function(31)and control input (4) into the performance index (1) yields an optimization problem. In order to find the solution of the resultant problem, a well-known algorithm can be applied. We use the “Matlab 2012” software to simulate the numerical results. By applying the optimization toolbox and fminunc function, the unknown coefficients are determined. Table 4 shows the performance index improvement versus the polynomial order. Figure 3 depicts the noise function imposed on the system dynamics. In Figure 4, the control and state functions for both of the exact [without considering any noise function] and the computed solutions [with the noise function] are plotted.

Table 4: Optimal performance index for Example 2.

Polynomial Order	$m = 1$	$m = 3$
Optimal cost functional	8.094×10^{-3}	1.744×10^{-5}

5 Conclusion

This paper shows a numerical scheme to find a solution to a class of FOCPs having a noise function in the system dynamics. The motivation of the proposed problem is its application in

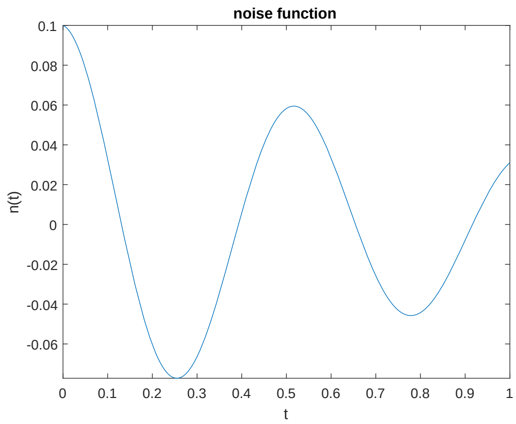

Figure 3: The imposed noise function to the system dynamics for Example 2.

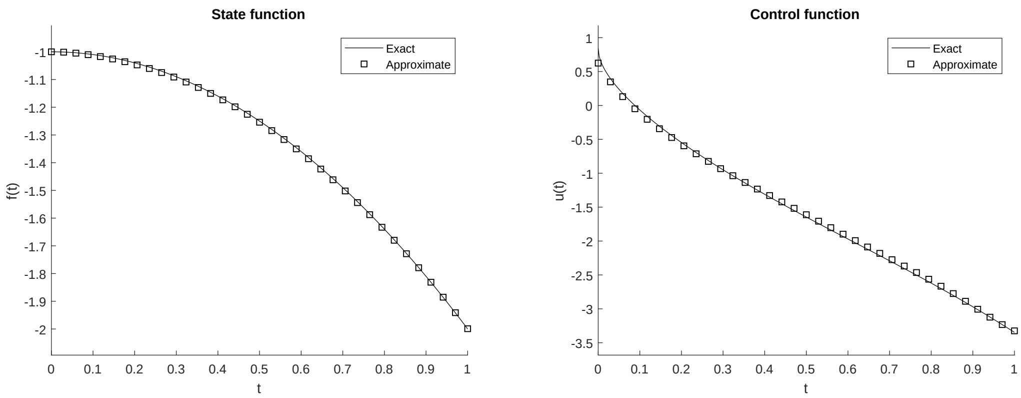

Figure 4: Exact [without noise] and estimated [with the noise] state and control functions in presence of the noise function $n(t)$ for Example 2 ($m = 6$).

the real system modeling. To proceed wiht the numerical method, the Bernstein polynomial basis is used. The fractional derivatives of this Bernstein basis can be easily computed. Additionally, a novel fractional operational matrix is formed to simplify the numerical complexity. A theoretical analysis for the operational matrix convergence is provided as well. Our numerical examples verify theoretical results. Moreover, by taking small numbers of the basis, good results for the proposed FOCP in the presence of the noise function, are obtained.

References

[1] Magin R. L. (2006). “Fractional calculus in bioengineering”, Chicago: Begell House.

[2] Jumarie G. (2008). “Modeling fractional stochastic systems as non-random fractional dynamics driven by Brownian motions”, Applied Mathematical Modelling, 32(5), 836–859.

[3] Tricaud C., Chen Y. Q. (2012). "Optimal Mobile Sensing and Actuation Policies in Cyber-physical Systems", Londo: Springer-Verlag.

[4] Oldham K. B., Spanier J. (1974). "The fractional calculus", Academic Press, New York.

[5] Monje C. A., Chen Y., Vinagre B. M., Xue D., Feliu-Batlle V. (2010). "Fractional-order Systems and Controls: Fundamentals and Applications", Springer-Verlag, London.

[6] Jajarmi A., Baleanu D. (2017). "Suboptimal control of fractional-order dynamic systems with delay argument", Journal of Vibration and Control, 24(12), 2430–2446.

[7] Kilbas A. A., Srivastava, H. M., Trujillo J. J. (2006). "Theory and Applications of Fractional Differential Equations", 204, Elsevier Science.

[8] Kowalski E. (2006). "Bernstein Polynomials and Brownian Motion", The American Mathematical Monthly, 113(10), 865–886.

[9] Hasan M. M., Tangpong X. W., Agrawal O. P. (2011). "Fractional optimal control of distributed systems in spherical and cylindrical coordinates", Journal of Vibration and Control, 18, 1506–1525.

[10] Özdemir N., Agrawal O. P., Karadeniz D., İskender B. B. (2009). "Fractional optimal control problem of an axis-symmetric diffusion-wave propagation", Physica Scripta, T136 014024.

[11] Nemati A., Yousefi S. A., Soltanian F., Saffar-Ardabili J. (2016). "An efficient numerical solution of fractional optimal control problems by using the Ritz method and Bernstein operational matrix", Asian Journal of Control, 18(6), 2272–2282.

[12] Mamehrashi K., Nemati A. (2019). "A new approach for solving infinite horizon optimal control problems using Laguerre functions and Ritz spectral method", International Journal of Computer Mathematics.

[13] Bhrawy A. H., Doha, E. H., Tenreiro Machado, J. A., Ezz-Eldien, S. S. (2015). "An Efficient Numerical Scheme for Solving Multi-Dimensional Fractional Optimal Control Problems With a Quadratic Performance Index. Asian Journal of Contro"l, 17(6), 2389–2402.

[14] Nemati A., Mamehrashi, K. (2019). "The Use of the Ritz Method and Laplace Transform for Solving 2D Fractional Order Optimal Control Problems Described by the Roesser Model", Asian Journal of Control, 21(4), 1–13.

[15] Taherpour V., Nazari M., Nemati A. (2020). "A new numerical Bernoulli polynomial method for solving fractional optimal control problems with vector components", Computational Methods for Differential Equations, Accepted paper.

[16] Agrawal O. P., Baleanu D. (2007). "A Hamiltonian Formulation and a Direct Numerical Scheme for Fractional Optimal Control Problems", Journal of Vibration and Control, 13(9-10), 1269–1281.

[17] Maleki M., Hashim I., Abbasbandy S., Alsaedi A. (2015). "Direct solution of a type of constrained fractional variational problems via an adaptive pseudospectral method", Journal of Computational and Applied Mathematics, 283, 41–57.

[18] Nemati A., Yousefi S. A. (2016). "A numerical method for solving fractional optimal control problems using Ritz method", Journal of Computational and Nonlinear Dynamics, 11, 1–7.

[19] Lotfi A., Yousefi S. A., Dehghan M. (2013). "Numerical solution of a class of fractional optimal control problems via the Legendre orthonormal basis combined with the operational matrix and the Gauss quadrature rule", Journal of Computational and Applied Mathematics, 250, 143–160.

[20] Doha E. H., Bhrawy A. H., Baleanu D., Ezz-Eldien S. S., Hafez R. M. (2015). "An efficient numerical scheme based on the shifted orthonormal Jacobi polynomials for solving fractional optimal control problems", Advances in Difference Equations, 2015(1), 15.

[21] Lotfi A., Dehghan M., Yousefi S. A. (2011). "A numerical technique for solving fractional optimal control problems", Computers & Mathematics with Applications, 62(3), 1055–1067.

[22] Trujillo J. J., Ungureanu V. M. (2018). "Optimal control of discrete-time linear fractional-order systems with multiplicative noise", International Journal of Control, 91(1), 57–69.

[23] Ahadpour S., Nemati A., Mirmasoudi F., Hematpour N. (2018). "Projective synchronization of piecewise nonlinear Chaotic maps", Theoretical and Mathematical Physics, 197(3), 1856–1864.

[24] FujishlGe S., Sawaragl Y. (1975). "Optimal control for linear continuous-time systems with general noises based upon sampled data. International Journal of Systems Science", 6(12), 1111–1118.

[25] Saadatmandi A. (2014). "Bernstein operational matrix of fractional derivatives and its applications", Applied Mathematical Modelling, 38(4), 1365–1372.

[26] Nemati A. (2017). "Numerical solution of 2D fractional optimal control problems by the spectral method along with Bernstein operational matrix", International Journal of Control,

[27] Farin G. (1993). "Curves and Surfaces for CAGD", Academic Press, Boston.

[28] R. T. F., T. N. T. G. (2003). "Construction of orthogonal bases for polynomials in Bernstein form on triangular and simplex domains".

[29] G. M. P. (2003). "Interpolation and Approximation by Polynomials".

[30] Samko S. G., Kilbas A. A., Marichev O. I. (1993). "Fractional Integrals and Derivatives. Theory and Applications", Gordon and Breach Science Publishers, Yverdon.

[31] Wu Z., Sun X., Ma L. (2013). "Sampling scattered data with Bernstein polynomials: stochastic and deterministic error estimates. Advances in Computational Mathematics", 38(1), 187–205.

[32] Rivlin T. J. (1981). "An Introduction to the Approximation of Functions", Courier Corporation, New York.

[33] Jüttler B. (1998). “The dual basis functions for the Bernstein polynomials. Advances in Computational Mathematics”, 8(4), 345–352.

[34] Rudas I. J., Tar J. K., Patkai B. (2006). “Compensation of dynamic friction by a fractional order robust controller”, Paper presented at the IEEE International Cconference on Computational Cybernetics, Budapest.

[35] Petrovacki N., Jelicic Z. D. (2006). “Optimal transient response of erbium-doped fiber amplifiers Paper presented at the IEEE international conference on numerical simulation of semiconductor optoelectronic devices”, Singapore.

[36] Kreyszig E. (1978). “Introductory Functional Analysis with Applications”, John Wiley and Sons,Inc., New York.

[37] Shakoor P., Ricardo A., Delfim F. M. T. (2013). “Fractional order optimal control problems with free terminal time”, Journal of Industrial & Management Optimization, 10(2), 363–381.

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چکیده

این مقاله، روشی عددی برپایه روش طیفی ریتز را برای حل دسته‌ای از مسائل کنترل بهینه کسری در دامنه محدود دارای نویز بکار می‌برد. پایه‌های چندجمله‌ای برنشتاین به همراه ماتریس عملیاتی مشتق برای تخمین توابع نامعین حالت و کنترل مورد استفاده قرار می گیرند. با جایگذاری توابع تخمینی کنترل و حالت در تابع هدف، یک مساله بهینه‌سازی نامقید ایجاد می‌شود. برای حل و شبیه‌سازی آن از نرم‌افزار Matlab و جعبه ابزارهای آن استفاده خواهد شد. در مسائل مورد بحث، تابع هدف شامل توابع کنترل و توابع حالت می‌باشد. روش پیشنهادی با توجه به سازگاری آن در مدیریت شرایط اولیه و مرزی و نویز، قابل اعتماد و اطمینان است. برای نشان دادن کارآیی و کاربردی بودن روش، چند مثال کاربردی در انتهای مقاله آورده شده است.

کلمات کلیدی

بهینه سازی، روش طیفی، تابع نویز، مساله کنترل بهینه کسری، ماتریس عملیاتی.