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Research Article

Approximate Orthogonally Higher Ring Derivations

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Abstract. In this paper, we prove that every orthogonally higher ring derivation is a higher ring derivation. Also we find the general solution of the pexider orthogonally higher ring derivations

$$\begin{cases} f_n(x+y) = g_n(x) + h_n(y),\ \langle x, y\rangle = 0, \\ f_n(xy) = \sum_{i+j=n} g_i(x)h_j(y). \end{cases}$$

Then we prove that for any approximate pexider orthogonally higher ring derivation under some control functions $\varphi(x, y)$ and $\psi(x, y)$, there exists a unique higher ring derivation $D = \{d_n\}_{n=0}^{\infty}$, near $\{f_n\}_{n=0}^{\infty}$, $\{g_n\}_{n=0}^{\infty}$ and $\{h_n\}_{n=0}^{\infty}$ estimated by φ and ψ.

Keywords. Approximation, Control function, Estimation, Higher derivation.

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1 Introduction

Let $\mathcal{A}$ be an algebra. A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a *derivation* if it satisfies the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. Let $\mathbb{N}_0$ be the set of all nonnegative integers. If we define a sequence $D = \{d_n\}_{n\in\mathbb{N}_0}$ of linear mappings on $\mathcal{A}$ by $d_0 = I$ and $d_n = \frac{\delta^n}{n!}$, where I is the identity mapping on $\mathcal{A}$, then the Leibniz rule ensures that d_n's satisfy the condition

$$d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y),$$

for every $x, y \in \mathcal{A}$ and each non-negative integer n. Such a sequence D is called a *higher derivation*. Note that d_1 is a derivation, if D is a higher derivation. Higher derivations were introduced by Hasse and Schmidt [3], and algebraists sometimes call them *Hasse-Schmidt derivations*. For an account on higher derivations the reader is referred to the book [2].

Mirzavaziri [6] proved that there exists a one-to-one correspondence between higher derivations and the family of sequences of derivations on torsion free algebras. He showed that for each higher derivation $D = \{d_n\}_{n=0}^{\infty}$ on a torsion free algebra $\mathcal{A}$, there is a unique sequence $\Delta = \{\delta_n\}_{n=1}^{\infty}$ of derivations on $\mathcal{A}$ such that

$$d_n = \sum_{i=1}^{n}\Big(\sum_{\sum_{j=1}^{i} r_j = n} \Big(\prod_{j=1}^{i} \frac{1}{r_j + \ldots + r_i} \Big)\, \delta_{r_1} \ldots \delta_{r_i} \Big),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$.

Let $x \in M_n(\mathbb{R})$ be fixed. For inner derivation

$$\delta_x(a) = ax - xa \quad (a \in M_n(\mathbb{R})),$$

on $M_n(\mathbb{R})$, the ordinary higher derivation $D = \{d_n\}_{n=0}^{\infty}$ on $M_n(\mathbb{R})$ is defined as follows.

$$d_0 = I, \quad d_n(a) = \frac{\delta_X^n(a)}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{n!} \binom{n}{k} x^k a x^{n-k} \quad (a \in M_n(\mathbb{R})).$$

For example, consider $x \in M_2(\mathbb{R})$ given by

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, the ordinary higher derivation corresponding to x, is defined by $d_0 = I$ and

$$\begin{aligned} d_n(a) &= \sum_{k=0}^{n} \frac{(-1)^k}{n!} \binom{n}{k} x^k a x^{n-k} \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{n!} \binom{n}{k} \begin{pmatrix} a_{11} & (-1)^{n-k} a_{12} \\ (-1)^k a_{21} & (-1)^n a_{22} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \frac{a_{11}}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k} & \frac{a_{12}}{n!}(-1)^n\sum_{k=0}^{n}\binom{n}{k} \\ \frac{a_{21}}{n!}\sum_{k=0}^{n}\binom{n}{k} & \frac{a_{22}}{n!}(-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k} \end{pmatrix}$$

$$= \frac{2^n}{n!}\begin{pmatrix} 0 & (-1)^n a_{12} \\ a_{21} & 0 \end{pmatrix},$$

for all $a = [a_{ij}] \in M_2(\mathbb{R})$ and each $n \in \mathbb{N}$.

A functional equation is said to be *stable*, if any approximate solution of it, is near to a true solution of it. The stability problem of functional equations originated from the following question of Ulam [13]: *Under what condition does there exist an additive mapping near an approximately additive mapping?* Hyers [5] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. A generalized version of the theorem of Hyers for approximately linear mapping was given by Th. M. Rassias [9]. After that, several functional equations have been extensively investigated by a number of authors (for instances, see [7, 8, 10, 11]).

In this paper, we prove that any orthogonally higher ring derivation on an inner product algebra (an algebra equipped with an inner product) is a higher ring derivation. Also, we find the general solution and prove the generalized Hyers-Ulam stability of the pexider orthogonally higher ring derivations on inner product Banach algebras (a Banach algebra equipped with an inner product).

2 Pexider Orthoganally Higher Ring Derivations

In this section, we first show that any orthogonally higher ring derivation on an inner product algebra is a higher ring derivation.

Definition 1. Let $\mathcal{A}$ be an inner product algebra. A sequence $D = \{d_n\}_{n=0}^{\infty}$ of mappings form $\mathcal{A}$ into $\mathcal{A}$ with $d_0 = I$ is called an *orthogonally higher ring derivation* if for each $n \in \mathbb{N}$,

$$d_n(x+y) = d_n(x) + d_n(y), \tag{1}$$

for all $x, y \in \mathcal{A}$ with $\langle x, y\rangle = 0$ and

$$d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y), \tag{2}$$

for all $x, y \in \mathcal{A}$.

Note that an orthogonally additive mapping cannot be additive or linear in general. Ratz in Corollary 10 of [12] showed that if $(X, \perp)$ is an inner product space and $(Y, +)$ a uniquely 2-divisible abelian group, then a function $d : X \to Y$ is orthogonally additive, if and only if there exist additive mappings $a : \mathbb{R} \to Y, \quad b : X \to Y$ such that $d(x) = a(\|x\|^2) + b(x)$, for every $x \in X$.

Using this corollary, in the next theorem we characterize the orthogonally higher ring derivations.

Theorem 1. Let $\mathcal{A}$ be an inner product algebra. Any orthogonally higher ring derivation on $\mathcal{A}$ is a higher ring derivation.

Proof. Let $D = \{d_n\}_{n=0}^{\infty}$ be an orthogonally higher ring derivation on $\mathcal{A}$. Since for any $n \in \mathbb{N}$, d_n is an orthogonally additive mapping on the inner product space $\mathcal{A}$, by Corollary 10 of [12], it follows that d_n is of the form

$$d_n(x) = a_n(\|x\|^2) + b_n(x) \quad (x \in \mathcal{A}),$$

in which $a_n : \mathbb{R} \to \mathcal{A}$ and $b_n : \mathcal{A} \to \mathcal{A}$ are additive mappings. Using this, it follows from (2) that

$$\begin{aligned} &a_n(\|xy\|^2) + b_n(xy) \\ &= \sum_{i+j=n} (a_i(\|x\|^2) + b_i(x))(a_j(\|y\|^2) + b_j(y)) \\ &= \sum_{i+j=n} \Big(a_i(\|x\|^2)a_j(\|y\|^2) + a_i(\|x\|^2)b_j(y) + b_i(x)a_j(\|y\|^2) + b_i(x)b_j(y)\Big), \end{aligned}$$

for all $x, y \in \mathcal{A}$. Let $k \in \mathbb{N}$, replacing x by $2^k x$ and y by $2^k y$ in the above equation, we obtain

$$\begin{aligned} &2^{4k}a_n(\|xy\|^2) + 2^{2k}b_n(xy) \\ &= \sum_{i+j=n} (2^{4k}a_i(\|x\|^2)a_j(\|y\|^2) + 2^{3k}a_i(\|x\|^2)b_j(y) + 2^{3k}b_i(x)a_j(\|y\|^2) + 2^{2k}b_i(x)b_j(y)), \end{aligned}$$

for all $x, y \in \mathcal{A}$. Dividing the above equation by 2^{4k} and letting $k \to \infty$, we get

$$a_n(\|xy\|^2) = \sum_{i+j=n} a_i(\|x\|^2)a_j(\|y\|^2) \quad (x, y \in \mathcal{A}), \tag{3}$$

which implies that

$$2^{2k}b_n(xy) = \sum_{i+j=n} (2^{3k}a_i(\|x\|^2)b_j(y) + 2^{3k}b_i(x)a_j(\|y\|^2) + 2^{2k}b_i(x)b_j(y)) \quad (x, y \in \mathcal{A}).$$

Also, dividing the above equation by 2^{3k} and letting $k \to \infty$, we get

$$\sum_{i+j=n} (a_i(\|x\|^2)b_j(y) + b_i(x)a_j(\|y\|^2)) = 0 \quad (x, y \in \mathcal{A}),$$

which implies that

$$b_n(xy) = \sum_{i+j=n} b_i(x)b_j(y) \quad (x, y \in \mathcal{A}).$$

Since $d_0 = I$ (by definition of higher derivation), it follows that $b_0 = I$ and $a_0 = 0$. Therefore $\{b_n\}_{n=0}^{\infty}$ is a higher ring derivation. By induction on n we show that $a_n = 0$ for each $n \in \mathbb{N}$. Assume that $a_k = 0$ for all $k < n$. From Equation (3) we deduce that

$$a_n(\|xy\|^2) = a_n(\|x\|^2)a_0(\|y\|^2) + \sum_{\substack{i+j=n \\ 0\le i\le n-1}} a_i(\|x\|^2)a_j(\|y\|^2),$$

which implies $a_n(\|xy\|^2) = 0$ for all $x, y \in \mathcal{A}$. For each $t \geq 0$ define $xy = \sqrt{t}\|z\|^{-1}z \quad (0 \neq z \in \mathcal{A})$. Then $a_n(t) = a_n(\|xy\|^2) = 0$. Since a_n is additive, it is odd. Thus, for each $t < 0$, $a_n(t) = -a_n(-t) = 0$ and so $a_n = 0$ on $\mathbb{R}$. This completes the proof.

□

Corollary 1. Let $\mathcal{A}$ be an inner product algebra. Any orthogonally ring derivation on $\mathcal{A}$ is a ring derivation.

Definition 2. Let $\mathcal{A}$ be an inner product algebra. If $F = \{f_n\}_{n=0}^{\infty}$, $G = \{g_n\}_{n=0}^{\infty}$ and $H = \{h_n\}_{n=0}^{\infty}$ are sequences of mappings from $\mathcal{A}$ into $\mathcal{A}$ such that $f_0 = g_0 = h_0 = I$ and for each $n \in \mathbb{N}_0$,

$$f_n(x+y) = g_n(x) + h_n(y), \tag{4}$$

for all $x, y \in \mathcal{A}$ with $\langle x, y\rangle = 0$ and

$$f_n(xy) = \sum_{i+j=n} g_i(x)h_j(y), \tag{5}$$

for all $x, y \in \mathcal{A}$, then the triple (F, G, H) is siad to be a *pexider orthogonally higher ring derivation* on $\mathcal{A}$.

To see a similar aspect of the above definition, the reader is referred to [1, 4].

Remark 1. In the framework of unital inner product algebras, each pexider orthogonally higher ring derivation is in fact a higher ring derivation. To see this, let $\mathcal{A}$ be a unital inner product algebra with the identity element $\mathbf{e}$. Letting $y = 0$ in equation (4) for $n = 1$ we obtain

$$f_1(x) = g_1(x) + h_1(0), \quad (x \in \mathcal{A}). \tag{6}$$

On the other hand, letting $y = \mathbf{e}$ in equation (5) for $n = 1$ we have

$$f_1(x) = g_1(x)h_0(\mathbf{e}) + g_0(x)h_1(\mathbf{e}) = g_1(x)\mathbf{e} + xh_1(\mathbf{e}), \quad (x \in \mathcal{A}). \tag{7}$$

Comparing (6) and (7) we get $h_1(0) = xh_1(\mathbf{e})$ for all $x \in \mathcal{A}$. Putting $x = 0$ in this equation we have $h_1(0) = 0$. So, it follows from (6) that $f_1(x) = g_1(x)$ for all $x \in \mathcal{A}$.

Reasoning like above, we get that $g_1(0) = 0$ and so $f_1(x) = h_1(x)$ for all $x \in \mathcal{A}$. Hence $f_1 = g_1 = h_1$, and consequently, f_1 is an ordinary derivation on $\mathcal{A}$. By continuing this process, we can prove that $f_n = g_n = h_n$ for all $n \in \mathbb{N}$, which means that $\{f_n\}_{n=0}^{\infty}$ is an orthogonally higher ring derivation and so by Theorem 1, is a higher ring derivation on $\mathcal{A}$.

Proposition 1. Let $\mathcal{A}$ be an inner product algebra. If the triple (F, G, H) is a pexider orthogonally higher ring derivation on $\mathcal{A}$, then there exists a higher ring derivation $D = \{d_n\}_{n=0}^{\infty}$ on $\mathcal{A}$ such that for each $n \in \mathbb{N}_0$,

$$f_n(x) = d_n(x) + f_n(0), \quad g_n(x) = d_n(x) + g_n(0), \quad h_n(x) = d_n(x) + h_n(0),$$

for all $x \in \mathcal{A}$.

Proof. Putting $x = y = 0$ in (4), we conclude that $f_n(0) = g_n(0) + h_n(0)$. Putting $x = 0$ in (4), we get $f_n(y) = g_n(0) + h_n(y)$ for all $y \in \mathcal{A}$ and so

$$h_n(y) = f_n(y) - g_n(0) \quad (y \in \mathcal{A}). \tag{8}$$

Putting $y = 0$ in (4), we get $f_n(x) = g_n(x) + h_n(0)$ for all $x \in \mathcal{A}$ and so

$$g_n(x) = f_n(x) - h_n(0) \quad (x \in \mathcal{A}). \tag{9}$$

If we define the function $d_n : \mathcal{A} \to \mathcal{A}$ by $d_n(x) = f_n(x) - f_n(0)$, then using relations (8) and (9), we have

$$f_n(x) = d_n(x) + f_n(0), \quad g_n(x) = d_n(x) + g_n(0), \quad h_n(x) = d_n(x) + h_n(0), \tag{10}$$

for all $x \in \mathcal{A}$ and $n \in \mathbb{N}_0$. Substituting equations (10) into (4), we obtain

$$d_n(x+y) = d_n(x) + d_n(y),$$

for all $x, y \in \mathcal{A}$ with $\langle x, y \rangle = 0$ and $n \in \mathbb{N}_0$. That is, for any $n \in \mathbb{N}_0$, the function d_n is orthogonally additive.

Substituting equations (10) into (5), we obtain

$$\begin{aligned} d_n(xy) + f_n(0) &= \sum_{i+j=n} (d_i(x) + g_i(0))(d_j(y) + h_j(0)) \\ &= \sum_{i+j=n} (d_i(x)d_j(y) + d_i(x)h_j(0) + g_i(0)d_j(y) + g_i(0)h_j(0)) \end{aligned}$$

for all $x, y \in \mathcal{A}$. Putting $x = y = 0$ in the last equation, we get

$$f_n(0) = \sum_{i+j=n} g_i(0)h_j(0),$$

and so

$$d_n(xy) = \sum_{i+j=n} (d_i(x)d_j(y) + d_i(x)h_j(0) + g_i(0)d_j(y)), \tag{11}$$

for all $x, y \in \mathcal{A}$. Putting $y = 0$ and $x = 0$ in equation (11), we get respectively,

$$\sum_{i+j=n} d_i(x)h_j(0) = 0, \quad \sum_{i+j=n} g_i(0)d_j(y) = 0,$$

for all $x, y \in \mathcal{A}$, which implies that

$$d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y),$$

for all $x, y \in \mathcal{A}$. That is, $D = \{d_n\}_{n=0}^{\infty}$ is an orthogonally higher ring derivation and so by Theorem 1, is a higher ring derivation. □

In the next theorems, we will prove the generalized Hyers-Ulam stability of the pexider orthoganally higher ring derivations.

Theorem 2. Let $\mathcal{A}$ be an inner product Banach algebra and $\varphi, \psi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be functions such that for all $x, y \in \mathcal{A}$,

$$\lim_{k\to\infty} 2^k \varphi(2^{-k}x, 2^{-k}y) = 0, \lim_{k\to\infty} 2^{2k} \psi(2^{-k}x, 2^{-k}y) = 0. \tag{12}$$

Suppose that $\varphi(x,0), \varphi(0,y) \le \varphi(x,y)$ for all $x, y \in \mathcal{A}$ and there exists $M > 0$ such that for all $x, y \in \mathcal{A}$ with $\langle x, y\rangle = 0$ and $\langle x+y, x-y\rangle = 0$,

$$\max\{\varphi(x,y), \varphi(x,-y), \varphi(x+y, x-y)\} \le M\varphi(x,x),$$

and the limit

$$\widetilde{\varphi}(x) = \sum_{k=1}^{\infty} 2^{k-1} \varphi(2^{-k}x, 2^{-k}x), \tag{13}$$

exists for all $x \in \mathcal{A}$. If $F = \{f_n\}_{n=0}^{\infty}$, $G = \{g_n\}_{n=0}^{\infty}$ and $H = \{h_n\}_{n=0}^{\infty}$ are sequences of mappings from $\mathcal{A}$ into $\mathcal{A}$ such that for any $n \in \mathbb{N}_0$, f_n is odd, $g_n(0) = h_n(0) = 0$ and

$$\|f_n(x+y) - g_n(x) - h_n(y)\| \le \varphi(x,y), \tag{14}$$

for all $x, y \in \mathcal{A}$ with $\langle x, y\rangle = 0$ and

$$\left\| f_n(xy) - \sum_{i+j=n} g_i(x) h_j(y) \right\| \le \psi(x,y), \tag{15}$$

for all $x, y \in \mathcal{A}$, then there exists a unique higher ring derivation $D = \{d_n\}_{n=0}^{\infty}$ from $\mathcal{A}$ into $\mathcal{A}$ such that

$$\begin{aligned} &\|f_n(x) - d_n(x)\| \le 9M\widetilde{\varphi}(x), \\ &\|g_n(x) - d_n(x)\| \le 9M\widetilde{\varphi}(x) + \varphi(x,0), \\ &\|h_n(x) - d_n(x)\| \le 9M\widetilde{\varphi}(x) + \varphi(0,x), \end{aligned} \tag{16}$$

for all $x \in \mathcal{A}$ and $n \in \mathbb{N}_0$.

Proof. Since for all $x \in \mathcal{A}$, $\langle x, 0\rangle = 0$, so putting $y = 0$ in (14), we get

$$\|f_n(x) - g_n(x)\| \le \varphi(x,0), \tag{17}$$

for all $x \in \mathcal{A}$. Also putting $x = 0$ in (14), we get

$$\|f_n(y) - h_n(y)\| \le \varphi(0,y), \tag{18}$$

for all $y \in \mathcal{A}$. Using (14), (17) and (18), we have

$$\|f_n(x+y) - f_n(x) - f_n(y)\| \le 3\varphi(x,y), \tag{19}$$

for all $x, y \in \mathcal{A}$ with $\langle x, y\rangle = 0$.

Let $n \in \mathbb{N}_0$ and $x \in \mathcal{A}$ be fixed. There exists $y \in \mathcal{A}$ such that $\langle x, y\rangle = 0$ and $\langle x+y, x-y\rangle = 0$; then, moreover, $\langle x, -y\rangle = 0$. Applying inequality (19) for the orthogonal vectors $(x, -y)$ and $(x+y, x-y)$, we get

$$\|f_n(x-y)-f_n(x)+f_n(y)\| \le 3\varphi(x,-y), \tag{20}$$

$$\|f_n(2x)-f_n(x+y)-f_n(x-y)\| \le 3\varphi(x+y,x-y). \tag{21}$$

Using (19), (20) and (21), we have

$$\|f_n(2x)-2f_n(x)\| \le 3(\varphi(x,y)+\varphi(x,-y)+\varphi(x+y,x-y)) \le 9M\varphi(x,x). \tag{22}$$

From the above inequality we get

$$\begin{aligned}\|f_n(x)-2^k f_n(2^{-k}x)\| &\le \sum_{m=1}^{k} \|2^{m-1} f_n(2^{-(m-1)}x) - 2^m f_n(2^{-m}x)\| \\ &\le 9M \sum_{m=1}^{k} 2^{m-1}\varphi(2^{-m}x, 2^{-m}x).\end{aligned}$$

It follows from (13) and the above inequality that the sequence $\{2^k f_n(2^{-k}x)\}$ is Cauchy in Banach algebra $\mathcal{A}$ and so is convergent. If we define the mapping $d_n : \mathcal{A} \to \mathcal{A}$ by

$$d_n(x) := \lim_{k\to\infty} 2^k f_n(2^{-k}x) \quad (x \in \mathcal{A}),$$

then

$$\|f_n(x)-d_n(x)\| \le 9M\widetilde{\varphi}(x),$$

for all $x \in \mathcal{A}$.

It follows from (17) and (18) that for all $x \in \mathcal{A}$ and $n \in \mathbb{N}_0$

$$\|2^k f_n(2^{-k}x) - 2^k g_n(2^{-k}x)\| \le 2^k \varphi(2^{-k}x, 0),$$

and

$$\|2^k f_n(2^{-k}x) - 2^k h_n(2^{-k}x)\| \le 2^k \varphi(0, 2^{-k}x).$$

So taking limit, we get

$$\lim_{k\to\infty} 2^k g_n(2^{-k}x) = \lim_{k\to\infty} 2^k h_n(2^{-k}x) = d_n(x),$$

for all $x \in \mathcal{A}$ and $n \in \mathbb{N}_0$ and also using (17) and (18) we get

$$\|g_n(x)-d_n(x)\| \le 9M\widetilde{\varphi}(x)+\varphi(x,0),$$

$$\|h_n(x)-d_n(x)\| \le 9M\widetilde{\varphi}(x)+\varphi(0,x).$$

Let $x, y \in \mathcal{A}$ with $\langle x, y\rangle = 0$. It follows from (14) that for all $x \in \mathcal{A}$ and $n \in \mathbb{N}_0$

$$\begin{aligned}\|d_n(x+y)-d_n(x)-d_n(y)\| &= \lim_{k\to\infty} \|2^k f_n(2^{-k}(x+y)) - 2^k g_n(2^{-k}x) - 2^k h_n(2^{-k}y)\| \\ &\le \lim_{k\to\infty} 2^k \varphi(2^{-k}x, 2^{-k}y) = 0.\end{aligned}$$

That is, the mappings d_n are orthogonally additive.

Let $x, y \in \mathcal{A}$ and $n \in \mathbb{N}_0$. Replacing x by $2^{-k}x$ and y by $2^{-k}y$ in (15) and multiplying by 2^{2k}, we have

$$2^{2k}\left\|f_n((2^{-k}x)(2^{-k}y)) - \sum_{i+j=n} g_i(2^{-k}x)h_j(2^{-k}y)\right\| \leq 2^{2k}\psi(2^{-k}x, 2^{-k}y),$$

which tends to zero as $k \to \infty$. Since the sequence $\{2^k g_i(2^{-k}x)\}$ converges for all $x \in \mathcal{A}$, it is bounded. Thus for each $x \in \mathcal{A}$ there is $C_x > 0$ such that $\|2^k g_i(2^{-k}x)\| \leq C_x$. Therefore

$$\begin{aligned}
&\left\|d_n(xy) - \sum_{i+j=n} d_i(x)d_j(y)\right\| \\
&\leq \|d_n(xy) - 2^{2k} f_n(2^{-2k}xy)\| + \left\|2^{2k} f_n(2^{-2k}xy) - \sum_{i+j=n}(2^k g_i(2^{-k}x))(2^k h_j(2^{-k}y))\right\| \\
&+ \left\|\sum_{i+j=n}(2^k g_i(2^{-k}x) - d_i(x))d_j(y)\right\| + \left\|\sum_{i+j=n}(2^k g_i(2^{-k}x))(2^k h_j(2^{-k}y) - d_j(y))\right\| \\
&\leq \|d_n(xy) - 2^{2k} f_n(2^{-2k}xy)\| + 2^{2k}\left\|f_n((2^{-k}x)(2^{-k}y)) - \sum_{i+j=n} g_i(2^{-k}x)h_j(2^{-k}y)\right\| \\
&+ \sum_{i+j=n} \|2^k g_i(2^{-k}x) - d_i(x)\|\|d_j(y)\| + \sum_{j=0}^{n} C_x\|2^k h_j(2^{-k}y) - d_j(y)\|,
\end{aligned}$$

which tends to zero as $k \to \infty$. Therefore the sequence $D = \{d_n\}_{n=0}^{\infty}$ is an orthogonally higher ring derivation and so by Theorem 1 it is a higher ring derivation.

If $D' = \{d'_n\}_{n=0}^{\infty}$ is another higher ring derivation satisfying (16), then for each $n \in \mathbb{N}_0$ we have

$$\|d_n(x) - d'_n(x)\| \leq \|d_n(x) - f_n(x)\| + \|f_n(x) - d'_n(x)\| \leq 18M\widetilde{\varphi}(x),$$

for all $x \in \mathcal{A}$. Therefore

$$\|d_n(x) - d'_n(x)\| = \lim_{k\to\infty} 2^{-k}\|d_n(2^k x) - d'_n(2^k x)\| \leq \lim_{k\to\infty} 2^{-k}(18M\widetilde{\varphi}(2^k x)) = 0.$$

So we obtian that $d_n(x) = d'_n(x)$ for all $x \in \mathcal{A}$ and $n \in \mathbb{N}_0$ and then $D = D'$. Thus the higher ring derivation D is unique and this completes the proof. □

Theorem 3. Let $\mathcal{A}$ be an inner product Banach algebra and $\varphi, \psi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be functions such that for all $x, y \in \mathcal{A}$

$$\lim_{k\to\infty} 2^{-k}\varphi(2^k x, 2^k y) = 0, \lim_{k\to\infty} 2^{-2k}\psi(2^k x, 2^k y) = 0.$$

Suppose that $\varphi(x, 0), \varphi(0, y) \leq \varphi(x, y)$ for all $x, y \in \mathcal{A}$ and there exists $M > 0$ such that for all $x, y \in \mathcal{A}$ with $\langle x, y\rangle = 0$ and $\langle x+y, x-y\rangle = 0$,

$$\max\{\varphi(x, y), \varphi(x, -y), \varphi(x+y, x-y)\} \leq M\varphi(x, x),$$

and the limit

$$\widetilde{\varphi}(x) = \sum_{k=1}^{\infty} 2^{-k}\varphi(2^{k-1}x, 2^{k-1}x),$$

exists for all $x \in \mathcal{A}$. If $F = \{f_n\}_{n=0}^{\infty}$, $G = \{g_n\}_{n=0}^{\infty}$ and $H = \{h_n\}_{n=0}^{\infty}$ are sequences of mappings from $\mathcal{A}$ into $\mathcal{A}$ such that for any $n \in \mathbb{N}_0$, f_n is odd, $g_n(0) = h_n(0) = 0$ and

$$\|f_n(x+y) - g_n(x) - h_n(y)\| \leq \varphi(x,y),$$

for all $x,y \in \mathcal{A}$ with $\langle x,y\rangle = 0$ and

$$\left\| f_n(xy) - \sum_{i+j=n} g_i(x)h_j(y) \right\| \leq \psi(x,y),$$

for all $x,y \in \mathcal{A}$, then there exists a unique higher ring derivation $D = \{d_n\}_{n=0}^{\infty}$ from $\mathcal{A}$ into $\mathcal{A}$ such that

$$\|f_n(x) - d_n(x)\| \leq 9M\widetilde{\varphi}(x),$$
$$\|g_n(x) - d_n(x)\| \leq 9M\widetilde{\varphi}(x) + \varphi(x,0),$$
$$\|h_n(x) - d_n(x)\| \leq 9M\widetilde{\varphi}(x) + \varphi(0,x),$$

for all $x \in \mathcal{A}$ and $n \in \mathbb{N}_0$.

Proof. As in the proof of Theorem 2, replacing x by $2^{k-1}x$ in (22) and multiplying by 2^{-k} we get

$$\|2^{-k}f_n(2^k x) - f_n(x)\| \leq 9M \sum_{m=1}^{k} 2^{-m}\varphi(2^{m-1}x, 2^{m-1}x),$$

for all $x \in \mathcal{A}$ and $n \in \mathbb{N}_0$. Thus the Cauchy sequence $\{2^{-k}f_n(2^k x)\}$ is convergent and so for each $n \in \mathbb{N}_0$ there exists a mapping $d_n : \mathcal{A} \to \mathcal{A}$ defined by

$$d_n(x) := \lim_{k\to\infty} 2^{-k}f_n(2^k x) \quad (x \in \mathcal{A}),$$

such that

$$\|f_n(x) - d_n(x)\| \leq 9M\widetilde{\varphi}(x),$$

for all $x \in \mathcal{A}$. The rest of proof is similar to that of Theorem 2. □

Corollary 2. Let $\mathcal{A}$ be an inner product Banach algebra and $\varepsilon > 0$ be a real number. If $F = \{f_n\}_{n=0}^{\infty}$, $G = \{g_n\}_{n=0}^{\infty}$ and $H = \{h_n\}_{n=0}^{\infty}$ are sequences of mappings from $\mathcal{A}$ into $\mathcal{A}$ such that for any $n \in \mathbb{N}_0$, f_n is odd, $g_n(0) = h_n(0) = 0$ and

$$\|f_n(x+y) - g_n(x) - h_n(y)\| \leq \varepsilon,$$

for all $x,y \in \mathcal{A}$ with $\langle x,y\rangle = 0$ and

$$\left\| f_n(xy) - \sum_{i+j=n} g_i(x)h_j(y) \right\| \leq \varepsilon,$$

for all $x,y \in \mathcal{A}$, then there exists a unique higher ring derivation $D = \{d_n\}_{n=0}^{\infty}$ from $\mathcal{A}$ into $\mathcal{A}$ such that

$$\|f_n(x) - d_n(x)\| \leq 9\varepsilon, \quad \|g_n(x) - d_n(x)\| \leq 10\varepsilon, \quad \|h_n(x) - d_n(x)\| \leq 10\varepsilon,$$

for all $x \in \mathcal{A}$ and $n \in \mathbb{N}_0$.

Proof. This follows from Theorem 3 by taking $\varphi(x,y) = \psi(x,y) = \varepsilon$ for all $x, y \in \mathcal{A}$. □

The next corollary follows from Theorem 2 (Theorem 3).

Corollary 3. Let $\mathcal{A}$ be an inner product Banach algebra and φ, ψ be functions satisfying the conditions of Theorem 2 (Theorem 3). If $F = \{f_n\}_{n=0}^{\infty}$ is a sequence of odd mappings from $\mathcal{A}$ into $\mathcal{A}$ such that for any $n \in \mathbb{N}_0$

$$\|f_n(x+y) - f_n(x) - f_n(y)\| \le \varphi(x,y),$$

for all $x, y \in \mathcal{A}$ with $\langle x, y\rangle = 0$ and

$$\left\| f_n(xy) - \sum_{i+j=n} f_i(x) f_j(y) \right\| \le \psi(x,y),$$

for all $x, y \in \mathcal{A}$, then there exists a unique higher ring derivation $D = \{d_n\}_{n=0}^{\infty}$ from $\mathcal{A}$ into $\mathcal{A}$ such that

$$\|f_n(x) - d_n(x)\| \le 3M\widetilde{\varphi}(x),$$

for all $x \in \mathcal{A}$ and $n \in \mathbb{N}_0$.

Example 1. Define the sequence of odd functions $F = \{f_n : M_2(\mathbb{R}) \to M_2(\mathbb{R})\}_{n=0}^{\infty}$ by

$$f_0(a) = \begin{pmatrix} a_{11} & a_{12} + \frac{a_{12}}{|a_{12}|+1} \\ a_{21} & a_{22} \end{pmatrix}, \quad f_n(a) = \frac{2^n}{n!} \begin{pmatrix} 0 & (-1)^n \left(a_{12} + \frac{a_{12}}{|a_{12}|+1}\right) \\ a_{21} & 0 \end{pmatrix},$$

for all $a = [a_{ij}] \in M_2(\mathbb{R})$ and each $n \in \mathbb{N}$. Then for all $a, b \in M_2(\mathbb{R})$ and each $n \in \mathbb{N}_0$, we have

$$\begin{aligned} \|f_n(a+b) - f_n(a) - f_n(b)\| &= \left\| \frac{2^n}{n!} \begin{pmatrix} 0 & (-1)^n \left(\frac{a_{12}+b_{12}}{|a_{12}+b_{12}|+1} - \frac{a_{12}}{|a_{12}|+1} - \frac{b_{12}}{|b_{12}|+1} \right) \\ 0 & 0 \end{pmatrix} \right\| \\ &= \frac{2^n}{n!} \left| (-1)^n \left(\frac{a_{12}+b_{12}}{|a_{12}+b_{12}|+1} - \frac{a_{12}}{|a_{12}|+1} - \frac{b_{12}}{|b_{12}|+1} \right) \right| \\ &< 6, \end{aligned}$$

and

$$\|f_n(ab) - \sum_{i=0}^{n} f_i(a)f_{n-i}(b)\| = \|f_n(ab) - f_0(a)f_n(b) - f_n(a)f_0(b) - \sum_{i=1}^{n-1} f_i(a)f_{n-i}(b)\|$$

$$= \Big\| \frac{2^n}{n!}\begin{pmatrix} 0 & (-1)^n\Big(a_{11}b_{12}+a_{12}b_{22}+\frac{a_{11}b_{12}+a_{12}b_{22}}{|a_{11}b_{12}+a_{12}b_{22}|+1}\Big) \\ a_{21}b_{11}+a_{22}b_{21} & 0 \end{pmatrix}$$

$$-\frac{2^n}{n!}\begin{pmatrix} a_{11} & a_{12}+\frac{a_{12}}{|a_{12}|+1} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 & (-1)^n(b_{12}+\frac{b_{12}}{|b_{12}|+1}) \\ b_{21} & 0 \end{pmatrix}$$

$$-\frac{2^n}{n!}\begin{pmatrix} 0 & (-1)^n(a_{12}+\frac{a_{12}}{|a_{12}|+1}) \\ a_{21} & 0 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12}+\frac{b_{12}}{|b_{12}|+1} \\ b_{21} & b_{22} \end{pmatrix}$$

$$-\sum_{i=1}^{n-1}\frac{2^i}{i!}\frac{2^{n-i}}{(n-i)!}\begin{pmatrix} 0 & (-1)^i(a_{12}+\frac{a_{12}}{|a_{12}|+1}) \\ a_{21} & 0 \end{pmatrix}\begin{pmatrix} 0 & (-1)^{n-i}(b_{12}+\frac{b_{12}}{|b_{12}|+1}) \\ b_{21} & 0 \end{pmatrix}\Big\|$$

$$= \frac{2^n}{n!}\Big\|\begin{pmatrix} 0 & (-1)^n(a_{11}b_{12}+a_{12}b_{22}+\frac{a_{11}b_{12}+a_{12}b_{22}}{|a_{11}b_{12}+a_{12}b_{22}|+1}) \\ a_{21}b_{11}+a_{22}b_{21} & 0 \end{pmatrix}$$

$$-\begin{pmatrix} (a_{12}+\frac{a_{12}}{|a_{12}|+1})b_{21} & (-1)^n a_{11}(b_{12}+\frac{b_{12}}{|b_{12}|+1}) \\ a_{22}b_{21} & (-1)^n a_{21}(b_{12}+\frac{b_{12}}{|b_{12}|+1}) \end{pmatrix}$$

$$-\begin{pmatrix} (-1)^n b_{21}(a_{12}+\frac{a_{12}}{|a_{12}|+1}) & (-1)^n b_{22}(a_{12}+\frac{a_{12}}{|a_{12}|+1}) \\ a_{21}b_{11} & a_{21}(b_{12}+\frac{b_{12}}{|b_{12}|+1}) \end{pmatrix}$$

$$-\sum_{i=1}^{n-1}\binom{n}{i}\begin{pmatrix} (-1)^i(a_{12}+\frac{a_{12}}{|a_{12}|+1})b_{21} & 0 \\ 0 & (-1)^{n-i}a_{21}(b_{12}+\frac{b_{12}}{|b_{12}|+1}) \end{pmatrix}\Big\|$$

$$= \frac{2^n}{n!}\Big\|\begin{pmatrix} -(a_{12}+\frac{a_{12}}{|a_{12}|+1})b_{12}(1+(-1)^n) & (-1)^n\Big(\frac{a_{11}b_{12}+a_{12}b_{22}}{|a_{11}b_{12}+a_{12}b_{22}|+1}-a_{11}(\frac{b_{12}}{|b_{12}|+1})-b_{22}(\frac{a_{12}}{|a_{12}|+1})\Big) \\ 0 & -a_{21}(b_{12}+\frac{b_{12}}{|b_{12}|+1})(1+(-1)^n) \end{pmatrix}$$

$$-\begin{pmatrix} (a_{12}+\frac{a_{12}}{|a_{12}|+1})b_{21}\sum_{i=1}^{n-1}(-1)^i\binom{n}{i} & 0 \\ 0 & a_{21}(b_{12}+\frac{b_{12}}{|b_{12}|+1})\sum_{i=1}^{n-1}(-1)^{n-i}\binom{n}{i} \end{pmatrix}\Big\|$$

$$= \frac{2^n}{n!}\Big\|\begin{pmatrix} 0 & (-1)^n\Big(\frac{a_{11}b_{12}+a_{12}b_{22}}{|a_{11}b_{12}+a_{12}b_{22}|+1}-a_{11}(\frac{b_{12}}{|b_{12}|+1})-b_{22}(\frac{a_{12}}{|a_{12}|+1})\Big) \\ 0 & 0 \end{pmatrix}\Big\|$$

$$= \frac{2^n}{n!}\Big|(-1)^n\Big(\frac{a_{11}b_{12}+a_{12}b_{22}}{|a_{11}b_{12}+a_{12}b_{22}|+1}-a_{11}\Big(\frac{b_{12}}{|b_{12}|+1}\Big)-b_{22}\Big(\frac{a_{12}}{|a_{12}|+1}\Big)\Big)\Big|$$

$$< 2(1+|a_{11}|+|b_{22}|)$$

$$\leq 2(1+\|a\|+\|b\|).$$

So, if we define

$$\varphi(a,b)=6, \quad \psi(a,b)=2(1+\|a\|+\|b\|),$$

then by Corollary 3 the higher ring derivation defined by

$$d_0 = I, \quad d_n(a) = \frac{2^n}{n!}\begin{pmatrix} 0 & (-1)^n a_{12} \\ a_{21} & 0 \end{pmatrix}$$

satisfies the inequality

$$\|f_n(a) - d_n(a)\| \le 18$$

for all $a \in M_2(\mathbb{R})$ and each $n \in \mathbb{N}_0$.

3 Conclusion

In this paper, we showed that any orthogonally higher ring derivation on an inner product algebra (an algebra equipped with an inner product) is a higher ring derivation. Also, we found the general solution of pexider orthogonally higher ring derivations on inner product algebras. Finally, we showed that for any approximate pexider orthogonally higher ring derivation on an inner product Banach algebra $\mathcal{A}$ under some control funtions $\varphi(x,y)$ and $\psi(x,y)$, there exists a unique higher ring derivation $D = \{d_n\}_{n=0}^{\infty}$ on $\mathcal{A}$, near the approximate pexider orthogonally higher ring derivation estimated by φ and ψ.

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