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Applying Duality Results to Solve the Linear Programming Problems with Grey Parameters

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Abstract. Linear programming problems have exact parameters. In most real-world, we are dealing with situations in which accurate data and complete information are not available. Uncertainty approaches such as fuzzy and random can be used to deal with uncertainties in real-life. Fuzzy and stochastic theories cannot be used if the number of experts and the level of experience is so low that it is impossible to extract membership functions or the number of samples is small. To solve these problems, the grey system theory is proposed. In this paper, a linear programming problem in a grey environment with resources in interval grey numbers is considered. Most of the proposed methods for solving grey linear programming problems become common linear programming problems. However, we seek to solve the problem directly without turning it into a standard linear programming problem for the purpose of maintaining uncertainty in the original problem data in the final solution. For this purpose, we present a method based on the duality theory for solving the grey linear programming problems. This method is more straightforward and less complicated than previous methods. We emphasize that the concept presented is beneficial for real and practical conditions in management and planning problems. Therefore, we shall illustrate our method with some examples in different situations.

Keywords. Grey number, Grey linear programming, Duality theory, Uncertainty.

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1 Introduction

Managers need to use optimization methods to make the right decisions and solve their problems. Operational research is one of the optimization methods that help make the right decisions and solve management problems. One of the standard methods to optimization a goal due to various constraints is Linear Programming (LP) problem [1]. To represent an optimization problem as a linear programming problem, assumptions such as proportionality, additivity, divisibility, and deterministicity of all model parameters are required in the problem formulation [1]. In real cases, these assumptions rarely apply. The LP model is used to select some future activities. As a result, it will inevitably include some degree of uncertainty.

Lack of information, inaccuracy in information, and inaccuracy of forecasting are the characteristics of inaccurate systems. During the decision-making process, the presence of uncertainty in the data and the problem's situation usually confronts the decision-maker with conditions of doubt and uncertainty and makes it difficult to decide and choose the best option [4]. Due to differences in the type and characteristics of uncertainty systems, different theories, methods, and techniques have been used, including statistics and probability, fuzzy set theory, and grey systems theory [16]. Fuzzy set theory is based on the definition of fuzzy numbers, and fuzzy numbers, in turn, depend on the definition of membership functions, which is based on the number of experts and their level of experience. Statistics and probability are also the uncertainty that arises from a completely random process and requires high distribution and sampling functions. In other words, if the decision-maker fails to reduce the uncertainty by obtaining more data, we will face uncertainty from a completely random process.

If the number of experts and level of experience is low and membership functions cannot be extracted, or the number of samples is small i.e., we cannot use fuzzy uncertainty theory or random uncertainty theory; we use grey system theory. This theory, proposed by Deng, provides a very effective way to deal with uncertainty, providing desirable outcomes using low and volatile information [10]. Grey systems theory studies topics that have a definite range and scope and an uncertain nature. With the development of grey systems theory, today, this theory has become a new branch of science whose theoretical structure includes systems analysis, modeling, forecasting, decision making, control, and optimization techniques. Due to this theory's advantages over other methods of dealing with the system of uncertainty, its application is expanding in recent years [2, 5, 34, 35, 38]. LP problems with interval grey numbers have been studied by several authors [3, 9, 14, 15, 19, 23, 27, 36]. For example, Nasseri et al. [24] have proposed a Simplex algorithm-based method for solving Grey Linear Programming (GLP) problem (grey parameter in the objective function) by using grey arithmetic concepts and grey number ranking. Their proposed method has the advantage over previous methods in that it is no longer necessary to whiten GLP problem parameters. Since the problem is solved directly, the input data's uncertainty will be better reflected in the final solution.

The duality theory for inexact LP problems was studied by Soyster [30] and Thuente [32]. Rohn [29] discussed the duality in an interval LP problem with a real-valued objective function. Duality theory was developed by Rodder and Zimmermann [28] for solving fuzzy parameter LP problems using the aspiration level approach. Several researchers solved fuzzy programming problems using fuzzy duality theory [12, 18, 31]. Ramik [26] introduced some new concepts, and results, possibilities, and necessary relations of duality in fuzzy linear programming. Nasseri and Darvishi [23] gave duality theory for solving the GLP problems. Darvishi [7] has studied some of the duality results in the GLP problem. One of the weaknesses of LP with grey parameters problems is solving the problem without converting the parameters to display the uncertainty of the input data in the output solution. To solve this problem, in this research, we tried to use a dual LP problem with grey data, to present a method without whitening grey parameters to solve the LP problem with grey data.

This research is formatted as follows. Section 2, presents some necessary notations and definitions of grey systems. The definition of the GLP problem is given in Section 3. In Section 4, a dual LP problem with grey parameters and a new algorithm based on dual results to solve it are introduced. In Section 5, by presenting three different examples, the efficiency and numerical analysis of the proposed method is presented. Finally, Section 6 consists of conclusions.

2 Preliminaries

In this section, we describe the definitions and concepts needed to study and analyze the mathematical of grey systems and grey number calculations to solve the GLP problem [10, 13, 17]. Grey systems theory is one of the most critical methods for studying and analyzing systems with incorrect parameters and incomplete information [16]. One of the main concepts of grey systems theory that plays a significant role in studying uncertainty is grey numbers. There are different kinds of grey numbers [16] which we use interval grey numbers in this article.

Definition 1. An interval grey number is the one whose exact value is unknown but whose range is known [6].

$$\otimes x \in [\underline{x}, \overline{x}] = \{t \mid \underline{x} \le t \le \overline{x}\},\ \underline{x} \le \overline{x}, \tag{1}$$

where, t is grey number information, $\underline{x}$ lower limit, and $\overline{x}$ upper limit.

Remark 1. [6] We show the set of grey numbers with $R(\otimes)$ the symbol.

Definition 2. [21] Let $\otimes x_1 \in [\underline{x}_1, \overline{x}_1]$ and $\otimes x_2 \in [\underline{x}_2, \overline{x}_2]$ be two grey numbers. The following operations can be defined:

$$\otimes x_1 + \otimes x_2 = [\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2],$$
$$\otimes x_1 - \otimes x_2 = \otimes x_1 + (-\otimes x_2) = [\underline{x}_1 - \overline{x}_2, \overline{x}_1 - \underline{x}_2],$$
$$\otimes x_1 \times \otimes x_2 = \left[\min\{\underline{x}_1\underline{x}_2, \overline{x}_1\overline{x}_2, \overline{x}_1\underline{x}_2, \underline{x}_1\overline{x}_2\}, \max\{\underline{x}_1\underline{x}_2, \overline{x}_1\overline{x}_2, \overline{x}_1\underline{x}_2, \underline{x}_1\overline{x}_2\}\right],$$
$$k \cdot \otimes x \in \begin{cases} [k\underline{x}, k\overline{x}], & k > 0, \\ [k\overline{x}, k\underline{x}], & k < 0. \end{cases}$$

Remark 2. [13] Let $\otimes x \in [\underline{x}, \overline{x}]$ be a grey number. Then we have $\otimes x \div \otimes x = 1$, and $\otimes x - \otimes x = 0$.

Remark 3. [13] For any real number x, we have $\otimes x \in [x, x]$.

Definition 3. [27] For any grey number $\otimes x \in [\underline{x}, \overline{x}]$, the kernel, $\otimes\widehat{x}$ of the grey number is defined as $\otimes\widehat{x} = \dfrac{\underline{x}+\overline{x}}{2}$.

Definition 4. [37] The length of the grey number $\otimes x \in [\underline{x}, \overline{x}]$ is defined as $\mu(\otimes x) = \overline{x} - \underline{x}$.

Ranking of grey numbers is very important in grey decision making and optimization problems. Several methods and more details can be seen in Dervishi et al.[8].

Definition 5. [37] Suppose that the background, which makes a grey number $\otimes x$ come into being, is Ω and $\mu(\Omega)$ is the value of Ω. Then $g^{\circ}(\otimes x) = \dfrac{\mu(\otimes x)}{\mu(\Omega)}$ is called the degree of greyness of $\otimes x$ (denoted as g° for short).

Definition 6. [11] Suppose $\otimes x_1$ and $\otimes x_2$ are two grey numbers and $\otimes\widehat{x_1}$, $\otimes\widehat{x_2}$ are the center of $\otimes x_1$ and $\otimes x_2$ respectively, $g^{\circ}(\otimes x_1)$ and $g^{\circ}(\otimes x_2)$ are the degree of greyness of $\otimes x_1$ and $\otimes x_2$, respectively. So, if $\otimes\widehat{x_1} < \otimes\widehat{x_2} \Rightarrow \otimes x_1 <_G \otimes x_2$

$$\text{if} \qquad \otimes\widehat{x_1} = \otimes\widehat{x_2} \Rightarrow \begin{cases} \text{if} & g^{\circ}(\otimes x_1) = g^{\circ}(\otimes x_2) \Rightarrow \otimes x_1 =_G \otimes x_2, \\ \text{if} & g^{\circ}(\otimes x_1) < g^{\circ}(\otimes x_2) \Rightarrow \otimes x_1 >_G \otimes x_2, \\ \text{if} & g^{\circ}(\otimes x_1) > g^{\circ}(\otimes x_2) \Rightarrow \otimes x_1 <_G \otimes x_2. \end{cases} \tag{2}$$

For further study on the grey theory systems see [13].

3 Grey Linear Programming

The GLP problem is one of the appropriate approaches to deal with uncertainty in real-life problems. Here, we present the general model of LP problems, including grey numbers

$$\begin{aligned} \text{Maximize} \otimes \mathrm{Z} =_{\mathrm{G}} & \sum_{j=1}^{n} \otimes c_j \otimes x_j \\ \textbf{s.t} \qquad & \sum_{j=1}^{n} \otimes \otimes a_{ij} x_j \leq_G b_i, \\ & \otimes x_j \geq_G \otimes 0, \end{aligned} \tag{3}$$

where $\otimes c_j, \otimes a_{ij}, \otimes x_j, \otimes b_i \in R(\otimes), i = 1, 2, ..., m,\ j = 1, 2, ..., n.$

Definition 7. Linear programming problem with grey right-hand sides' parameters is defined as follows:

$$\begin{aligned}
&\text{Minimize} \otimes \mathrm{Z} =_{\mathrm{G}} \sum_{j=1}^{n} \otimes c_j \otimes x_j \\
&\textbf{s.t} \qquad \sum_{j=1}^{n} \otimes \otimes a_{ij} x_j \geq_G b_i, \\
&\qquad\quad \otimes x_j \geq_G \otimes 0,
\end{aligned} \tag{4}$$

where $\otimes c_j, \otimes a_{ij} \in R, \otimes x_j, \otimes b_i \in R(\otimes), i = 1, 2, ..., m,\ j = 1, 2, ..., n.$

Because in the process, we need the notion of a feasible grey solution and grey optimal solution, we consider the following definitions.

Definition 8. The set $\{\otimes x_j,\ j = 1, 2, ..., n\}$ is called to be a feasible solution (4), if they satisfy into the model constraints.

Definition 9. A feasible solution $\{\otimes x_j,\ j = 1, 2, ..., n\}$ of the problem (4), is said to be an optimal feasible solution, if $\sum_{j=1}^{n} \otimes c_j \otimes x_0 \leq \sum_{j=1}^{n} \otimes c_j \otimes x_j$, $\otimes x_j \in D$, $j = 1, 2, ..., n$.

Definition 10. (Grey basic feasible solution): Consider the system $\sum_{j=1}^{n} a_{ij} \otimes x_j =_G \otimes b_i\ i = 1, 2, ..., m$ and $\otimes x_j \geq_G 0$, where $[a_{ij}]_{m\times n}$ is a $m \times n$ matrix and $[\otimes b_i] i = 1, 2, ..., m$ is an m vector. Suppose that $rank[a_{ij}, \otimes b_i]_{m\times n+1} = rank[a_{ij}]_{m\times n} = m$ partition, $A = [a_{ij}]_{m\times n}$, $j = 1, 2, ..., n$, $i = 1, 2, ..., m$, after possibly rearranging the columns of A, as $[B, N]$, where B is $m \times m$ non-singular matrix. It is apparent that

$$\otimes x_B =_G (\otimes x_{B_1}, \otimes x_{B_2}, ..., \otimes x_{B_m})^T =_G B^{-1} \otimes b, \quad \otimes x_N =_G \otimes 0,$$

is a solution of $\sum_{j=1}^{n} a_{ij} \otimes x_j =_G \otimes b_i$, $i = 1, 2, ..., m$ the vector $\otimes x =_G (\otimes x_B^T, \otimes x_N^T)^T$ where $\otimes x_N =_G \otimes 0$ is called a basic grey solution of the system.

If $\otimes x_B \geq_G \otimes 0$, then $\otimes x$ is called a grey basic feasible solution of the system, and the corresponding grey objective value is $\otimes z =_G c_B \otimes x_B$, where $c_B = \left(c_{B_1}, c_{B_2}, ..., c_{B_m}\right)$.

For all $j = 1, 2, ..., n$, define $y_j = B^{-1} a_j$, $z_j = c_B y_j = c_B B^{-1} a_j$ and for any primary index $j = B_i$, $j = 1, 2, ..., m$, we have $z_j - c_j = c_B B^{-1} a_j - c_j = 0$.

Theorem 1. Let problem (4) be non-degenerate. A basic feasible solution, $\otimes x_B =_G (\otimes x_{B_1}, \otimes x_{B_2}, ..., \otimes x_{B_m})^T =_G B^{-1} \otimes b$, $\otimes x_N =_G \otimes 0$ is optimal to (4) if and only if, $z_j - c_j = c_B B^{-1} a_j - c_j \leq 0$, for any non-basic variable.

Proof. Let that $\otimes x =_G (\otimes x_B^T, \otimes x_N^T)^T$, $\otimes x_B =_G (\otimes x_{B_1}, \otimes x_{B_2}, ..., \otimes x_{B_m})^T =_G B^{-1} \otimes b$, $\otimes x_N =_G \otimes 0$ is a solution of $\sum_{j=1}^{n} a_{ij} \otimes x_j =_G \otimes b_i$, $i = 1, 2, ..., m$. So that, the optimal grey value of the objective function is $\otimes Z =_G c_B \otimes x_B =_G c_B B^{-1} \otimes b$. On the other hand, for any grey basic feasible solution $\otimes x$ to (4), we have $\otimes b =_G A \otimes x =_G B \otimes x_B + N \otimes x_N$. Hence, $B \otimes x_B =_G \otimes b - N \otimes x_N \Rightarrow \otimes x_B =_G B^{-1} \otimes b - B^{-1} N \otimes x_N$.

Thus, for any basic grey feasible solution to (4), we have

$$\otimes Z =_G c \otimes x =_G c_B \otimes x_B + c_N \otimes x_N,$$

$$
\begin{aligned}
&=_G c_B B^{-1} \otimes b - c_B B^{-1} N \otimes x_N + c_N \otimes x_N, \\
&=_G c_B B^{-1} \otimes b - \left(c_B B^{-1} N - c_N\right) \otimes x_N, \\
&=_G c_B B^{-1} \otimes b - \sum_{j \neq B_i} \left(c_B B^{-1} a_j - c_j\right) \otimes x_j, \\
&=_G c_B B^{-1} \otimes b - \sum_{j \neq B_i} \left(z_j - c_j\right) \otimes x_j.
\end{aligned}
$$

Now, consider the following three cases:
1) If for all $j = 1, 2, ..., n$, we have:

$$\left(z_j - c_j\right) \geq 0 \Rightarrow \left(z_j - c_j\right) \otimes x \geq_G \otimes 0 \Rightarrow \sum_{j \neq B_i} \left(z_j - c_j\right) \otimes x_j \geq_G \otimes 0, \text{then} \otimes Z \leq_G \otimes Z^*.$$

This is a contradiction to $\otimes Z^*$ being optimal.
2) If for all $j = B_i, 1 \leq i \leq m$, we have $\left(z_j - c_j\right) = 0$, then $\otimes Z^* =_G \otimes Z$.
3) If for all $j = 1, 2, ..., n$, we have:

$$\left(z_j - c_j\right) \leq 0 \Rightarrow \left(z_j - c_j\right) \otimes x \leq_G \otimes 0 \Rightarrow \sum_{j \neq B_i} \left(z_j - c_j\right) \otimes x_j \leq_G \otimes 0, \text{then} \otimes Z \geq_G \otimes Z^*.$$

Hence, we have $z_j - c_j = c_B B^{-1} a_j - c_j \leq 0, j = 1, 2, ..., n$. □

4 Duality in Linear Programming Problem with Grey Parameters

Duality is an essential concept in linear algebra and mathematical programming that derives from examining a problem from two different perspectives. Specifically, a device is interpreted from linear relationships defined in terms of a matrix in terms of column space or its row space. These two different perspectives lead us to some real results, such as the equation of the row and column rank of a matrix, the equation of optimal values of the initial problem, and the dual problem of LP [20]. The duality concept is one of the most essential and exciting concepts in linear programming. The basic idea in this theory is that every LP problem has a corresponding dual problem. So, whenever an LP problem is given, by solving it in the simplex method, we have two problems solution synchronous. For each LP problem with grey parameters, there is a corresponding problem named dual, which satisfies the duality results and the crisp and fuzzy environments [23]. Therefore, we try to find a solution to the LP problems with grey parameters using the duality concept. If we determine the dual of the problem (4), then we can present the mathematical model of the LP problem by grey right-hand sides as follows:

$$
\begin{aligned}
&\text{Minimize } \otimes \mathrm{Z} =_{\mathrm{G}} \sum_{j=1}^{m} \otimes y_i \otimes b_i \\
&\textbf{s.t} \qquad \sum_{j=1}^{m} \otimes \otimes y_i a_{ij} \geq_G c_j, \; j = 1, 2, ..., n, \qquad (5)
\end{aligned}
$$

$$\otimes y_i \geq_G \otimes 0 \;\; i = 1,2,...,m,$$

where $c_j, a_{ij} \in R,\ y_i, b_i \in R(\otimes), i = 1,2,...,m,\ \ j = 1,2,...,n.$

Theorem 2. [23] (The weak duality property) If $\otimes x^0 = \left(\otimes x_1^0, \otimes x_2^0, ..., \otimes x_n^0\right) \geq_G \otimes 0$ is any feasible solution to the primal GLP problem (4) and $\otimes y^0 = (\otimes y_1^0, \otimes y_2^0, ..., \otimes y_m^0) \geq_G \otimes 0$ is any feasible solution to the dual of the problem (4) i.e. (5), then $\sum_{i=1}^{m} \otimes y_i^0 \otimes b_i, \leq_G \sum_{j=1}^{n} c_j \otimes x_j^0$.

Corollary 1. [23] If $\otimes x^*$ and $\otimes y^*$ are, respectively, feasible solutions to primal (4) and dual (5), and $c \otimes x^* =_G \otimes y^{*T} \otimes b$, then $\otimes x^*$ and $\otimes y^*$are optimal solutions to their respective problems.

4.1 The proposed algorithm (Duality Method)

Now we propose an a Algorithm based on the duality method for finding the optimal solution. Consider the following problem:

$$\begin{aligned} \text{Minimize}\ \ & \otimes \mathrm{Z} =_{\mathrm{G}} C \otimes X \\ \textbf{s.t}\ \ & A \otimes X \geq_G \otimes b, \\ & \otimes X \geq_G \otimes 0. \end{aligned} \tag{6}$$

1) Write a dual problem of grey linear programming.

$$\begin{aligned} \text{Maximize}\ \ & \otimes \mathrm{u} =_{\mathrm{G}} \otimes y^T \otimes b \\ \textbf{s.t}\ \ & \otimes y^T A \leq_G C, \\ & \otimes y^T \geq_G \otimes 0. \end{aligned} \tag{7}$$

2) Suppose that a basic feasible solution and a corresponding simplex table are available.

3) The basic feasible solution is given by $\otimes y_B^T =_G B^{-1} c =_G \otimes f_0$ and $\otimes y_N^T =_G \otimes 0$. In this case, the value of the grey target function will be as: $\otimes u =_G \otimes y^T B^{-1} C =_G \otimes f_{00}$.

4) Compute $\otimes f_{0j} =_G \otimes u_j - \otimes b_j$, for all $j = 1,2,...,m;\ j \neq B_i,\ i = 1,2,...,n.$

5) Let $\otimes f_{0k} =_G \min\left\{\otimes f_{ij}\right\},\ \ j = 1,2,...,m.$

6) If $\otimes f_{0k} \geq_G \otimes 0$, then stop, the current solution is optimal.

7) If $\otimes f_{0k} <_G \otimes 0$ and, $f_{ik} \leq \otimes 0,\ i = 1,2,...,n$, then stop, the problem is unbounded.

8) If $\otimes f_{0k} <_G \otimes 0$ and, $f_{ik} > \otimes 0,\ i = 1,2,...,n$, then, determine an index r corresponding to a variable y_{B_r} leaving the basis as follows:

$$\frac{f_{r0}}{f_{rk}} = \min\left\{\frac{f_{i0}}{f_{ik}} | f_{ik} \geq 0\right\}, i = 1,2,...,n.$$

9) Pivot on f_{rk}an element and update the simplex tableau to go to the second step.

In the following, the efficiency of the proposed method is shown by providing practical examples.

5 Numerical Examples

In this section, by presenting three different examples of problem (4), including different types of constraints and decision variables that can occur in real life, the proposed method's efficiency to solve them is evaluated. Here, we give GLP problems and solve these by the proposed method described in the last section.

Example 1. A livestock company is willing to provide the feed required by its livestock at a minimum cost. The number of nutrients in each kilogram of these substances (in terms of the number of units of nutrients in the substance), the number of nutrients needed per day, and each ingredient's cost are listed below.

Table 1: The nutrients, cost, and required daily amount of a Livestock company.

Amount of daily necessities	Alfalfa	Corn	Nutrients
$\otimes[4,6]$	3	2.5	Vitamin
$\otimes[3,4]$	4	1	Protein
	4	2	Cost

Let
$\otimes x_1$: Number of packages needed for Corn;
$\otimes x_2$: Number of packages needed for Alfalfa.
Consider the following GLP problem.

$$\begin{aligned} \text{Minimize } \ \otimes Z =_G 2\otimes x_1 + 4\otimes x_2 & \\ \textbf{s.t} \qquad 2.5\otimes x_1 + 3\otimes x_2 \geq_G \otimes[4,6], & \\ \otimes x_1 + 4\otimes x_2 \geq_G \otimes[3,4], & \\ \otimes x_1, \otimes x_2 \geq_G \otimes 0, & \end{aligned} \tag{8}$$

and its dual problem:

$$\begin{aligned} \text{Maximize } \ \otimes u =_G \otimes[4,6]\otimes y_1 + \otimes[3,4]\otimes y_2 & \\ \textbf{s.t} \qquad 2.5\otimes y_1 + \otimes y_2 \leq_G 2, & \\ 3\otimes y_1 + 4\otimes y_2 \leq_G 4, & \\ \otimes y_1, \otimes y_2 \geq_G \otimes 0. & \end{aligned} \tag{9}$$

In this example, we describe a primal model to the constraints of the "greater than or equal to" type, the decision variables are "≥ 0", and so, duality model has the constraints of the “less than or equal to” type. Also, the decision variables are “≥ 0”. Now, we solve the problem by using the proposed Algorithm 4.1.

Table 2: The simplex tableau of the problem (9).

Basic variables	$\otimes u$	$\otimes y_1$	$\otimes y_2$	$\otimes s_1$	$\otimes s_2$	R.H.S
$\otimes u_0$	[1,1]	-[4,6]	-[3,4]	[0,0]	[0,0]	[0,0]
$\otimes s_1$	[0,0]	2.5	1	[1,1]	[0,0]	2
$\otimes s_2$	[0,0]	3	4	[0,0]	[1,1]	4
$\otimes u_0$	[1,1]	[0,0]	-[0.6,2.4]	[1.6,2.4]	[0,0]	[3.2,4.8]
$\otimes y_1$	[0,0]	[1,1]	0.4	0.4	[0,0]	0.8
$\otimes s_2$	[0,0]	[0,0]	2.8	-1.2	[1,1]	1.6
$\otimes u_0$	[1,1]	[0,0]	[0,0]	[0.57,2.14]	[0.21,0.85]	[3.54,6.17]
$\otimes y_1$	[0,0]	[1,1]	[0,0]	0.57	0.14	0.571
$\otimes y_2$	[0,0]	[0,0]	[1,1]	0.43	0.36	0.571

Table 2 is the optimal table of the duality problem, so by using the coefficients of the slack variables of the duality problem in the first line of the optimal table, we extract the optimal answer to the primal problem. Grey optimal solution to the primal problem will be as follows.

$$\otimes x_1 = \otimes[0.57, 2.14], \quad \otimes x_2 = \otimes[0.21, 0.85], \quad \otimes z = \otimes[3.54, 6.17].$$

Example 2. Consider the following GLP problem:

$$\begin{aligned} \text{Minimize } \ \otimes Z =_G & \otimes x_1 + 2 \otimes x_2 + \otimes x_3 \\ \textbf{s.t} \quad & 2 \otimes x_1 + 3 \otimes x_2 + \otimes x_3 \leq_G \otimes[4, 8], \\ & 2 \otimes x_1 + 3 \otimes x_2 + \otimes x_3 \geq_G \otimes[0.5, 1.5], \\ & \otimes x_1, \otimes x_2, \otimes x_3 \geq_G \otimes 0. \end{aligned} \tag{10}$$

and its dual problem:

$$\begin{aligned} \text{Maximize } \ \otimes \mathrm{u} =_G & \otimes[4, 8] \otimes y_1 + \otimes[0.5, 1.5] \otimes y_2 \\ \textbf{s.t} \quad & -2 \otimes y_1 + 2 \otimes y_2 \leq_G 1, \\ & 3 \otimes y_1 + 3 \otimes y_2 \leq_G 2, \\ & -\otimes y_1 + \otimes y_2 \leq_G 1, \\ & \otimes y_1, \otimes y_2 \geq_G \otimes 0. \end{aligned} \tag{11}$$

In this example, we describe a primal model contain some constraints of the “less than or equal to” type, some of the “greater than or equal to” type, the decision variables are “≥ 0”. So, the duality model has the constraints of the “less than or equal to” type, and the decision variables are “≥ 0”. Now, we solve the problem according to the Algorithm 4.1.

Table 3 gives the optimal table of the duality problem, so by using the coefficients of the slack variables of the duality problem in the first line of the optimal table, we

Table 3: The simplex tableau of the problem (11).

Basic variables	$\otimes u$	$\otimes y_1$	$\otimes y_2$	$\otimes s_1$	$\otimes s_2$	$\otimes s_3$	R.H.S
$\otimes u_0$	[1,1]	[4,8]	-[0.5,1.5]	[0,0]	[0,0]	[0,0]	[0,0]
$\otimes s_1$	[0,0]	-2	2	1	0	0	1
$\otimes s_2$	[0,0]	3	3	0	1	0	2
$\otimes s_3$	[0,0]	-1	1	0	0	1	1
$\otimes u_0$	[1,1]	[2.5,7.5]	[0,0]	[0.25,0.75]	[0,0]	[0,0]	[0.25,0.75]
$\otimes y_2$	[0,0]	-1	1	0.5	0	0	0.5
$\otimes s_2$	[0,0]	6	0	-1.5	1	0	0.5
$\otimes s_3$	[0,0]	-1	1	-0.5	0	1	0.5

extract the optimal solution of the primal problem. The solution of the primal problem will be as follows:

$$\otimes x_1 = \otimes[0.25, 0.75], \ \otimes x_2 = \otimes[0,0], \quad \otimes x_3 = \otimes[0,0], \quad \otimes Z = \otimes[0.25, 0.75].$$

Example 3. Consider the following GLP problem.

$$\begin{aligned} \text{Minimize} \ \ & \otimes \mathrm{Z} =_{\mathrm{G}} 10 \otimes x_1 + 50 \otimes x_2 + 20 \otimes x_3 \\ \textbf{s.t} \ \ & \otimes x_1 + 2 \otimes x_2 + \otimes x_3 \geq_G \otimes[400, 600], \\ & \otimes x_1 + 8 \otimes x_2 \geq_G \otimes[100, 300], \\ & \otimes x_2, \otimes x_3 \geq_G \otimes 0, \ \otimes x_1 \text{ is unrestricted,} \end{aligned} \tag{12}$$

and its dual problem:

$$\begin{aligned} \text{Maximize} \ \ & \otimes \mathrm{u} =_{\mathrm{G}} \otimes[400, 600] \otimes y_1 + \otimes[100, 300] \otimes y_2 \\ \textbf{s.t} \ \ & \otimes y_1 + \otimes y_2 =_G 10, \\ & 2 \otimes y_1 + 8 \otimes y_2 \leq_G 50, \\ & \otimes y_1 \qquad\qquad \leq_G 20, \\ & \otimes y_1, \otimes y_2 \geq_G \otimes 0. \end{aligned} \tag{13}$$

In this example, we describe a primal model to constraints of the "greater than or equal to" type, the decision variables are "$\geq_G 0$", or "unrestricted". So, the duality model has the constraints of the "less than or equal to" type, and some of the "equal to" type, and the decision variables are "$\geq_G 0$". Now, we solve the problem according to Algorithm 4.1.

Table 4 presents the optimal table of the duality problem, so by using the coefficients of the slack variables of the duality problem in the first line of the optimal table, we extract the optimal answer to; the primal problem. The solution to the primal problem will be as follows.

$$x_1 = \otimes[400, 600], \quad \otimes x_2 = \otimes[0,0], \quad \otimes x_3 = \otimes[0,0], \ \otimes Z = \otimes[4000, 6000].$$

Different methods have been proposed to solve GLP problems, most of which have been using whitening problem parameters. GLP problems were studied by several authors

Table 4: The simplex tableau of the problem (13).

Basic variables	$\otimes u$	$\otimes y_1$	$\otimes y_2$	$\otimes R_1$	$\otimes s_2$	$\otimes s_3$	R.H.S
$\otimes u_0$	[1,1]	-[400,600]	-[100,300]	[M,M]	[0,0]	[0,0]	[0,0]
$\otimes R_1$	[0,0]	1	1	[1,1]	[0,0]	[0,0]	10
$\otimes s_2$	[0,0]	2	8	[0,0]	[1,1]	[0,0]	50
$\otimes s_3$	[0,0]	1	0	[0,0]	[0,0]	[1,1]	20
$\otimes u_0$	[1,1]	-[M,M] -[400,600]	-[M,M] -[100,300]	[0,0]	[0,0]	[0,0]	-30[M,M]
$\otimes R_1$	[0,0]	1	1	[1,1]	[0,0]	[0,0]	10
$\otimes s_2$	[0,0]	2	8	[0,0]	[1,1]	[0,0]	50
$\otimes s_3$	[0,0]	1	0	[0,0]	[0,0]	[1,1]	20
$\otimes u_0$	[1,1]	[0,0]	[100,500]	[M,M] +[400,600]	[0,0]	[0,0]	+10[M,M] +[400,600]
$\otimes y_1$	[0,0]	[1,1]	1	1	[0,0]	[0,0]	10
$\otimes s_2$	[0,0]	[0,0]	6	-2	[1,1]	[0,0]	30
$\otimes s_3$	[0,0]	[0,0]	-1	-1	[0,0]	[1,1]	10

[9, 14, 15, 19, 27, 33, 36]. Nasseri et al. [24] presented the first method of solving GLP problems without whitening, which had grey objective function coefficients for GLP problems. In this paper, we have presented for the first time a method for solving GLP problems with the right-hand side grey parameters without whitening grey numbers. In this method, we use the dual GLP problem that has not been done before, and we do so for constraints in different states. The presented Algorithm for this method is straightforward and will show the uncertainty of the input data in the results.

6 Conclusion

Grey systems theory is a fundamental methodology for dealing with inexact conditions. The problem of LP with grey resources is beneficial for real and practical problems. In this research, we fixed a new concept of duality for LP minimization problems with grey parameters. We argue the grey basic feasible solution notions for LP minimization problems with grey parameters. With the use of arithmetic operations between interval grey numbers, we have proved the optimal basic feasible solution and the weak duality property for LP minimization problems with grey data. These results would be useful for establishing a new Algorithm. Finally, by giving different examples, we demonstrate the proposed method's efficiency for solving GLP in different modes. In the most proposed methods for find LP problems solution with grey parameters, the GLP problems transformed into one or a series of the classical LP problems and then obtained an optimal solution, but with the use of the above results, we proposed a new Algorithm (duality method) for the find LP problems solution with grey data. The proposed method is less sophisticated than other methods. One of the essential advantages of this method is that it does not use whitening to solve GLP problems, resulting presented uncertainty in the input data in the output data.

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کاربرد نتایج دوالیتی برای حل مساله برنامه‌ریزی خطی با پارامترهای خاکستری

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چکیده

مسائل برنامه‌ریزی خطی دارای پارامترهای دقیق هستند. در دنیای واقعی، ما با شرایطی روبرو هستیم که اطلاعات دقیق و کامل در دسترس نیست. در این شرایط می‌توان از رویکردهای عدم قطعیت مانند فازی و تصادفی برای مقابله با عدم اطمینان در زندگی واقعی استفاده کرد. اگر تعداد متخصصان و سطح تجربه به قدری کم باشد که استخراج توابع عضویت غیرممکن باشد یا تعداد نمونه ها کم باشد، نمی‌توان از نظریه‌های فازی و تصادفی استفاده کرد. برای حل این مشکل، نظریه سیستم خاکستری ارائه شد. در این مقاله ، یک مساله برنامه‌ریزی خطی در یک محیط خاکستری با منابعی به صورت اعداد خاکستری بازه‌ای در نظر گرفته شده است. بیشتر روش‌های پیشنهادی برای حل مسائل برنامه‌ریزی خطی خاکستری بر مبنای تبدیل آن به یک مساله برنامه‌ریزی خطی معمولی می‌باشند. با این حال، ما به دنبال حل مستقیم مساله و بدون تبدیل آن به یک مساله برنامه‌ریزی خطی استاندارد به منظور حفظ عدم قطعیت داده‌های ورودی در جواب‌های بدست آمده نهایی هستیم. برای این منظور، ما یک روش، بر اساس دوال مساله برنامه‌ریزی خطی با منابع خاکستری ارائه می‌دهیم. روش پیشنهادی ساده‌تر از روش‌های قبلی بوده و از پیچیدگی کمتری نسبت به آنها برخوردار است. تأکید می‌شود که روش ارائه شده برای شرایط واقعی و عملی در مسائل مدیریت و برنامه‌ریزی می تواند مفید باشد. بنابراین، کارایی روش پیشنهادی با ارائه چند مثال در شرایط مختلف نشان داده شده است.

کلمات کلیدی

عدد خاکستری، برنامه‌ریزی خطی خاکستری، نظریه دوگان، عدم قطعیت.