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Weakly Perfect Graphs of Modules

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Abstract. In this study, R and M are assumed to be a commutative ring with non-zero identity M and an R-module, respectively. Scalar Product Graph of M, denoted by $G_R(M)$, is a graph with the vertex-set M and two different vertices a and b in M are connected if and only if there exists r belong to R such that $a = rb$ or $b = ra$. This paper studies some properties of such weakly perfect graphs.

Keywords. Scalar Product, Graph join, Weakly Perfect, Module.

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1 Introduction

Throughout this paper, let G be a simple graph with a set of vertices $V = V(G)$ and a set of edges $E = E(G)$. Let $u_1, u_j \in V(G)$. If u_i is adjacent to u_j, we have $u_i \sim u_j$. The k-coloring of G is an appropriate form of k colors to $V(G)$ such that no two adjacent vertices have the same color. The smallest number k with this property, denoted by $\chi(G)$, is called the chromatic number of G. A clique of G is a complete subgraph of G and the cardinality of the largest clique of G is called the clique number of G and denoted by $\omega(G)$. A graph G is called weakly perfect if $\chi(G) = \omega(G)$. Maimani and et al. [4] introduced a class of such graphs. Nikandish and et al. [5] presented a graph of ideals that are weakly perfect. The graph is perfect if every induced subgraph is weakly perfect. Hence, every perfect graph is weakly perfect and there are several classes that indicate that the converse may not hold in general. Fander [3] introduced a new class of perfect graphs.

Let $S \subseteq V(G)$. Herein, S is an independent set if the maximum degree of the subgraph induced by $V(G)$ is zero. Independent number, denoted by $\alpha(G)$, is the maximum cardinality of any independent set. It is trivial that vertex S is a clique of G if and only if it is an independent set of $\bar{G}$. Thus, $\alpha(G) = \omega(\bar{G})$.

A topological index is a numerical quantity that is invariant under automorphisms of the graph. The topological index based on the distance function was first used by H. Wiener [7]. If $u, v \in V(G)$ are two different vertices, then $d(u,v)$ is the length of the shortest path between u and v. Therefore, the Wiener index of G is:

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d(u,v) \tag{1}$$

Suppose that R is a commutative ring with identity and $W(R)$ is a set of non-unit elements of R. Afkhami et al. [1] defined the Cozero-divisor graph of R, denoted by $\Gamma'(R)$, with vertices $W(R)^* = W(R)\backslash\{0\}$ and $x, y \in W(R)^*(x \neq y)$; then, $x \sim y$ if and only if $x \notin Ry$ and $y \notin Rx$ where Rc is an ideal generated by $c \in R$. Suppose that M is an R-module and $W_R(M) = \{x \in M \mid Rm \neq M\}$. With R as R-module, $W_R(R)$ is a set of all non-unit elements of R. Alibemani et al. [2] introduced Cozero-divisor graphs in relation to R-module M in which vertices are $W_R(M)^* = W_R(M) \setminus \{0\}$ and $m, n \in W_R(M)^*(m \neq n)$ and then, $m \sim n$ if and only if $m \notin Rn$ and $n \notin Rm$. The mentioned authors studied the properties of this graph.

The next section introduces a new class of graphs arising from weakly perfect modules. Moreover, a formula is presented for $\chi(G)$ and $\omega(G)$ of such graphs. In Section 3, The Wiener index of such graphs is calculated.

2 Weakly Perfect Graphs of Modules

This section defines a scalar product graph of the module and shows that it is weakly perfect in some cases. The definition of the join of two graphs needs to be noted here. Suppose that X and Y are two separate graphs. $X+Y$ is join of X and Y with a set of vertices $V(X+Y) = V(X) \cup V(Y)$

and a set of edges $E(X+Y) = E(X) \cup E(Y) \cup \{xy : x \in V(X), y \in V(Y)\}$.
In addition, $\mid V(X+Y) \mid = \mid V(X) \mid + \mid V(Y) \mid$ and $\mid E(X+Y) \mid = \mid E(X) \mid + \mid E(Y) \mid + \mid V(X) \| V(Y) \mid$.
Also, for two graphs X and Y we have $\chi(X+Y) = \chi(X) + \chi(Y)$.

Lemma 1. Let G and H be separate graphs. Then $\alpha(G+H) = max\{\alpha(G), \alpha(H)\}$ and $\omega(G+H) = \omega(G) + \omega(H)$.

Proof. Suppose that $max\{\alpha(G), \alpha(H)\} = \alpha(G)$ and $S = \{u_1, u_2, \ldots, u_{\alpha(G)}\}$ is the maximum independent number of G. For any (u_i, u_j), $1 \leq i, j \leq \alpha(G)$, $i \neq j$, and the edge $u_i u_j$ is not in $E(G)$; thus, $u_i u_j \notin E(G+H)$. It is implied that S is an independent set of $G+H$. Indeed, $\alpha(G+H) \geq max\{\alpha(G), \alpha(H)\}$. Now, for the converse, suppose that S' is the maximum independent number and the sum of graphs G and H. Then, S' is not the subset of $V(G)$ and $V(H)$ contemporary. Suppose that $S' \subset V(G)$ and therefore, $\alpha(G+H) \leq \alpha(G)$ and $\alpha(G+H) \leq \alpha(H)$. Hence, $\alpha(G+H) \leq max\{\alpha(G), \alpha(H)\}$. Suppose that C is an arbitrary clique of $G+H$. It can be assumed that $C = C_1 \cup C_2$ in which $C_1 \subseteq V(G)$ and $C_2 \subseteq V(H)$. It is quite trivial that $\mid C1 \mid \leq \omega(G)$ and $\mid C2 \mid \leq \omega(H)$. Therefore, $\omega(G+H) \leq \omega(G) + \omega(H)$. Thus, We have $\omega(G+H) \geq \omega(G) + \omega(H)$. □

Definition 1. [6] Suppose that R is a commutative ring with non-zero identity and M be an R-module. We define the Scalar-product graph of R-module M, namely $G_R(M)$, in which the vertices of $G_R(M)$ are elements of M and $x, y \in M (x \neq y)$ then, $x \sim y$ is adjacent if and only if there exists r belonging to R such that $x = ry$ or $y = rx$.

Remark 1. Let $G_R(M)$ be a Scalar-product graph of R-module M. If $x, y \in M$ then x is adjacent to y if and only if $Rx \subseteq Ry$ or $Ry \subseteq Rx$.

Remark 2. According to the definition of the cozero-divisor graph over modules, we have the followings:
(1) If M is an R-module, the subgraph of $G_R(M)$ in which vertices are $W_R(M)^*$ is the complement of the cozero-divisors graph of M.
(2) We have $G_R(M) = G_1 + G_2$ where G_1 is a complete graph with $\mid W_R(M)^* \mid$ vertices and G_2 is the complement of the the cozero-divisor graph of M.

In the following, if $G_R(M)$ is the scalar product graph of some R- module M, we compute $\chi(G_R(M))$ and $\omega(G_R(M))$.

Lemma 2. Suppose that M is an R-module. Then, the scalar product graph $G_R(M)$ is complete if and only if the cyclic submodules of M are linearly ordered by inclusion relation.

Proof. Let M be an R-module and $N_1 =< a >, N_2 =< b >$ be two cyclic submodules of M in which $a \neq b$ in M. Since the scalar product graph $G_R(M)$ is complete, a and b are adjacent. We have $< a > \subseteq < b >$ or $< b > \subseteq < a >$ and $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. Conversely, Let M be an R-module in which the cyclic submodules are linearly ordered by inclusion relation. If $a \neq b$ represents two vertices of $G_R(M)$ then $< a > \subseteq < b >$ or $< b > \subseteq < a >$. Therefore, a and b are adjacent in $G_R(M)$. Hence, $G_R(M)$ is complete. □

Suppose that M is R-module and A, B are two non-zero submodules of M.Then, M is called uniserial if $A \subseteq B$ or $B \subseteq A$. Clearly, the valuation ring R is uniserial as a module over itself. Also, submodules and quotients of uniserial modules are again uniserial.

Lemma 3. Let $\mathbb{Z}_n$ be a $\mathbb{Z}$-module. If p, m are prime and positive integer numbers, then for $n = 1, p, p^m$, the scalar product graph $G_{\mathbb{Z}}(\mathbb{Z}_n)$ will be complete.

Proof. Let M be a simple module. Then, submodules of M are linearly ordered by inclusion. Hence, submodules of $\mathbb{Z}_p$ are uniserial. Through Lemma 2.5, the scalar product graph of $\mathbb{Z}_p$ is complete.
Also, $\mathbb{Z}_{p^n} = \frac{1\mathbb{Z}}{p^n\mathbb{Z}} \supset \frac{p\mathbb{Z}}{p^n\mathbb{Z}} \supset \frac{p^2\mathbb{Z}}{p^n\mathbb{Z}} \supset \ldots \supset \frac{p^{n-1}\mathbb{Z}}{p^n\mathbb{Z}} \supset \frac{p^n\mathbb{Z}}{p^n\mathbb{Z}} = 0$, here $\mathbb{Z}_{p^n}$ is uniserial. Therefore its scalar product graph is complete. $\square$

Theorem 1. Suppose that p is a prime number. Then, the edge number of $G_{\mathbb{Z}}(\mathbb{Z}_{2p})$ is $2p^2 - 2p + 1$.

Proof. In Remark 2.5, we have $G_{\mathbb{Z}}(\mathbb{Z}_{2p}) = K_p + G_2$ such that K_p is a complete graph with p vertices and G_2 is the complement of the cozero-divisor graph of $\mathbb{Z}_{2p}$ which is $\overline{K_{1,p-1}}$. By definition 2.1, we have:
$| E(G_{\mathbb{Z}}(\mathbb{Z}_{2p})) |= \frac{p(p-1)}{2} + \frac{(p-1)(p-2)}{2} + p^2 = 2p^2 - 2p + 1$ $\square$

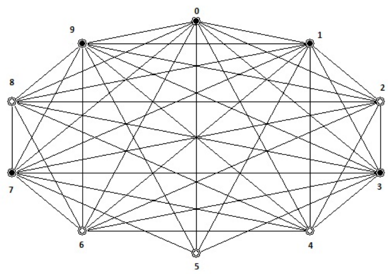

Figure 1: Scalar Product of $\mathbb{Z}$-module $\mathbb{Z}_{10}$.

Theorem 2. Let $\mathbb{Z}_n$ be a $\mathbb{Z}$-module. If $n = 1, p, p^m$ and $n = 2p$, then the graph $G_{\mathbb{Z}}(\mathbb{Z}_n)$ is weakly perfect. Also, if $n = 2p$, we have $\chi(G_{\mathbb{Z}}(\mathbb{Z}_n)) = \omega(G_{\mathbb{Z}}(\mathbb{Z}_n)) = 2p - 1$.

Proof. By Lemma 2.7, $G_{\mathbb{Z}}(\mathbb{Z}_n)$ is a complete graph with n vertices. Hence, It is weakly perfect. If $n = 2p$, then by Remark 2.5, we have $G_{\mathbb{Z}}(\mathbb{Z}_{2p}) = K_p + G_2$ such that K_p is a complete graph with p vertices and G_2 is the complement of cozero-divisor graph of $\mathbb{Z}_{2p}$ which is $\overline{K_{1,p-1}}$. Also, $\chi(K_p) = \omega(K_p) = p$ and $\chi(G_2) = \omega(G_2) = p - 1$. Therefore, by Lemma 2.2, we have $\chi(G_{\mathbb{Z}}(\mathbb{Z}_{2p})) = \omega(G_{\mathbb{Z}}(\mathbb{Z}_{2p})) = 2p - 1$. $\square$

Table 1 show clique, chromatic and edge number of the scalar-product graph of $\mathbb{Z}_{2p}$:

Theorem 3. Suppose that p is a prime number. Then, the edge number of $G_{\mathbb{Z}}(\mathbb{Z}_{3p})$ is $\frac{9}{2}p^2 - \frac{7}{2}p + 2$.

Proof. By Remark 2.5, we have $G_{\mathbb{Z}}(\mathbb{Z}_{3p}) = K_{2p-1} + G_3$ such that K_{2p-1} is a complete graph with p vertices and G_3 is the complement of cozero-divisor graph of $\mathbb{Z}_{3p}$ which is $\overline{K_{2,p-1}}$. By Definition 2.1, we have:
$| E(G_{\mathbb{Z}}(\mathbb{Z}_{3p})) |= \frac{(2p-1)(2p-2)}{2} + 1 + \frac{(p-1)(p-2)}{2} + (2p-1).(p+1) = \frac{9}{2}p^2 - \frac{7}{2}p + 2$ $\square$

Table 1: clique, chromatic and edge number of $G_{\mathbb{Z}}(\mathbb{Z}_{2p})$

p	$\chi(G)$	$\omega(G)$	$\mid E(G)\mid$
3	5	5	13
5	9	9	41
7	13	13	85
11	21	21	145

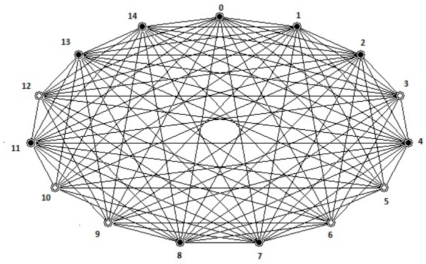

Figure 2: Scalar Product of $\mathbb{Z}$-module $\mathbb{Z}_{15}$.

Theorem 4. Let $\mathbb{Z}_n$ be a $\mathbb{Z}$-module. If $n = 3p$, then the graph $G_{\mathbb{Z}}(\mathbb{Z}_n)$ is weakly perfect. Also $\chi(G_{\mathbb{Z}}(\mathbb{Z}_n)) = \omega(G_{\mathbb{Z}}(\mathbb{Z}_n)) = 3p-2$.

Proof. If $n = 3p$, then by Remark 2.5, we have $G_{\mathbb{Z}}(\mathbb{Z}_{3p}) = K_{2p-1} + G_3$ where K_{2p-1} is a complete graph with $2p-1$ vertices and G_3 is the complement of the cozero-divisor graph of $\mathbb{Z}_{3p}$ which is $\overline{K_{2,p-1}}$. Also, $\chi(K_{2p-1}) = \omega(K_{2p-1}) = 2p-1$ and $\chi(G_3) = \omega(G_3) = p-1$. Therefore, by Lemma 2.2, we have $\chi(G_{\mathbb{Z}}(\mathbb{Z}_{3p})) = \omega(G_{\mathbb{Z}}(\mathbb{Z}_{3p})) = 3p-2$. □

Table 2 shows the clique, chromatic and edge number of the scalar-product graph of $\mathbb{Z}_{3p}$:

Table 2: clique, chromatic, and edge number of $G_{\mathbb{Z}}(\mathbb{Z}_{3p})$

p	$\chi(G)$	$\omega(G)$	$\mid E(G)\mid$
5	13	13	97
7	19	19	198
11	31	31	508
13	37	37	717

Theorem 5. Suppose that p is a prime number. Then, the edge number of $G_{\mathbb{Z}}(\mathbb{Z}_{5p})$ is $\frac{25}{2}p^2 - \frac{13}{2}p + 4$.

Proof. By Remark 2.5, we have $G_{\mathbb{Z}}(\mathbb{Z}_{5p}) = K_{4p-3} + G_5$ such that K_{4p-3} is a complete graph with $4p-3$ vertices and G_3 is the complement of the cozero-divisor graph of $\mathbb{Z}_{5p}$ which is $\overline{K_{4,p-1}}$. By Definition 2.1, we have:
$\mid E(G_{\mathbb{Z}}(\mathbb{Z}_{5p}))\mid = \frac{(4p-3)(4p-4)}{2} + 6 + \frac{(p-1)(p-2)}{2} + (4p-3).(p+3) = \frac{25}{2}p^2 - \frac{13}{2}p + 4.$ □

Theorem 6. Let $\mathbb{Z}_n$ be a $\mathbb{Z}$-module. If $n = 5p$, then the graph $G_{\mathbb{Z}}(\mathbb{Z}_n)$ is weakly perfect. Also, $\chi(G_{\mathbb{Z}}(\mathbb{Z}_n)) = \omega(G_{\mathbb{Z}}(\mathbb{Z}_n)) = 5p - 4$.

Proof. If $n = 5p$, then by Remark 2.5, we have $G_{\mathbb{Z}}(\mathbb{Z}_{5p}) = K_{4p-3} + G_5$ such that K_{4p-3} is a complete graph with $4p-3$ vertices and G_5 is the complement of the cozero-divisor graph of $\mathbb{Z}_{5p}$ which is $\overline{K_{4,p-1}}$. Also, $\chi(K_{4p-3}) = \omega(K_{4p-3}) = 4p-3$ and $\chi(G_5) = \omega(G_5) = p-1$. Therefore, by Lemma 2.2, we have $\chi(G_{\mathbb{Z}}(\mathbb{Z}_{5p})) = \omega(G_{\mathbb{Z}}(\mathbb{Z}_{5p})) = 5p-4$. □

3 Wiener Index of $G_R(M)$

Suppose that G is a graph. The Wiener index of G is half of the sum of the distance between two distinct vertices. For example, we have $W(K_n) = \frac{1}{2}n(n-1)$ and $W(K_{1,n-1}) = (n-1)^2$.
This section computes Wiener indices of $G_{\mathbb{Z}}(\mathbb{Z}_{2p})$ and $G_{\mathbb{Z}}(\mathbb{Z}_{3p})$ for some prime p. Similar to what we had before, the Scalar product graphs of $\mathbb{Z}$-module $\mathbb{Z}_{2p}$ and $\mathbb{Z}_{3p}$ are the join of complete graph and complement of a cozero-divisor graph. Therefore, we seek a formula for the Wiener index of the join of two graphs.

Theorem 7. [8] For any two graphs X_1 and X_2, we have:

$$\begin{aligned} W(X_1+X_2) = &\mid V(X_1)\mid^2 - \mid V(X_1)\mid + \mid V(X_2)\mid^2 - \mid V(X_2)\mid \\ &+ \mid V(X_1)\mid\mid V(X_2)\mid - \mid E(X_1)\mid - \mid E(X_2)\mid. \end{aligned}$$

Now, we have the following propositions.

Proposition 1. Suppose that p is a prime number. Then, we have $W(G_{\mathbb{Z}}(\mathbb{Z}_{2p})) = 2p^2 - 1$.

Proof. By Proof 1, the scalar product graph of $\mathbb{Z}_{2p}$ is the join of K_p and $\overline{K_{1,p-1}}$. Thus, from Theorem 7, we have

$$\begin{aligned} W(K_p + \overline{K_{1,p-1}}) = &\mid V(K_p)\mid^2 - \mid V(K_p)\mid + \mid V(\overline{K_{1,p-1}})\mid^2 - \mid V(\overline{K_{1,p-1}})\mid \\ &+ \mid V(K_p)\mid\mid V(\overline{K_{1,p-1}})\mid - \mid E(K_p)\mid - \mid E(\overline{K_{1,p-1}})\mid \\ =&p^2 - p + p^2 - p + p^2 - \frac{1}{2}p(p-1) - \frac{1}{2}(p-1)(p-2) \\ =&2p^2 - 1. \end{aligned}$$

□

Proposition 2. Suppose that p is a prime number. Then, we have $W(G_{\mathbb{Z}}(\mathbb{Z}_{3p})) = \frac{9}{2}p^2 + \frac{1}{2}p - 2$.

Proof. By Proof 3, the scalar product graph of $\mathbb{Z}_{3p}$ is the join of K_{2p-1} and $\overline{K_{2,p-1}}$. Thus, according to Theorem 7, we have:

$$\begin{aligned} W(K_{2p-1} + \overline{K_{2,p-1}}) = &\mid V(K_{2p-1})\mid^2 - \mid V(K_{2p-1})\mid + \mid V(\overline{K_{2,p-1}})\mid^2 - \mid V(\overline{K_{2,p-1}})\mid \\ &+ \mid V(K_{2p-1})\mid\mid V(\overline{K_{2,p-1}})\mid - \mid E(K_{2p-1})\mid - \mid E(\overline{K_{2,p-1}})\mid \\ =&(2p-1)^2 - (2p-1) + (p+1)^2 - (p+1) \end{aligned}$$

$$
\begin{aligned}
&+(2p-1)(p+1)-\frac{1}{2}(2p-1)(2p-2)-[1+\frac{1}{2}(p-1)(p-2)]\\
=&\frac{9}{2}p^2+\frac{1}{2}p-2.
\end{aligned}
$$

□

Proposition 3. Suppose that p is a prime number. Then, we can have $W(G_{\mathbb{Z}}(\mathbb{Z}_{5p}))=\frac{25}{2}p^2+\frac{3}{2}p-4$.

Proof. By Proof 5, the scalar product graph of $\mathbb{Z}_{5p}$ is the join of K_{4p-3} and $\overline{K_{4,p-1}}$. Thus, from Theorem 7, we have:

$$
\begin{aligned}
W(K_{4p-3}+\overline{K_{4,p-1}}) =& \mid V(K_{4p-3})\mid^2-\mid V(K_{4p-3})\mid+\mid V(\overline{K_{4,p-1}})\mid^2-\mid V(\overline{K_{4,p-1}})\mid\\
&+\mid V(K_{4p-3})\mid\mid V(\overline{K_{4,p-1}})\mid-\mid E(K_{4p-3})\mid-\mid E(\overline{K_{4,p-1}})\mid\\
=&(4p-3)^2-(4p-3)+(p+3)^2-(p+3)\\
&+(4p-3)(p+3)-\frac{1}{2}(4p-3)(4p-4)-[6+\frac{1}{2}(p-1)(p-2)]\\
=&\frac{25}{2}p^2+\frac{3}{2}p-4.
\end{aligned}
$$

□

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گراف‌های به طور ضعیف تام روی مدول‌ها

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چکیده

در این مطالعه، R و M به ترتیب یک حلقه‌ی جابجایی با همانی غیر صفر و یک R-مدول است. گراف ضرب اسکاری روی M، را که با $G_R(M)$ نشان می‌دهیم، گرافی با مجموعه رئوس M است و دو راس متمایز a و b در M مجاورند اگر و تنها اگر r متعلق به R وجود داشته باشد به طوری که $a = rb$ یا $b = ra$. این مقاله برخی از خواص این گراف‌های به طور ضعیف تام را مطالعه می‌کند.

کلمات کلیدی

ضرب اسکالری، گراف متصل، به طور ضعیف تام، مدول.