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Research Article

Finding the Most Efficient DMU in DEA: A Model-Free Procedure

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Abstract. Data envelopment analysis models are able to rank decision-making units (DMUs) based on their efficiency scores. In spite of the fact that there exists a unique ranking of inefficient DMUs, ranking efficient DMUs is problematic. However, rather than ranking methods, another way to choose one of the efficient units is to determine the most efficient DMU. Up to the present, many models have been proposed to rank DMUs and determine the most efficient one. These models require solving nonlinear or integer programs, which are NP-hard and time-consuming. Considering efficient DMU's characteristics, this paper proposes a procedure to find the most efficient DMU through some simple operations. The validity of the proposed approach is verified and tested via some numerical examples.

Keywords. Data envelopment analysis, Most efficient DMU, Input and output weights, Mathematical model.

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1 Introduction

Data Envelopment Analysis (DEA), as a nonparametric method in economics and operations research, was proposed by Charnes et al. [8]. It is used for evaluating the relative performance of a set of similar decision-making units (DMUs) with multiple inputs and multiple outputs while estimating the production frontier. The DEA has been proved as an effective tool for performance evaluation in a variety of fields, with interesting applications in health care, education, banking, manufacturing, and so on. The DMUs are assumed to be comparable [12, 13]. However, they may have their own unique conditions. The DEA method was originally introduced according to constant returns to scale (CRS) and has then developed to variable returns to scale [6], nonincreasing returns to scale, and nondecreasing returns to scale [31]. Dealing with fuzzy data and network structures can be considered as another development in this field [29].

Efficiency scores in the DEA are estimated using the most desirable input/output weights. Based on the efficiency score, a DMU is identified as efficient or inefficient. In the second case, the DEA models present a related set of efficient units to be applied as a benchmark. It allocates efficiency scores of less than 1 to inefficient DMUs and 1 to efficient DMUs.

To select a set of weights instead of choosing different weights for each DMU, a common set of weights (CSW) has been suggested for assessing the performance of DMUs. Comparison and ranking of the performance of DMUs based on the same benchmark can be possible using a CSW. Using a CSW in DEA is equivalent to a ranking method such as the analytical hierarchy process in multiple criteria decision-making literature [20]. To generate a CSW in DEA, several alternative approaches were proposed in [11, 24, 28, 35].

Traditional DEA models cannot distinguish between efficient DMUs, while they can rank inefficient ones. In the last four decades, different theoretical developments have been done, according to the original CCR model, to deal with the occurring practical problems. Adler et al. [1] reviewed the main ranking approaches and divided them into six main groups: cross-efficiency [33], super-efficiency [5], benchmark [42], multivariate statistics [17, 18, 35], inefficient DMUs and multi-criteria decision-making (MCDM) [19].

The most efficient DMU (which, in some cases is referred to as “the best”) can be specified using ranking approaches. However, it is unnecessary to rank all efficient DMUs to find a unique DMU as the most efficient. It is significant that ranking methods usually use different sets of optimal weights for each DMU separately, while to determine the most efficient DMU, a common set of optimal weights must simultaneously be applied. While in ranking efficient DMUs, at least one optimization model for each DMU must be solved, the most efficient DMU can be found just by solving an integrated model. Ranking methods allow each DMU to determine its own optimal weights to get the highest rank. Therefore, the majority of these approaches use different sets of weights to rank efficient units and to attain these weights. Different optimization problems must be solved. Although there are some approaches based on a CSW that rank DMUs by solving only one model, the issue of selecting a Pareto solution would be problematic [11, 45].

During some real-life applications, the decision-maker usually wants to select only one DMU among the set of considered DMUs. Obviously, there is always intense competition among efficient DMUs. Therefore, it is reasonable to evaluate these units through CSW. There have been several developments to extend some integrated DEA models for finding the most efficient DMU. Li and Reeves [23] presented a multiple criteria DEA model that could be used for improving the discrimination power of classical DEA. Based on minimizing the maximum deviation and minimizing the sum of deviations of DMUs, their model effectively yields more acceptable input and output weights without former information. To show the applicability of the proposed MCDM model, some examples of previous works were utilized in [32, 33, 34, 43].

Karsak and Ahiska [21] used benchmarks to introduce a multiple-criteria decision-making model and claimed their proposed model resulted in the most efficient DMU. However, Amin et al. [4] showed that this model was unable to determine the most efficient DMU in some specific cases. Ertay and Ruan [14] proposed a cross-efficiency approach to determine the most efficient unit. Next, Ertay et al. [15] proposed a min-max model to find the most efficient DMU and applied it to a real data set consisting of 19 facility layout alternatives. The related objective function contains a parameter that needs to be selected on a trial-and-error method. Amin and Toloo [3] introduced an integrated DEA model in order to detect the most efficient DMU. They claimed that their proposed model could find the most CCR-efficient DMU without solving a series of linear programs (LPs) and therefore was less time-consuming. Moreover, their model eliminated the requirement of using the parameter mentioned in [15]. Amin [2] showed that the model proposed in [3] might result in more than one efficient DMU. He proposed a mixed-integer nonlinear programming model to obtain a single most efficient DMU, especially in the CRS technology. Toloo and Nalchigar [39] developed the previous models to find the most BCC-efficient DMU. Toloo [36] addressed some problems of applying the model in [39] and introduced a new mixed-integer linear programming (MILP) model to determine the most BCC-efficient DMU. Foroughi [16] claimed that the mixed-integer nonlinear model proposed in [2] was infeasible in some cases. Then he proposed an MILP model to find the most efficient DMU using the super-efficiency perspective. Toloo and Salahi [41] extended a nonlinear model to deal with improving the discriminating power of DEA models, and it was shown that the proposed model identified the most efficient unit. Özsoy et al. [27] introduced another model based on mixed-integer programming to determine the most efficient DMU in two-stage systems and sub-stages. An approach considering user's subjective opinions was developed by Toloo et al. [40] to find the most efficient information system projects.

Wang and Jiang [44] suggested a set of MILP models for identifying the most efficient DMU under various returns to scale technologies. They claimed that the proposed approaches were simpler than Foroughi's. Toloo [37] formulated an MILP model to determine the most efficient unit without any explicit inputs. He also utilized the model to detect the most efficient professional tennis player. Lam [22] introduced a new MILP model that had an objective function similar to that of the super-efficiency model. Moreover, Salahi and Toloo [30] suggested a model to find the maximum epsilon in the determination of the most efficient DMU. Özsoy et al. [26] proposed a model excluding the non-Archimedean epsilon and calculating the CSW to choose the most efficient unit. As an alternative to all MILP models for identifying the most efficient unit under the CRS assumption, Toloo [38] formulated a min-max model.

While all the aforementioned approaches may introduce various units as the most efficient ones, they could be time-consuming and NP-hard and are based on solving one or more than one mathematical programs. Furthermore, it is not possible to claim that one of them is more reasonable than the others. Therefore, a different approach is needed to find the most efficient unit. In this paper, an easy-to-use algorithm is presented to determine the most efficient unit without solving any mathematical programming problem. The proposed method is based on the definition of the most efficient DMU and the properties of efficient DMUs.

Table 1: Comparison of different methods proposed for finding the most efficient DMU according to their models

Reference	Method - Model type
Etray & Ruan [14]	**LP-Cross efficiency**
Etray et al. [15]	**Min-max model**
Karsak & Ahiska [21]	**MCDM**
Amin & Toloo [3]	**Integrated DEA model**
Amin [2]	**Mixed-Integer Nonlinear Programming**
Foroughi [16]	**MILP model**
Toloo [36]	**MILP model**
Wang & Jiang [44]	**MILP model**
Toloo [38]	**Min-max model**
Toloo & Salahi [41]	**Nonlinear model**
Ozsoy et al. [27]	**MIP (two stage)**
Proposed Model	**No Mathematical Model**

The structure of the paper is organized as follows. Section 2 gives preliminaries on the subject. In Section 3, a new algorithm will be proposed to generate the real most efficient DMU. Subsequently, in Section 4, a discussion is given after solving two numerical examples to show the potential applications of the proposed algorithm. Conclusions appear in Section 5.

2 Preliminaries

Consider a set of n DMUs ($\text{DMU}_j : j = 1, \ldots, n$) such that each one uses the input vector $x_j = (x_{1j}, \ldots, x_{mj})$ to produce the output vector $y_j = (y_{1j}, \ldots, y_{rj})$.

Charnes et al. [8] proposed the basic DEA model, called the CCR model. They solved the following fractional programming problem to obtain an optimal solution regarding values for the input weights $v = (v_1, \ldots, v_m)$ and the output weights $u = (u_1, \ldots, u_s)$ as variables:

$$
\begin{array}{lll}
\max & \theta_p = \dfrac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip}} & \\
\text{s.t.} & \dfrac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1 & j = 1,\dots,n \\
& v_i \geq 0 & i = 1,\dots,m \\
& u_r \geq 0 & r = 1,\dots,s
\end{array} \tag{1}
$$

where $\mathrm{DMU}_p \quad (p \in \{1,\dots,n\})$ is the DMU under evaluation.

Definition 1. DMU_p is CCR-efficient if the model (1) has at least one optimal solution (u^*, v^*) with $v^* > 0$ and $u^* > 0$ in which $\theta_p^* = 1$. Otherwise, DMU_p is CCR-inefficient. (Here, the notation $*$ shows the optimality).

The model mentioned in (1) can be converted into a linear programming model as follows [7]:

$$
\begin{array}{lll}
\max & \theta = u y_p & \\
\text{s.t.} & v x_p = 1 & \\
& u y_j - v x_j \leq 0 & j = 1,\dots,n \\
& u \geq 0, \quad v \geq 0 &
\end{array} \tag{2}
$$

According to Definition 1, all DMUs can be partitioned into efficient and inefficient sets. The ranking of the inefficient DMUs is based on their efficiency scores; however, the efficient DMUs must be ranked based on some other criteria. This has led to many ranking models in the literature. Although ranking models can sort these units, all the efficient units are related to a model to be solved. Furthermore, based on each ranking method, the result may differ.

It should also be noted that a specific applicable definition of the most efficient DMU does not exist. Moreover, all existing models randomly choose one of the efficient DMUs as the best while it is not unique. Thus, the performances of these models are not comparable. Finally, all these models use integer and nonlinear programming to determine the most efficient DMU, which is time-consuming and NP-hard. In the next section, the most efficient DMU is defined, and then the most efficient unit is determined accordingly.

In order to overcome the mentioned drawbacks, in the first step, the definition of the most efficient DMU is presented as follows.

Definition 2. [38] DMU_p is called the most efficient DMU if there is a common set of optimal weights $(u^*, v^*) > 0$ such that $u^* y_p - v^* x_p = 0$ and $u^* y_j - v^* x_j < 0$ for each $j \neq p$.

Example 1. Consider four DMUs whose data are shown in Table 2, and their corresponding production possibility set (PPS) is shown in Figure 1.

The hyperplane $\Delta : 6y - 3x_1 - x_2 = 0$ satisfies Definition 2, and as it can be observed, DMU_A is the most efficient.

In the next section, an approach will be proposed to find the most efficient unit without solving any nonlinear, binary, or even linear models.

PPS is the set of all (x, y)'s such that the input vector x can produce the output vector y. For more details on the mathematical definition, see [10].

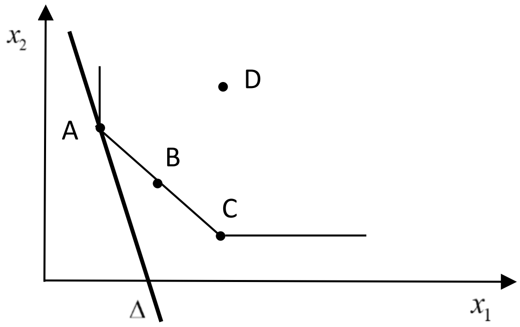

Figure 1: The PPS of Example 1.

Table 2: Data of Example 1.

	A	**B**	**C**	**D**
x_1	1	2	3	3
x_2	3	2	1	4
y	1	1	1	1

3 The Proposed Approach

3.1 Requirements

It is known that the optimal value of model (1) is 1. The following theorem represents the relationship between efficiency and the most efficient DMU.

Theorem 1. Inefficient DMUs cannot be the most efficient.

Proof. Suppose that DMU_p is inefficient, that is, either (a) $\theta^* < 1$ or (b) $\theta^* = 1$ and that at least one component of the vector (u^*, v^*) equals zero at optimal solutions to model (2).

(a). If $\theta^* < 1$, then the model (2) implies that

$$\theta^* = u^* y_p < v^* x_p = 1,$$
$$u^* y_p - v^* x_p < 0.$$

Therefore, DMU_p does not satisfy Definition 2.

(b). In this case, the DMU_p cannot be the most efficient obviously, since by Definition 2, the vector (u^*, v^*) must be positive.

□

Theorem 2. If there exist $u_r \geq 0 \quad (r = 1, \ldots, s)$ and $v_i \geq 0 \quad (i = 1, \ldots, m)$ such that

$$\frac{\sum_r u_r y_{rp}}{\sum_i v_i x_{ip}} = \text{Max}_j \left\{ \frac{\sum_r u_r y_{rj}}{\sum_i v_i x_{ij}} \right\}, \tag{3}$$

then DMU_p is efficient [9].

Theorem 2 makes it possible to find the efficient DMUs by choosing various combinations of input and output weights.

Corollary 1. DMU_p, where

$$\frac{\sum_r y_{rp}}{\sum_i x_{ip}} = \text{Max}_j \left\{ \frac{\sum_r y_{rj}}{\sum_i x_{ij}} \right\},$$

is an efficient unit.

Proof. By setting $u = (1, \ldots, 1)$ and $v = (1, \ldots, 1)$ in Theorem 2, the result is trivial. □

Now, according to Definition 1 and Corollary 1, the conditions for finding the most efficient DMU are expressed in the following theorem.

Theorem 3. If there exists a unique index p satisfying the condition

$$\frac{\sum_r y_{rp}}{\sum_i x_{ip}} = \text{Max}_j \left\{ \frac{\sum_r y_{rj}}{\sum_i x_{ij}} \right\}, \tag{4}$$

then, DMU_p is the most efficient unit.

Proof. In order to show that DMU_p is the most efficient unit, it suffices to show that DMU_p satisfies Definition 1. For this purpose, first, by assuming $\frac{\sum_r y_{rp}}{\sum_i x_{ip}} = k$, we obtain $\sum_r y_{rp} - k\left(\sum_i x_{ip}\right) = 0$. Then, for each $\text{DMU}_j \quad (j = 1, \ldots, n;\ j \neq p)$, we have

$$\frac{\sum_r y_{rp}}{\sum_i x_{ip}} = k > \frac{\sum_r y_{rj}}{\sum_i x_{ij}}.$$

Therefore, $\sum_r y_{rj} - k\left(\sum_i x_{ij}\right) < 0$ for all j $(j = 1, \ldots, n,\ j \neq p)$. Hence, according to Definition 1, by choosing $\bar{v} = (k, \ldots, k)$ and $\bar{u} = (1, \ldots, 1)$, DMU_p is the most efficient unit. □

Now, since the conditions for finding the most efficient unit are fully provided, the most efficient unit can be easily found through the following algorithm without solving any models.

3.2 The Algorithm

In this section, an algorithm is proposed to find the most efficient unit without solving any mathematical programming problem. The proposed algorithm is based on Definition 1 and the characteristics of efficient DMUs.

The algorithm starts with a set of initial weights ($u = (1, \ldots, 1)$ and $v = (1, \ldots, 1)$). If there exists a unique DMU such that these weights satisfy Definition 2 and Theorem 2, then it is the most efficient one. Otherwise, the initial weights are changed by using the specific method (i.e., the hyperplane is rotated). Eventually, such changes lead to the adjustment of the initial hyperplane so that the resulted hyperplane satisfies Definition 2. The steps of the algorithm are stated in the following steps.

Step 0. If some DMUs are multiples of each other (i.e., the vectors of inputs and outputs of a DMU are multiples of each other), then eliminate all these DMUs except one.

Step 1. Set $t = 1$, $v^{(t)} = (1,\ldots,1)$, $u^{(t)} = (1,\ldots,1)$, and $M_0 = \{1,\ldots,n\}$.

Step 2. Find $k = \max\left\{\frac{\sum_r u_r^{(t)} y_{rj}}{\sum_i v_i^{(t)} x_{ij}} \Big| \ j \in M_{t-1}\right\}$ and set

$$M_t = \left\{\underset{j}{\arg\max}\left\{\frac{\sum_r u_r^{(t)} y_{rj}}{\sum_i v_i^{(t)} x_{ij}},\ j \in M_{t-1}\right\}\right\}.$$

Step 3. If M_t is a singleton, then $\text{DMU}_p\,(p \in M_t)$ is the most efficient unit. Set $(\bar{u},\bar{v}) = \left(u_r^{(t)}, kv_i^{(t)}\right)$ and stop. Otherwise, go to **Step 4**.

Step 4. Let $P, Q \in M_t$. Since P and Q are different DMUs, at least one component of the input vector or one component of the output vector is different.

Case I: If there exists an index l such that $x_{lp} \neq x_{lq}$, then we define

$$\begin{cases} v_l^{(t+1)} = 2v_l^{(t)}, & \\ v_i^{(t+1)} = v_i^{(t)}, & i = 1,\ldots,m; \quad i \neq l, \\ u_r^{(t+1)} = u_r^{(t)}, & r = 1,\ldots,s. \end{cases} \tag{5}$$

Case II: If there exists an index l such that $y_{lp} \neq y_{lq}$, then we define

$$\begin{cases} v_i^{(t+1)} = v_i^{(t)}, & i = 1,\ldots,m, \\ u_l^{(t+1)} = 0.5u_l^{(t)}, & \\ u_r^{(t+1)} = u_r^{(t)}, & r = 1,\ldots,s; \quad r \neq l. \end{cases} \tag{6}$$

Set $t \leftarrow t+1$ and go to **Step 2**.

Using the predefined weights at the first step, a set of efficient DMUs is found. These DMUs belong to M_1. If M_1 is a singleton, then the most efficient DMU is found. Otherwise, the weights are changed such that the hyperplane $uy_j - vx_j \leq 0$ is slightly turned, resulting in a reduction of the cardinality of M_t. This process should be repeated until M_t is a singleton. In order to maintain the above relation, the output weights are decreased, or input weights are increased. Finally, after finding the most efficient DMU, a set of weights is needed to satisfy Definition 2. Thus, in the final step, the related weights are set as $(\bar{u},\bar{v}) = \left(ur^{(t)}, kv_i^{(t)}\right)$.

3.3 Validity of the Algorithm

The validity of the algorithm is presented via the following theorem.

Theorem 4. Suppose that the algorithm produces a vector $(\overline{u},\overline{v})$ in the final step of Algorithm. This vector satisfies $\bar{u}y_j - \bar{v}x_j \le 0$ and $(\overline{u},\overline{v}) > 0$ for all $j = 1,\dots,n$.

Proof. In the first step, the algorithm uses the default vector $(\overline{u},\overline{v}) = (1,\dots,1,1,\dots,1)$. In the next steps, the algorithm reduces a component of u_r $(r = 1,\dots,s)$ into halves or increases a component of v_i $(i = 1,\dots,m)$ by making it double. This allows us to conclude that u_r's and v_i's satisfy Definition 2, that is, $\overline{u}y_j - \overline{v}x_j \le 0$ and $(\overline{u},\overline{v}) > 0$ for $j = 1,\dots,n$. It is obvious that $\bar{u}y_p - \bar{v}x_p = 0$ for $P \in M_t$. Therefore, in the final step, we have $\bar{u}y_p - \bar{v}x_p = 0$. □

The next theorem shows that the procedure of the algorithm concludes a set M_t that is a singleton.

Theorem 5. The procedure of the algorithm leads to a singleton M_t in finite iterations.

Proof. Suppose that in an iteration, M_t is not a singleton; that means there exist at least two DMUs, for example, P and Q such that $P, Q \in M_t$ and P is not a multiple coefficient of Q. If $\bar{u}$ and $\bar{v}$ are calculated according to (5) or (6), it is obvious that

$$\frac{\sum_r \bar{u}_r y_{rp}}{\sum_i \bar{v}_i x_{ip}} \neq \frac{\sum_r \bar{u}_r y_{rq}}{\sum_i \bar{v}_i x_{iq}}.$$

Without loss of generality, suppose

$$\frac{\sum_r \hat{u}_r y_{rp}}{\sum_i \hat{v}_i x_{ip}} > \frac{\sum_r \hat{u}_r y_{rq}}{\sum_i \hat{v}_i x_{iq}},$$

then, in the next step, we have $Q \notin M_{t+1}$. This shows $M_{t+1} \subset M_t$, and hence in the finite steps of algorithm, the set Mwill be a singleton. □

Now, Theorems 4 and 5 guarantee that the algorithm finds the most efficient DMU in finite steps. In the next section, the procedure of finding the most efficient DMU is illustrated by two examples.

The following theorem shows that the supporting hyperplane relies on PPS and establishes some conditions for the most efficient definition.

Theorem 6. The hyperplane constructed in Theorem 3 is the support of the PPS.

Proof. Suppose that for an arbitrary DMU_p, there exists (u^*, v^*) such that

$$u^* y_p - v^* x_p = 0, \tag{7}$$
$$u^* y_j - v^* x_j < 0, \qquad j \neq p, \tag{8}$$
$$u^* > 0,\ v^* > 0.$$

Inequalities (7) and (8) are multiplied by $\lambda_j \ge 0$, and the results are summed as follows:

$$\sum_{j=1}^{n} \lambda_j (u^* y_j) - \sum_{j=1}^{n} \lambda_j (v^* x_j) \le 0,$$

$$u^*\Big(\sum_{j=1}^{n}\lambda_j y_j\Big) - v^*\Big(\sum_{j=1}^{n}\lambda_j x_j\Big) \leq 0.$$

Based on the definition of CRS of PPS (see [10]), it follows

$$T_c = \{(x,y)|x \geq \sum \lambda_j x_j, y \leq \sum \lambda_j y_j, \lambda_j \geq 0\},$$

we get $u^*y - v^*x \leq 0$. Hence $(x,y) \in T$.

On the other hand, since $u^*y_p - v^*x_p = 0$, $u^*y - v^*x = 0$ supports the PPS. □

4 Numerical Examples

As it was explained, up to now, all the existing models for finding the most efficient DMU have used mathematical models such as linear, nonlinear, integer, or even mixed-integer programming. Thus, many of them can be NP-hard [25], while the proposed algorithm only uses a few simple operations. The next example shows the steps of the algorithm in detail.

Example 2. Consider the data set of Example 1.

Set $t := 1$, $u^{(1)} := 1$, $v^{(1)} := (1,1)$, and $M_0 := \{A,B,C,D\}$. Now, set

$$k = \max\left\{\frac{1}{1+3}, \frac{1}{2+2}, \frac{1}{3+1}, \frac{1}{3+4}\right\} = \frac{1}{4},$$

and then $M_1 = \{A,B,C\}$. Since M_1 is not a singleton, then the vector $(u^{(1)}, v^{(1)})$ must be changed. Considering the components of the input vector x, we have

$$x_{1A} \neq x_{1B} \neq x_{1C}.$$

By using (5), we obtain

$$\begin{cases} u^{(2)} = u^{(1)} = 1, \\ v_1^{(2)} = 2v_1^{(1)} = 2, \\ v_1^{(2)} = v_1^{(1)} = 1. \end{cases}$$

In the next iteration, $t = 2$ and

$$k = \max\left\{\frac{1}{2+3}, \frac{1}{4+2}, \frac{1}{6+1}, \frac{1}{6+4}\right\} = \frac{1}{5},$$

and $M_2 = \{A\}$. Since M_2 is a singleton, the algorithm terminates, and DMU_A is the most efficient DMU. By

$$(\bar{u}, \bar{v}) = (u^{(2)}, kv^{(2)}) = (1, \frac{1}{5}(2), \frac{1}{5}(1)) = (1, \frac{2}{5}, \frac{1}{5}),$$

the conditions of Definition 2 are satisfied. The supporting hyperplane is $y - \frac{2}{5}x_1 - \frac{1}{5}x_2 = 0$ and

$$\begin{aligned} A: &\quad 1-\frac{2}{5}(1)-\frac{1}{5}(3)=0,\\ B: &\quad 1-\frac{2}{5}(2)-\frac{1}{5}(2)=-\frac{1}{5}<0,\\ C: &\quad 1-\frac{2}{5}(3)-\frac{1}{5}(1)=-\frac{2}{5}<0,\\ D: &\quad 1-\frac{2}{5}(3)-\frac{1}{5}(4)=-1<0. \end{aligned}$$

Example 3. Here, we examine a data set from [44]. Consider 30 OECD countries with three inputs (unemployment ratio (x_1), rate of inflation (x_2), and baby death rate (x_3)) and five outputs (national income per capita (y_1), human development index (y_2), education index (y_3), contribution rate to labor force of woman population (y_4), and health expenditure per capita (y_5)) that are listed in Table 3.

Now, the proposed algorithm is applied to determine the most efficient DMU. The algorithm starts with the initial weights. At the first iteration, the value of k is determined as 8023.110874 and M_1 is a singleton. So, the final weights are driven as

$$\begin{cases} \bar{u}_r = u_r^{(1)} = 1, & r=1,\ldots,5,\\ \bar{v}_i = kv_i^{(1)} = 8023.110874, & i=1,2,3. \end{cases}$$

Our proposed approach determined DMU17 as the best efficient one. As mentioned in [44], different methods may determine various DMU as the most efficient ones. For instance, DMU27 was distinguished as the most efficient one in [44].

5 Conclusion

In order to distinguish and recognize the best performance of efficient DMUs, determining the most efficient DMU is crucial. Various approaches have been proposed to find these DMUs, which are NP-hard and time-consuming. Furthermore, there is no clue that one of them is more accurate than the others. In this paper, we introduced an easy-to-use method based on setting weights. The proposed procedure enabled us to find the most efficient unit only through some elementary operations and comparisons. This was done by using an algorithm that altered the weights of inputs and outputs to obtain a hyperplane satisfying the most efficient condition. As it was shown, the validity of the algorithm was confirmed by two theorems. Finally, the approach was applied to some examples. The way of setting the initial weights would be a topic for future research. Here, we set all weights equal to unity, but the effect of choosing random weights or selecting appropriate weights according to inputs and outputs could be surveyed. Also, researchers can develop some methods for updating weights in the algorithm so that the number of iterations is reduced.

Table 3: Data on 30 OECD countries [44]

DMU	Countries	x_1	x_2	x_3	y_1	y_2	y_3	y_4	y_5
1	Australia	5.1	3	6	34740	80.9	0.993	67.4	2036
2	Austria	7.2	1.8	5	37117	79.4	0.966	63.8	1968
3	Belgium	12.1	1.6	6	35712	78.8	0.977	57.3	2081
4	Canada	6.8	2.2	6	35133	80.3	0.991	72.8	2312
5	Czech	8.9	1.8	5	12152	75.9	0.936	64	930
6	Denmark	5.6	2.4	4	47984	77.9	0.993	74.2	2133
7	Finland	8.4	1.7	4	37504	78.9	0.993	72.8	1502
8	France	9.1	1.9	4	33918	80.2	0.982	62.4	2055
9	Germany	9.2	2.3	5	33854	79.1	0.953	67.4	2424
10	Greece	9.9	4.6	5	20327	78.9	0.97	56	1167
11	Hungary	7.2	5.3	8	10814	72.9	0.958	53.5	705
12	Iceland	1.8	4.8	4	52764	81.5	0.978	82.9	2103
13	Ireland	4.3	4.7	6	48604	78.4	0.993	62.2	1436
14	Italy	7.7	2.5	6	30200	80.3	0.958	50.1	1783
15	Japan	4.4	1	4	35757	82.3	0.946	60.5	1822
16	South Korea	3.7	2.8	5	16308	79	0.904	49.9	730
17	Luxembourg	4.2	1.1	5	80288	78.4	0.942	55.7	2215
18	Mexico	3.6	5	25	7298	75.6	0.863	42.6	356
19	Netherland	4.3	3.5	5	38618	79.2	0.988	69.5	2070
20	New Zealand	3.7	2.7	6	26464	79.8	0.993	71.2	1424
21	Norway	3.5	1.3	4	64193	79.8	0.991	77.3	2330
22	Poland	18.2	1.9	9	7946	75.2	0.951	57.6	496
23	Portugal	7.6	3.5	6	17456	77.7	0.925	67.8	1237
24	Slovak	11.7	3.3	8	8775	74.2	0.921	62.4	930
25	Spain	9.2	3.1	5	27226	80.5	0.987	57.2	1218
26	Sweden	5.8	2.2	3	39694	80.5	0.978	74.9	1746
27	Switzerland	3.8	0.9	3	50532	81.3	0.946	75.3	2794
28	Turkey	10.3	13.7	3.8	5816	71.4	0.812	26.5	255
29	England	2.8	1.6	6	37023	79	0.97	69.3	1461
30	USA	5.1	1.6	7	4200	77.9	0.971	70.1	4178

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