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Enlarging the Region of Attraction for Nonlinear Systems through the Sum-of-Squares Programming

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Abstract. In the present study, a novel methodology is developed to enlarge the Region of Attraction (ROA) at the point of equilibrium of an input-affine nonlinear control system. Enlarging the ROA for non-polynomial dynamical systems is developed by designing a nonlinear state feedback controller through the State-Dependent Riccati Equation (SDRE). Consequently, its process is defined in the form of a Sum-of-Squares (SOS) optimization problem with control and non-control constraints. Of note, the proposed technique is effective in estimating the ROA for a nonlinear system functioning on polynomial or non-polynomial dynamics. In the present study, the application of the proposed scheme is shown by numerical simulations.

Keywords. State-dependent Riccati equation, Region of attraction, Sum-of-squares programming, Lyapunov function.

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1 Introduction

Obtaining the ROA for a nonlinear system at a point of equilibrium is challenging for the control theory. The ROA at a point of equilibrium comprises a set of initial conditions the state trajectories of which converge on the equilibrium again. With a small ROA, a disturbance may quickly take the system out of the region and the system can not get back to the stable point of equilibrium. Therefore, the extent of the ROA is one of the criteria for the stability of nonlinear systems around the respective point of equilibrium. Computing the ROA is not an easy task and hence, its estimation has turned into a crucial problem.

In the present study, a new methodology for the enlargement of the ROA of input-affine nonlinear systems is developed by designing a nonlinear state feedback controller through the SDRE. The equation gives a shape factor that expands the ROA. Since the choice of the shape factor is affected by the system dynamics, a non-uniform extension of the ROA is established. The main advantage of the proposed approach is incorporating both polynomial and non-polynomial dynamics. Moreover, the proposed method deals with both bound and unbound ROAs. Accordingly, the ROA enlargement problem is defined as an SOS optimization problem. The present method is evaluated in comparison with the existing methods by its application to various nonlinear systems that have already been addressed in the previous studies. The methods proposed in the existing literature for the ROA estimation can be classified into categories, namely Lyapunov [1]-[9] and non-Lyapunov [10]-[13]. In the Lyapunov-based methods, a Lyapunov Function (LF) is sought that is positive definite with negative definite time derivative for its largest sublevel set. The mentioned largest sublevel set is considered as an estimate of the ROA. Reference [1] presented a new design to maximize the ROA for a saturated supercavitating vehicle. To cope with the immeasurability of the vertical speed, the development of the presented design was made possible through output feedback control schemes. The conditions required by the achieved controller to locally and asymptotically stabilize the closed-loop system were represented by LMI presentations.

Reference [3] developed a method for ROA enlargement in nonlinear systems by trajectory reversing. Numerical simulations were performed to validate the suggested method employing modeling systems of Van der Pol oscillator and Hahn.

In [5], stability analysis of polynomial systems was carried out based on LFs. For ROA enlargement, a region with variable size was used as a shape factor. The purpose was to achieve the biggest sublevel set of LFs with the biggest shape factor. In [6], similar to [5], by adopting a polynomial as a shape factor, the ROA was enlarged and the use of bilinear SOS programming with polynomial LFs was suggested. Also, due to their higher richness than the quadratic LFs, the sublevel sets of higher-order polynomial LFs were used. However, it should be noted that as soon as the degree of LF increases, the number of decision variables is considerably enlarged. Hence, to lower the number of decision variables, employing the maximum or a minimum number of a group of polynomial functions was suggested as a solution in [7]. Khodadadi et al. [9] proposed a numerical methodology for ROA estimating in nonlinear polynomial systems through SOS programming. On the other hand, for estimation enlargement, an SOS optimization problem was solved by defining a subset of the invariant set through a shape factor.

Reference [10] proposed a non-Lyapunov method for the estimation of the Robust Domain of Attraction (RDA) and directional enlargement of the Domain of Attraction (DOA) by using Markov chains and an invariant measure. Their methodology formulated the estimation of RDA and directional enlargement of DOA as an infinite-dimensional linear problem. A major shortcoming of Markov modeling in the estimation of DOA is that it incorporates the real DOA into the estimated one; as a result, in boundary partitioning, it does not ensure achieving stability. To tackle this problem, they suggested refining any partition set that had a large invariant measure. In recent studies, control parameters have been employed in the enlargement of the ROA [14]-[22]. Enlarging ROA in non-linear systems is a substantial issue for the designers of non-linear controllers. For the systems with large ROAs around a point of equilibrium, the problems concerning tracking and disturbance can be systematically tackled. In particular, it would be very useful to characterize the controllers that maximize the ROA. In [15], a formulation for Model Predictive Control (MPC) was presented in order to enlarge the DOA without the requirement to enlarge the horizon of prediction. An array of contractive control invariant sets was employed to substitute the MPC terminal region. Therefore, computing the contractive array of control invariant sets was essential in the developed formulation, which was solved only for linear systems.

Haghighatnia and Moghaddam [18] offered a method to enlarge the robust ROA in uncertain systems by designing linear controllers. The problem of enlarging the robust ROA was formulated as a novel optimization at three levels, which sought the optimal controlling parameters of the linear controller. The present study is arranged as follows. The ROA is estimated and then, enlarged; the pseudo-linearization is discussed, and the SDRE approach is developed in Section 2. The achieved results are illustrated in Section 3. In Section 4, by illustrative numerical examples, the effectiveness of the proposed method is proven and a comparison is made with other published works in the literature. Finally, a conclusion is briefly made in Section 5.

2 Preliminaries

In the following, the notation utilized in this study is provided.
R^n : an n-dimensional vector space over the field of the real numbers,
$\aleph_n$: the set of all polynomials with real coefficients in n variables,
Σ_n: the SOS polynomials set in n variables,
Ψ : the state set that is a bounded open subset of R^n Euclidean space and contains the origin,
$C^k(\Psi)$: tthe class of functions continuously differentiable for k times in Ψ.
Assume a system in the following form:

$$\dot{x} = f(x(t)), \tag{1}$$

where $x(t) \in R^n; x(0) = x_0$ is the initial state at $t = 0$ and f is an n-vector of elements of $\aleph_n$ with $f(0) = 0$.

Definition 1 (point of equilibrium)**.** $x_e \in R^n$ is a point of equilibrium in system (1) if $f(x_e) = 0$. The points of equilibrium in system (1) correlate with the intersection of its nullclines, i.e., the curves represented by $f(x) = 0$. Without loss of generality, we suppose that the point of equilibrium under investigation concurs with the origin of the state space of $R^n, (x_e = 0)$ in the sequel.

Definition 2 (ROA)**.** If the origin is a point of equilibrium for (1), the ROA of the origin is defined as follows:

$$\Omega = \{x_0 \in R^n \mid lim_{t\to 0}\phi(x_0, t) = 0\},$$

where $\phi(x_0, t)$ denotes the solution starting from the initial state.

Definition 3 (SOS polynomials)**.** A multivariate polynomial $P(x_1, x_2, ..., x_n) \triangleq P(x)$ is an SOS if polynomials $f_1(x), ..., f_m(x)$ exist such that

$$P(x) = \sum_{i=1}^{m} f_i^2(x). \tag{2}$$

From the definition, it can be deduced that the SOS polynomials set in n variables is a convex cone. It can be indicated that the existence of an SOS decomposition (2) is equivalent to the existence of a positive semidefinite matrix Q such that

$$P(x) = Z^T(x)QZ(x), \tag{3}$$

where $Z(x)$ is some properly chosen monomials vector. It is undisputed that an SOS polynomial is globally nonnegative. As a major characteristic of SOS polynomials, this is decisive for many applications in the field of control; in particular, the cases in which different polynomial inequalities are substituted with SOS conditions [23].

2.1 Estimating ROA

A Lyapunov-based method is developed for estimating the ROA. Enlarging the ROA provides more freedom in designing nonlinear controllers. Moreover, it can be considered as a way of improving the performance of nonlinear closed-loop systems. In this regard, we investigate a positive definite LF with a negative definite time derivative in the largest sublevel set. The mentioned largest sublevel set is an estimate of the ROA. We aim to achieve a provable ROA for the system such that all the points that start in this region will converge to the fixed point of origin.

Theorem 1 ([24])**.** If we have a function $V : R^n \to R$ that is continuously differentiable as follows:

$$V \quad is\ positive\ definite, \tag{4}$$

$$\Omega := \{x \in R^n \mid V(x) \le c\} \quad is \quad bounded, \quad and \tag{5}$$

$$\{x \in R^n \mid V(x) \le c\} \setminus \{0\} \subseteq \{x \in R^n \mid \frac{\partial V}{\partial t} f(x) < 0\}, \tag{6}$$

where c is a positive value. For all $x(0) \in \Omega$, a solution for (1) can be achieved and $lim_{t\to\infty} x(t) = 0$. Accordingly, Ω is the subset of the ROA for (1) and it is invariant. The continuously differentiable function $V(x)$ is called a local Lyapunov function. For arriving at a better estimate of the ROA, we need to acquire a $V(x)$ that leads to a larger Ω.

2.2 Enlarging ROA

One of the parameters that have a significant effect on the estimation of ROA is the shape factor. However, no systematic method for determining and selecting the shape factor has been presented so far. In the previous studies, since a fixed shape factor has always been used, the uniform extension of ROA has been addressed. In the present article, the shape factor is obtained through the SDRE approach and the dynamics of the system are effective in its selection. Therefore, the non-uniform extension of ROA is dealt with. With the aim of enlarging Ω, we consider a region with variable size $P_\beta = \{x \in R^n \mid P_0(x) \leq \beta\}$, where $P_0(x)$ is a positive convex polynomial called the shape factor. Maximizing β, as long as $P_\beta \subseteq \Omega$ is established, leads to an estimation of the ROA. With the application of theorem 1, we can formulate the problem of estimating the ROA as an optimization problem in the following form [5]:

$$\begin{aligned} &\max_{V\in R_n} \quad \beta \\ &s.t. \\ &\qquad V(x) > 0 \quad for\ all \quad x \in R^n \setminus \{0\} \ and \ V(0) = 0 \\ &\qquad the\ set\ \Omega \quad is \quad bounded, \\ &\qquad \{x \in R^n \mid P_0(x) \leq \beta\} \subseteq \Omega, \\ &\qquad \{x \in R^n \mid V(x) \leq c\} \setminus \{0\} \subseteq \{x \in R^n \mid \frac{\partial V}{\partial t} f(x) < 0\}. \end{aligned} \tag{7}$$

Given that the choice of the shape factor in the estimation of the ROA plays a fundamental role, in this paper, the shape factor is obtained based on the SDRE idea.

2.3 Pseudo Linearization

The pseudo-linearization methodology represents the nonlinear system as a linear-like system for which the matrix is dependent on state variables. Assume an input-affine nonlinear system in the following form:

$$\dot{x} = f(x) + B(x)u. \tag{8}$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the input vector, and $f: R^n \to R^n, \quad B: R^n \to R^{n\times m}$. Here it is assumed that $f(0) = 0$. In short, the concept of pseudo-linearization for Eq. (8) can be formulated as follows:

$$\dot{x} = A(x)x + B(x)u. \tag{9}$$

Matrices $A(x)$ and $B(x)$ are called State-Dependent Coefficients (SDC) matrices [30]. Assuming $f(0) = 0$ and $f(.) \in C^1(\Psi)$, there is always a function $A(x)$ that is continuous nonlinear and matrix-valued as follows:

$$f(x) = A(x)x, \tag{10}$$

where $A : \Psi \to R^{n\times n}$ is achieved by mathematical factorization; it is obviously nonunique with $n > 1$. Of note, the above-mentioned assumptions for $f(x)$ ensure that a global SDC parameterization of $f(x)$ exists on Ψ. The theorem below indicates that $f(x)$ can be reformulated as given in (10).

Theorem 2 ([30])**.** Let us have $f : \Psi \to R^n$ in a way that $f(0) = 0$ and $f(.) \in C^k(\Psi), k \geq 1$. For all $x \in R^n$, there is an SDC parameterization (10) of $f(x)$ for some $A : R^n \to R^{n\times n}$. An instance of such parameterization ensured by the given conditions is the following

$$A(x) = \int_{t=0}^{1} \frac{\partial f(x)}{\partial x}|_{x=\lambda x} d\lambda, \tag{11}$$

where λ is a dummy variable introduced in the integration.

Proof. (11) can be validated by assuming the functions set: $\hat{f} : R \to R^n$ established by $\hat{f}(\lambda) \triangleq f(\lambda x)$. Then, for each $x \in R^n$,

$$f(x) = \hat{f}(1) = \hat{f}(0) + \int_{t=0}^{1} \frac{d\hat{f}(\lambda)}{d\lambda} d\lambda.$$

$\hat{f}(0) = 0$, as assumed, and $\dfrac{d\hat{f}(\lambda)}{d\lambda} = (\frac{\partial f}{\partial x}|_{x=\lambda x})x$, thus

$$f(x) = \int_{t=0}^{1} \frac{\partial f(x)}{\partial x}|_{x=\lambda x} d\lambda x. \tag{12}$$

Comparing (12) with (10) gives the desired result (11). Using extended linearization, any input-affine nonlinear system (8) that meets the conditions for $f(x)$ given in theorem (2) can always be formulated as an SDC (9). Of note, pseudo-linearization has many advantages. First, unlike Jacobian linearization, it retains all nonlinearity properties of the system and second, its non-uniqueness creates extra degrees of freedom that can be used to enhance controller performance [30]-[31].

2.4 SDRE Method

The SDRE method originates in extending the Linear Quadratic Regulator (LQR) problem into nonlinear systems by maintaining all the nonlinear properties. It functions on pseudo-linearization of the system and entails factorization (i.e., parameterization) [30] of the nonlinear dynamics to the state vector and a matrix-valued function dependent on the state. In this regard, the SDRE algorithm completely incorporates the nonlinearities of the system, turning it into a (non-unique) linear structure having SDC matrices and minimizing a nonlinear performance index with a quadratic-like structure. Consider the system (9) with cost function

$$J = \frac{1}{2}\int_{t=0}^{\infty}(x^T(t)Q(x)x(t) + u^T(t)R(x)u(t))dt, \tag{13}$$

where $Q(x)$ and $R(x)$ are state-dependent weighting matrices to meet $Q(x) \geq 0$ and $R(x) > 0$ for all the values of x. It is clear that the design of the SDRE controller is similar to the LQR one with minor variations in matrices A, B, Q and R, which are state-dependent. However, Q and R do not necessarily require to be state-dependent. Therefore, in order to minimize (13), the algebraic SDRE in the following should be solved [31]:

$$A^T(x)P(x) + P(x)A(x) - P(x)B(x)R^{-1}(x)B^T(x)P(x) + Q(x) = 0. \tag{14}$$

The optimal control for minimization of the cost function (13) is

$$u(t) = -R^{-1}(x)B^T(x)P(x)x, \tag{15}$$

where $P(x)$ is the unique, symmetric, positive definite solution for (14). The major privilege of the SDRE method may be the non-uniqueness of the pseudo-linear representation of the system, which gives the designer more freedom. By replacing delay in the system matrix, Batmani and Khaloozadeh introduced a method to find a sub-optimal solution for a class of nonlinear time-delayed systems using the SDRE method [25]-[29].

3 Main Results

In the following, first, the nonlinear optimal control system is written in the pseudo-linear form. Then, the nonlinear state feedback control is obtained through SDER. The controller, while stabilizing the system and minimizing the cost function, leads to the expansion of the ROA by using the shape factor. Since the choice of this shape factor is affected by the system dynamics, a non-uniform extension of the ROA will be established. Consequently, the problem of enlarging ROA is formulated in the form of an SOS optimization problem. Assume the following class of input-affine nonlinear systems:

$$\dot{x} = f(x) + B(x)u. \tag{16}$$

where $f : R^n \to R^n, \quad B : R^n \to R^{n\times m}$, with $x \in R^n$ representing the state and $u \in R^m$ indicating the control input under the component-wise saturation constraints as follows

$$u \in U : \quad |u_i| \leq \bar{u}_i \quad , i = 1, ..., m. \tag{17}$$

with U is a compact subset of R^m comprising the origin as an interior point. The origin is considered as a point of equilibrium for (16) when $u = 0$ i.e. $f(0) = 0$ ([32]). We seek for a strategy of state feedback regulation $u = K(x)$ that asymptotically stabilizes (16) into the origin under (17), has the largest estimate of the ROA, and minimizes the cost function (13). Reference [33] sought to obtain an LF $V(x)$ which was an upper bound for the cost function: $J(K(x), x) \leq V(x)$. According to [5], when an SOS $V(x)$ and a polynomial state-dependent control law $u = K(x)$ can be achieved at a considered time instant t that are consistent with (17) such that $J(K(x), x) \leq V(x)$, the set $\Omega = \{x \in R^n | V(x) \leq c\}$ is a positive invariant region for the regulated input constrained plant. First, we write the system (16) in the following pseudo-linear form:

$$\dot{x} = A(x)x + B(x)u. \tag{18}$$

Applying the SDRE method, we calculate the feedback control $u = K(x)$. Replacing it in (18) we obtain:

$$\begin{cases} \dot{x} = A(x)x + B(x)K(x) = F(x) \\ K(x) \in U \end{cases} \tag{19}$$

Now the problem of estimating the ROA of (19) is written as the following optimization problem:

$$\begin{aligned} &\max_{V \in \mathcal{R}_n} \quad \beta \\ &s.t. \\ &\qquad V(0) = 0, V(x) > 0, \quad x \in R^n \\ &\qquad \{x \in R^n \mid P_0(x) \leq \beta\} \subseteq \{x \in R^n \mid V(x) \leq c\}, \\ &\qquad \{x \in R^n \mid V(x) \leq c\} \setminus \{0\} \subseteq \{x \in R^n \mid \frac{\partial V}{\partial x} F(x) < 0\} \\ &\qquad \{x \in R^n \mid V(x) \leq c\} \subseteq \{x \in R^n \mid \quad \mid K(x) \mid \leq \bar{u}_i\}. \end{aligned} \tag{20}$$

In order to enlarging the ROA, using the Positive Stellensatz theorem [34], the optimization problem (20) is converted to the SOS programming problem as follows:

$$\begin{aligned} &\max_{V \in \mathcal{R}_n, s_1, s_2, s_3, s_4 \in \Sigma_n} \quad \beta \\ &\qquad s.t. \\ &\qquad\qquad V - l_1 \in \Sigma_n \\ &\qquad\qquad (c - V - s_1(\beta - P_0)) \in \Sigma_n \\ &\qquad\qquad -\left(\frac{\partial V}{\partial x} F(x) + l_2 + s_2(c - V)\right) \in \Sigma_n \\ &\qquad\qquad (\bar{u}_i - K - s_3(c - V)) \in \Sigma_n \\ &\qquad\qquad (\bar{u}_i + K - s_4(c - V)) \in \Sigma_n \end{aligned} \tag{21}$$

where s_1, s_2, s_3 and s_4, are SOS polynomials. Also, $l_i(x)$ is a positive definite polynomial as $l_i(x) = \sum_{j=1}^{n} \varepsilon_{ij} x_j^2$ for $i = 1, 2$, in which ε_{ij} are numbers with positive values.

The previous description is summarized in the form of an algorithm below:

Step 1: linearize the system and compute the SDRE control.

Writing Equation (16) in the form of (18) and then, designing a nonlinear state feedback controller according to the SDRE and replacing it in (18), we obtain (19).

Step 2: To calculate the initial shape factor and LF, do:

$$A = \frac{\partial F}{\partial x}|_{x=0}. \tag{22}$$

Then, solve equation (23)

$$A^T P_1 + P_1 A = -I. \tag{23}$$

Using the matrix $P_1 \geq 0$, we define the initial shape factor and LF as follows:

$$P_0(x) = x^T P_1 x, \tag{24}$$

$$V = P_0(x). \tag{25}$$

Step 3: Set V to a fixed value and perform the SOS optimization for s_2, s_3, s_4

$$\begin{aligned} &\max_{c \in R, s_2, s_3, s_4 \in \Sigma_n} \quad c^* \\ &s.t. \\ &-\left(\frac{\partial V}{\partial x} F(x) + l_2 + s_2 (c - V)\right) \in \Sigma_n, \\ &(\bar{u}_i - K - s_3 (c - V)) \in \Sigma_n, \\ &(\bar{u}_i + K - s_4 (c - V)) \in \Sigma_n. \end{aligned} \tag{26}$$

Step 4: Set V and $P0$ to fixed values and perform the SOS optimization for s_1

$$\begin{aligned} &\max_{\beta \in R, s_1 \in \Sigma_n} \quad \beta^* \\ &s.t. \\ &(c - V - s_1 (\beta - P_0)) \in \Sigma_n. \end{aligned} \tag{27}$$

Step 5: Using $s_1, s_2, s_3, s_4, c^*, \beta^*, P_0$ from the previous steps, compute V as following:

$$\begin{aligned} &V - l_1 \in \Sigma_n \\ &(c - V - s_1 (\beta - P_0)) \in \Sigma_n, \\ &-\left(\frac{\partial V}{\partial x} F(x) + l_2 + s_2 (c - V)\right) \in \Sigma_n, \\ &(\bar{u}_i - K - s_3 (c - V)) \in \Sigma_n, \\ &(\bar{u}_i + K - s_4 (c - V)) \in \Sigma_n. \end{aligned} \tag{28}$$

Step 6: Use the quadratic part of the new LF as new P_0 and replace $V \; with \frac{V}{c^*}$
Using the newly achieved LF and shape factor, make repetitions to get convenient LF.

4 Simulation Results

The following instances are given to demonstrate capability of the method developed in the present study in ROA enlargement for the nonlinear optimal control system.

Example 1. Consider the following input-affine nonlinear system examined in reference [33]

$$\begin{cases} \dot{x}_1 = x_2(t), \\ \dot{x}_2 = -x_1(t) - \left(1 - x_1^2(t)\right) x_2(t) + u(t). \end{cases} \tag{29}$$

The weighting matrices $Q = diag([0.01 \quad 0.01]), R = 1$ and input saturation constraint $|u(t)| \leq 0.2.$ are assumed. In [33], the system's ROA was estimated by the fixed shape factor $p_0(x) = x^T x = x_1^2 + x_2^2$, as Figure 1.
In this paper, we search for the feedback control $u(t) = K(x(t))$ that stabilizes the system asymptotically, minimizes the cost function, and has the largest estimation of the ROA. We use the proposed method for Example 1.
Step 1: Note that $l_1 = l_2 = 10^{-6}(x_1^2 + x_2^2)$.
First, we reformulate the system (29) in the pseudo-linear form below:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -(1 - x_1^2(t)) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \tag{30}$$

Then, by applying the SDRE method (14), the matrix $P \geq 0$ is obtained. Matrix-array P is state-dependent:

$$P = \begin{bmatrix} 0.005x_1^2 + 0.015 & 0.005 \\ 0.005 & 0.01x_1^2 + 0.01 \end{bmatrix}, \tag{31}$$

$$u = -Kx = -R^{-1}B^T Px = -(0.005x_1 + 0.01x_1^2 x_2 + 0.01x_2). \tag{32}$$

The control obtained by the SDRE method leads to the minimum cost function. By replacing (32) in the control system (30) we obtain:

$$\begin{cases} \dot{x}_1 = x_2(t) \\ \dot{x}_2 = -1.005x_1(t) + 0.99x_1^2(t)x_2(t) - 1.01x_2(t) \end{cases} \Rightarrow \dot{x} = F(x). \tag{33}$$

Step 2: by using (22) and solving Equation (23), obtain:

$$P_1 = \begin{bmatrix} 1.4951 & 0.4975 \\ 0.4975 & 0.9876 \end{bmatrix}. \tag{34}$$

Using the matrix $P_1 \geq 0$, we define the initial shape factor as follows:

$$P_0(x) = x^T P_1 x = 1.4951x_1^2 + 0.99502x_1x_2 + 0.98764x_2^2, \tag{35}$$

$$V(x) = P_0(x), \tag{36}$$

Step 3: by using (26), obtain:

$$\begin{aligned} s_2 = {} & 0.14130x_1^4 - 0.05192x_1^3x_2 - 0.08065x_1^2x_2^2 + 0.33568x_1^2 \\ & 0.04097x_1x_2^3 + 0.09480x_1x_2 + 0.11980x_2^4 + 0.31168x_2^2 \end{aligned}$$

$$\begin{aligned} s_3 = {} & 0.04746x_1^4 + 0.01754x_1^3x_2 + 0.05305x_1^2x_2^2 + 0.01933x_1^2 \\ & + 0.01792x_1x_2^3 + 0.03783x_1x_2 + 0.03009x_2^4 + 0.00299x_2^2 + 0.05463 \end{aligned}$$

$$s_4 = 0.04746x_1^4 + 0.01754x_1^3x_2 + 0.05305x_1^2x_2^2 + 0.01933x_1^2$$

$$+0.01792x_1x_2^3+0.03783x_1x_2+0.03009x_2^4+0.00299x_2^2+0.05463$$

Step 4: by using (27), obtain:

$$\begin{aligned} s_1 &= 0.18938x_1^4+0.08673x_1^3x_2+0.27538x_1^2x_1^2-0.16025x_1^2 \\ &\quad +0.09146x_1x_2^3+0.15174x_1x_2+0.11653x_2^4-0.18607x_2^2+0.78816 \end{aligned}$$

Step 5: by using (28), obtain:

$$\begin{aligned} V &= 0.03230x_1^6-0.00278x_1^5x_2+0.00654x_1^4x_2^2-0.08683x_1^4 \\ &\quad -0.01464x_1^3x_2^3+0.00108x_1^3x_2+0.00576x_1^2x_2^4 \\ &\quad +0.04102x_1^2x_2^2+0.59692x_1^2+0.00581x_1x_2^54+0.02795x_1x_2^3 \\ &\quad +0.38961x_1x_2+0.01006x_2^6-0.03219x_2^4+0.40588x_2^2 \end{aligned}$$

Step 6: The new shape factor is: $P_0 = 0.59692x_1^2+0.38961x_1x_2+0.40588x_2^2$ Repeat the process in 35 iterations to obtain:

$$\begin{aligned} V &= 0.02717x_1^6-0.02229x_1^5x_2+0.03442x_1^4x_2^2-0.04244x_1^4 \\ &\quad +0.05791x_1^3x_2^3+0.09611x_1^3x_2+0.05382x_2^4+0.16735x_1^2x_2^2 \\ &\quad +0.16528x_1^2+0.01949x_1x_2^5+0.08147x_1x_2^3+0.10276x_1x_2 \\ &\quad +0.00546x_2^6+0.01261x_2^4+0.09084x_2^2. \end{aligned} \tag{37}$$

The level set of V in (37) is the estimated ROA for System (33), as illustrated in Figure 1. It has been compared with the estimates derived by the WK-SOS method in [33].

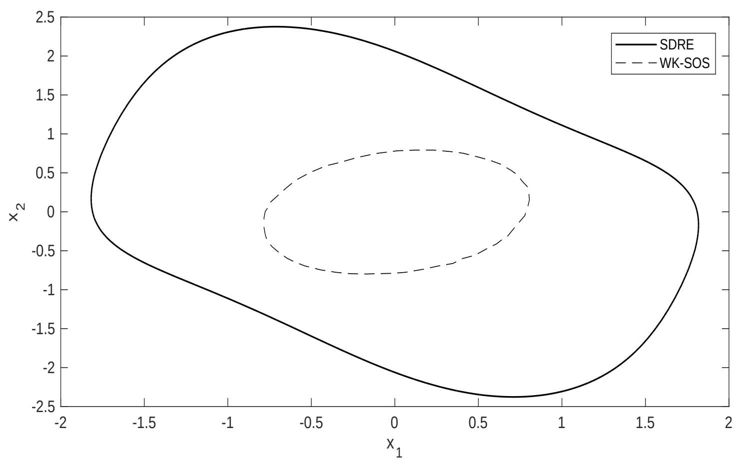

Figure 1: AR estimation provided in[33] (Dashed line) and estimated AR using proposed method (Continuous line).

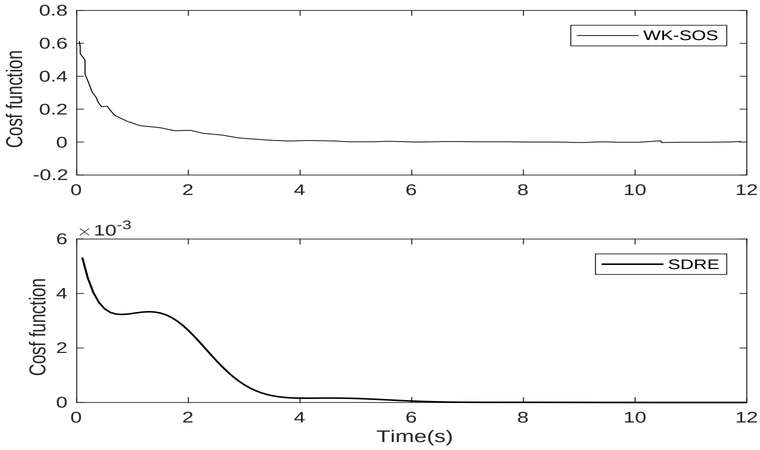

Figure 2: Cost function in the proposed method and reference [33] .

As shown in Figure 2, the cost function value in reference [33] starts from 0.7 and gradually decreases. In our method, this value is much better, starting at 0.004 and converging at a good speed. Compared to the method proposed in [33], which uses an uninspired shape factor, our method uses the shape factor by the selection of $P_0(x)$, which expands the ROA. Also, Figure 2 shows that the cost function of the proposed SDRE method is nonlinear, which indicates the efficiency of the proposed method. In Figure 3, the control function graph and state is shown.

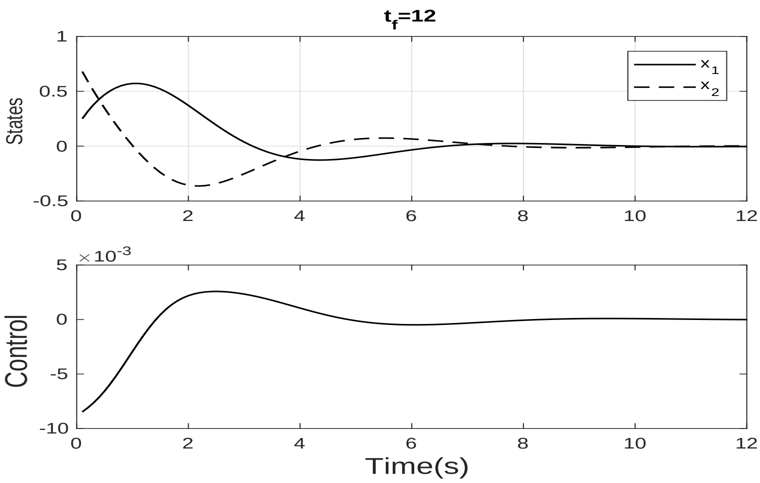

Figure 3: State trajectories and control signal by proposed method.

Remark 1. Given that SOS works with polynomials, if a system has non-polynomial dynamics, we need to find a polynomial approximation of the non-polynomial term. Then, we apply the method to the estimation of the ROA.

Example 2. For the non-linear system in [35] as given below:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -sin(x_1) - 0.5x_2(t), \end{cases} \tag{38}$$

by applying the Taylor series: $sin(x_1) = x_1 - x_1^3/6$ and replace it in (38) we obtain:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1^3/6 - x_1 - 0.5x_2(t). \end{cases} \tag{39}$$

Now, we are looking for an estimation of the ROA of the system (39). By using the proposed method, the estimated ROA is plotted in Figure 4. The ROA computed in [35] is shown in Figure 5. By comparing the Figures 4 and 5, it is observed that the estimation of ROA by the method developed in this study is considerably more efficient than the estimations carried out by the method in reference [35]. As shown in Example 2, it is evident that our method is appropriate for DOA estimation in systems of both polynomial and non-polynomial types.

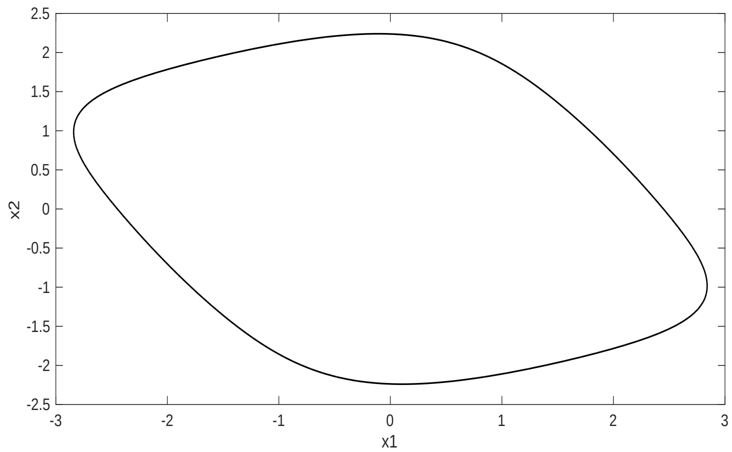

Figure 4: Estimated AR by using proposed method for Example 2.

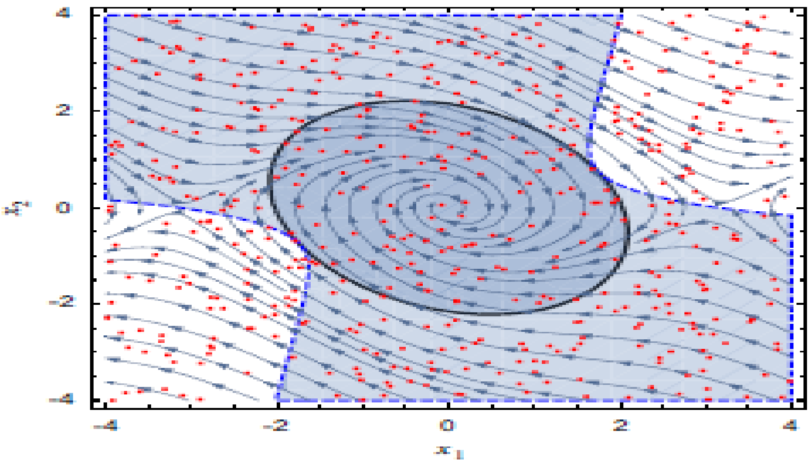

Figure 5: The black ellipsoid represents the RA estimated in [35] for Example 2. the dashed blue line (the boundary of the light blue area) demonstrates the $\dot{V}(x) < 0$, region; system trajectories are illustrated by the arrows; and the points in red indicate the randomly selected sampling states.

Example 3. Consider the following input-affine nonlinear system in [3]:

$$\begin{cases} \dot{x}_1 = -x_1 + 2x_1^2 x_2 + u, \\ \dot{x}_2 = -x_2. \end{cases} \tag{40}$$

It is further assumed that $Q = diag([1 \;\; 1]), R = 1$. First, we formulate the system in the pseudo-linear form below:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2x_1^2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \tag{41}$$

Then, we find the feedback control $u(t) = K(x(t))$ that asymptotically stabilizes the system, minimizes the cost function , and has the largest estimate of the ROA. By applying the SDRE method, the matrix P is obtained. The matrix-array is state-dependent: Then we find the feedback control $u(t) = K(x(t))$ that asymptotically stabilizes the system, the cost function minimum, and have the largest estimate of the AR. By applying the SDRE method, the matrix P is obtained. Matrix-array are state-dependent:

$$P = \begin{bmatrix} 0.414 & 0.343x_1^2 \\ 0.343x_1^2 & 0.627x_1^4 + 0.5 \end{bmatrix}, \tag{42}$$

$$u = -Kx = -R^{-1} B^T P x = -(0.414x_1 + 0.343x_1^2). \tag{43}$$

The control obtained by the SDRE method leads to the minimum cost function. By replacing (43) in the control system (41), we obtain:

$$\begin{cases} \dot{x}_1 = -1.414x_1 + 2x_1^2 x_2 - 0.343x_1^2, \\ \dot{x}_2 = -x_2. \end{cases} \tag{44}$$

Now, we are looking for an estimation of the ROA of the system (44). By solving the above problem, the estimated ROA is plotted in Figure 6.

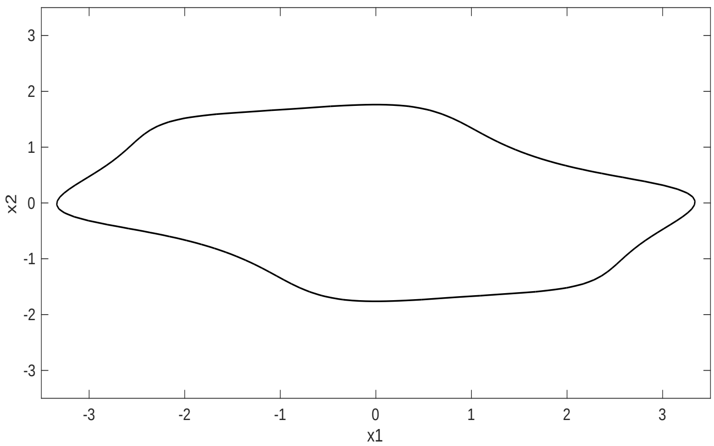

Figure 6: Estimated AR by using proposed method for Example 3.

The ROA computed in [3] is shown in Figure 7.

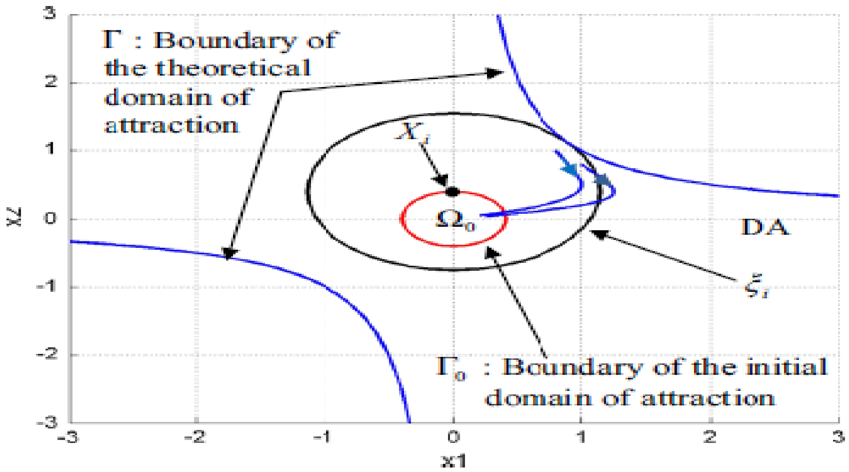

Figure 7: the RA estimated in [3] for Example 3 .

The value of the cost function converts to a constant number and equals zero (see Figures 8 and 9).

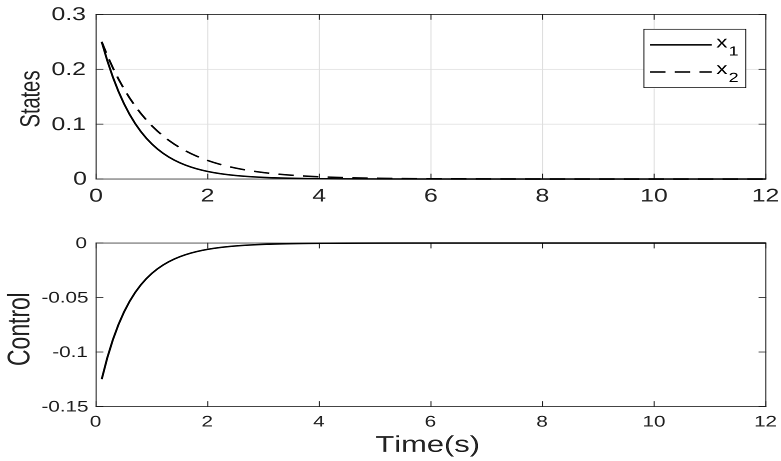

Figure 8: States trajectories and control signal by proposed method for Example 3.

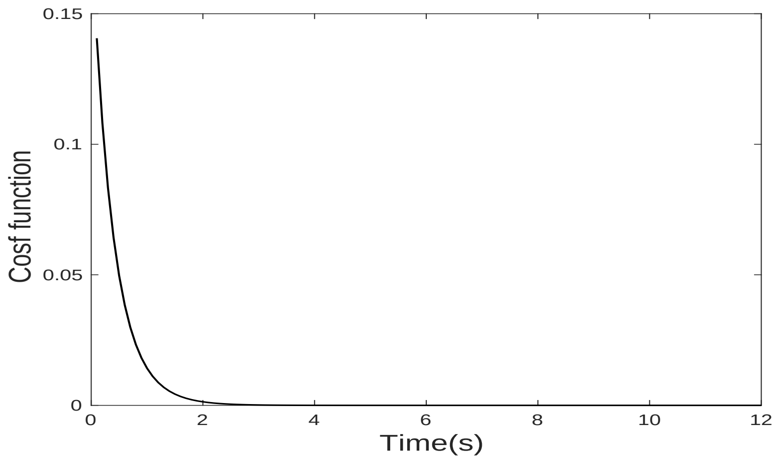

Figure 9: Cost function by proposed method for Example 3.

Figure 9 shows that the cost function of the presented methodology is nonlinear, which indicates the efficiency of the proposed method.

5 Conclusion

In the present research study, a new methodology was developed to estimate the ROA of nonlinear systems. Moreover, in order to enlarge the ROA, a combination of pseudo-linearization and SOS programming methods was employed. By designing a nonlinear state feedback controller through the SDRE, a shape factor was obtained that expanded the ROA. Since the choice of this shape factor was influenced by the dynamics of the system, we had a non-uniform extension of the ROA. The major privilege of the proposed approach was the incorporation of both the polynomial and non-polynomial dynamics. Moreover, the proposed method could deal with both the bound and unbound ROAs. Finally, to support the applicability as well as the superiority of the proposed method, numerical simulations were provided and the results were compared with other studies in the literature.

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توسیع ناحیه جذب سیستم‌های غیرخطی با استفاده از برنامه نویسی مجموع مربعات

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چکیده

در این پژوهش روشی جدید برای توسیع ناحیه جذب نقطه تعادل یک سیستم کنترل غیرخطی آفین ارائه شده است. برای توسیع ناحیه جذب سیستم‌های دینامیکی غیر چندجمله‌ای به کمک طراحی کنترل‌کننده فیدبک حالت غیرخطی از معادله ریکاتی وابسته به حالت استفاده شده است. فرایند بدست آوردن ناحیه جذب باعث ایجاد یک مسئله بهینه سازی مجموع مربعات با محدودیت‌های کنترلی و غیر کنترلی می‌شود. نکته قابل توجه این است که روش پیشنهادی برای تخمین ناحیه جذب سیستم‌های غیرخطی چندجمله‌ای و غیر چندجمله‌ای کارا است. همچنین در این مطالعه کاربرد روش پیشنهادی با شبیه سازی‌های عددی نشان داده شده است.

کلمات کلیدی

معادله ریکاتی وابسته به حالت، ناحیه جذب، فاکتور شکل، برنامه‌نویسی مجموع مربعات، تابع لیاپانوف.