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Research Article

A Cramer Method for Solving Fully Fuzzy Linear Systems Based on Transmission Average

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Abstract. Solving fuzzy linear systems has been widely studied during the last decades. However, there are still many challenges to solving fuzzy linear equations, as most of the studies have used the principle of extension, which suffers from shortcomings such as the lack of solution, achieving solutions under very strong conditions, large support of the obtained solutions, inaccurate or even incorrect solutions due to not utilizing all the available information, complicated process and high computational load. These problems motivated us to present a fuzzy Cramer method for solving fuzzy linear equations, which uses arithmetic operations based on the Transmission Average (TA). In this study, fully fuzzy linear systems in the form of $\tilde{A}\tilde{X} = \tilde{B}$, and dual fuzzy linear systems in the form of $\tilde{A}\tilde{X} + \tilde{B} = \tilde{C}\tilde{X} + \tilde{D}$ are solved using the proposed fuzzy Cramer method, and numerical examples are provided to confirm the effectiveness and applicability of the proposed method.

Keywords. Transmission average, Fuzzy arithmetic, Fuzzy approximation, Fuzzy Cramer method.

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1 Introduction

Fuzzy numbers are often used to represent and calculate parameter uncertainties in the procedure of mathematical modeling. Therefore, the analysis and calculation of linear systems with fuzzy numbers are principal in fuzzy mathematics. In recent decades, extensive research has been done in the field of fuzzy mathematics and its utilizations [5, 6, 10, 13, 15, 20, 24, 27, 32, 33, 34].

Friedman et al. have presented an embedding approach to solving general fuzzy linear systems [22]. Asady et al. have investigated solving general fuzzy linear systems and have developed a method to solve an $m \times n$ fuzzy linear system [16]. Iterative methods have been proposed to solve fuzzy systems in [2, 3, 7, 8].

The method of Buckley and Qu has been extended to fuzzy systems in the form of $A_1X + b_1 = A_2X + b_2$, where A_1, A_2, b_1 and b_2 are fuzzy matrices with fuzzy numbers. The classical solution seeks a fuzzy vector X that fulfills the system equation providing the exact equality between the fuzzy vectors X and b. In general, the solution to the system $A_1X + b_1 = A_2X + b_2$ is not identical with that of the system $AX = b$. However, in the case where the matrix $A = A_1 - A_2$ is non-singular, their solutions are identical. Consequently, the system $A_1X+b_1 = A_2X+b_2$ has been transformed into the fully fuzzy linear system $AX = b$, where $A = A_1 - A_2$ and $b = b_2 - b_1$, and has been solved using a new algorithm [19, 29]. In addition, a nonlinear programming method has been utilized to solve fuzzy linear systems [30].

In 2012, the algebraic solution of fuzzy linear systems has been investigated based on interval theory [11]. In [14, 17, 18], fuzzy systems have been solved using linear programming problems, and in [21, 25, 28], fuzzy system-solving methods with the input of complex numbers have been proposed.
Recently, Abbasi et al. have defined new arithmetic operations for fuzzy numbers [4]. Then, fuzzy equations have been solved using these defined arithmetic operations [1, 12].

Solving fuzzy linear systems has been extensively studied in recent decades, and many researchers have utilized the conventional extension principle. This principle defines the standard fuzzy arithmetic, which can lead to inaccurate solutions since it does not consider all the accessible information. Despite reasonable solutions for these methods, they are sometimes complicated with numerous and long techniques and considerable computation. These challenges and problems in solving fuzzy linear systems motivated us to propose a more efficient method. For this purpose, we have studied solving fully fuzzy and dual fuzzy linear systems using Transmission-Average (TA)-based fuzzy operations proposed in [4] and have proposed an analytical Cramer method to solve these systems, which is a more effective method compared to common methods and requires less computation.

The structure of the paper is as follows. In Section 2, the required preliminaries are presented. In Section 3, the fuzzy Cramer method is used to solve systems of the form $\tilde{A}\tilde{X} = \tilde{B}$ and $\tilde{A}\tilde{X} + \tilde{B} = \tilde{C}\tilde{X} + \tilde{D}$. In Section 4, numerical examples are presented to show the effectiveness of the proposed method. Finally, the conclusion ends the paper in Section 5.

2 Basic Definitions

Definition 1. [23] Let A be a fuzzy set in $R(A = \{(x, \mu_A(x))|x \in R\})$. Then,

i) A is called normal if there exists an $x \in R$ such that $\mu_{\tilde{A}}(x) = 1$. Otherwise, A is subnormal,

ii) The support of A, denoted by supp(A), is the subset of R whose elements all have non-zero membership grades in A. In other words, $\text{supp}(A)\{x \in R|\mu_A(x) > 0\}$,

iii) An α-level set (or α-cut) of a fuzzy set A in R is a non-fuzzy set denoted by A_α and defined by

$$A_\alpha = \begin{cases} \{x \in R \mid \mu_{\tilde{A}}(x) > 0\}, & \alpha > 0, \\ \text{cl}(\text{supp}(A)), & \alpha = 0, \end{cases} \tag{1}$$

where $\text{cl}(\text{supp}(A))$ denotes the closure of the support of A.

Definition 2. Let $\tilde{A}$ be a Normal, Convex, and Continuous (NCC) fuzzy set on the universal set U. Then, it can be defined from [26]:

$$ac\left(\tilde{A}\right) = \frac{1}{2}\left(\min core\left(\tilde{A}\right) + \max core\left(\tilde{A}\right)\right).$$

Definition 3. [26] A fuzzy number $\tilde{A}$ is called a pseudo-triangular fuzzy number if its membership function $\mu_{\tilde{A}}(x)$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} l_{\tilde{A}}(x), & \underline{a} \leqslant x \leqslant a, \\ r_{\tilde{A}}(x), & a \leqslant x \leqslant \overline{a}, \\ 0, & otherwise, \end{cases}$$

where $l_{\tilde{A}}(x)$ and $r_{\tilde{A}}(x)$ are non-decreasing and non-increasing functions respectively. The pseudo-triangular fuzzy number $\tilde{A}$ is denoted by the quintuplet $\tilde{A} = (\underline{a}, a, \overline{a}, l_{\tilde{A}}(x), r_{\tilde{A}}(x))$, and the triangular fuzzy number by the senary $(\underline{a}, a, \overline{a}, -, -)$.

Definition 4. [26] A fuzzy number $\tilde{A}$ is called a pseudo-trapezoidal fuzzy number if its membership function $\mu_{\tilde{A}}(x)$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} l_{\tilde{A}}(x), & \underline{a} \leqslant x \leqslant a_1, \\ 1, & a_1 \leqslant x \leqslant a_2, \\ r_{\tilde{A}}(x), & a_2 \leqslant x \leqslant \overline{a}, \\ 0, & otherwise, \end{cases}$$

where $l_{\tilde{A}}(x)$ and $r_{\tilde{A}}(x)$ are non-decreasing and non-increasing functions, respectively. The pseudo-trapezoidal fuzzy number $\tilde{A}$ is denoted by the senary $\tilde{A} = (\underline{a}, a_1, a_2, \overline{a}, l_{\tilde{A}}(x), r_{\tilde{A}}(x))$, and the trapezoidal fuzzy number $\tilde{A}$ is indicated by the senary $\tilde{A} = (\underline{a}, a_1, a_2, \overline{a}, -, -)$.

Definition 5. [12] Consider two pseudo-triangular fuzzy numbers:

$$\tilde{A} = (\underline{a}, a, \overline{a}, l_{\tilde{A}}(x), r_{\tilde{A}}(x)), \quad \tilde{B} = \left(\underline{b}, b, \overline{b}, l_{\tilde{B}}(x), r_{\tilde{B}}(x)\right),$$

with the following α-cut forms:

$$\tilde{A} = \bigcup_{\alpha} A_\alpha, \quad A_\alpha = \left[\underline{A}_\alpha, \overline{A}_\alpha\right], \quad \tilde{B} = \bigcup_{\alpha} B_\alpha, \quad B_\alpha = \left[\underline{B}_\alpha, \overline{B}_\alpha\right].$$

In what follows, fuzzy arithmetic operations are defined based on TA:

$$\begin{aligned} &\tilde{A} + \tilde{B} = \bigcup_{\alpha} \left(\tilde{A} + \tilde{B}\right)_\alpha, \\ &\left(\tilde{A} + \tilde{B}\right)_\alpha = \left[\frac{a+b}{2} + \left(\frac{\underline{A}_\alpha + \underline{B}_\alpha}{2}\right), \frac{a+b}{2} + \left(\frac{\overline{A}_\alpha + \overline{B}_\alpha}{2}\right)\right], \end{aligned} \tag{2}$$

$$\begin{aligned} &\tilde{A} - \tilde{B} = \bigcup_{\alpha} \left(\tilde{A} - \tilde{B}\right)_\alpha, \\ &\left(\tilde{A} - \tilde{B}\right)_\alpha = \left[\frac{a-3b}{2} + \left(\frac{\underline{A}_\alpha + \underline{B}_\alpha}{2}\right), \frac{a-3b}{2} + \left(\frac{\overline{A}_\alpha + \overline{B}_\alpha}{2}\right)\right], \end{aligned} \tag{3}$$

$$\begin{aligned} &\tilde{A}.\tilde{B} = \bigcup_{\alpha} \left(\tilde{A}.\tilde{B}\right)_\alpha, \\ &\left(\tilde{A}.\tilde{B}\right)_\alpha = \begin{cases} \left[\left(\frac{b}{2}\right)\underline{A}_\alpha + \left(\frac{a}{2}\right)\underline{B}_\alpha + \left(\frac{b}{2}\right)\overline{A}_\alpha + \left(\frac{a}{2}\right)\overline{B}_\alpha\right], & a > 0, b > 0, \\ \left[\left(\frac{b}{2}\right)\overline{A}_\alpha + \left(\frac{a}{2}\right)\underline{B}_\alpha + \left(\frac{b}{2}\right)\underline{A}_\alpha + \left(\frac{a}{2}\right)\overline{B}_\alpha\right], & a > 0, b < 0, \\ \left[\left(\frac{b}{2}\right)\overline{A}_\alpha + \left(\frac{a}{2}\right)\overline{B}_\alpha + \left(\frac{b}{2}\right)\underline{A}_\alpha + \left(\frac{a}{2}\right)\underline{B}_\alpha\right], & a < 0, b < 0, \\ \left[\left(\frac{b}{2}\right)\underline{A}_\alpha + \left(\frac{a}{2}\right)\overline{B}_\alpha + \left(\frac{b}{2}\right)\overline{A}_\alpha + \left(\frac{a}{2}\right)\underline{B}_\alpha\right], & a < 0, b > 0, \end{cases} \end{aligned} \tag{4}$$

$$\tilde{A}^{-1} = \bigcup_{\alpha} \left(\tilde{A}^{-1}\right)_\alpha, \left(\tilde{A}^{-1}\right)_\alpha = \left[\frac{1}{a^2}\underline{A}_\alpha, \frac{1}{a^2}\overline{A}_\alpha\right], \tag{5}$$

$$\begin{aligned} &\tilde{A}.\tilde{B}^{-1} = \bigcup_{\alpha} \left(\tilde{A}.\tilde{B}^{-1}\right)_\alpha, \\ &\left(\tilde{A}.\tilde{B}^{-1}\right)_\alpha = \begin{cases} \left[\left(\frac{1}{2b}\right)\underline{A}_\alpha + \left(\frac{a}{2b^2}\right)\underline{B}_\alpha + \left(\frac{1}{2b}\right)\overline{A}_\alpha + \left(\frac{a}{2b^2}\right)\overline{B}_\alpha\right], & a > 0, b > 0, \\ \left[\left(\frac{1}{2b}\right)\overline{A}_\alpha + \left(\frac{a}{2b^2}\right)\underline{B}_\alpha + \left(\frac{1}{2b}\right)\underline{A}_\alpha + \left(\frac{a}{2b^2}\right)\overline{B}_\alpha\right], & a > 0, b < 0, \\ \left[\left(\frac{1}{2b}\right)\overline{A}_\alpha + \left(\frac{a}{2b^2}\right)\overline{B}_\alpha + \left(\frac{1}{2b}\right)\underline{A}_\alpha + \left(\frac{a}{2b^2}\right)\underline{B}_\alpha\right], & a < 0, b < 0, \\ \left[\left(\frac{1}{2b}\right)\underline{A}_\alpha + \left(\frac{a}{2b^2}\right)\overline{B}_\alpha + \left(\frac{1}{2b}\right)\overline{A}_\alpha + \left(\frac{a}{2b^2}\right)\underline{B}_\alpha\right], & a < 0, b > 0. \end{cases} \end{aligned} \tag{6}$$

Definition 6. [4] Consider two pseudo-trapezoidal fuzzy numbers:

$$\tilde{A} = (\underline{a}, a_1, a_2, \overline{a}, l_{\tilde{A}}(x), r_{\tilde{A}}(x)), \quad \tilde{B} = \left(\underline{b}, b_1, b_2, \overline{b}, l_{\tilde{B}}(x), r_{\tilde{B}}(x)\right),$$

with the following α-cut forms:

$$\begin{aligned}
&\tilde{A} = \bigcup_{\alpha} A_{\alpha}, \quad A_{\alpha} = \left[\underline{A}_{\alpha}, \overline{A}_{\alpha}\right], \quad 0 \leqslant \alpha \leqslant 1,\\
&\tilde{B} = \bigcup_{\alpha} B_{\alpha}, \quad B_{\alpha} = \left[\underline{B}_{\alpha}, \overline{B}_{\alpha}\right], \quad 0 \leqslant \alpha \leqslant 1,\\
&B_1 = [b_1, b_2] \quad A_1 = [a_1, a_2].
\end{aligned}$$

Let

$$\varphi = \frac{a_1 + a_2}{2}, \quad \phi = \frac{b_1 + b_2}{2}.$$

In what follows, fuzzy arithmetic operations are defined based on TA:

$$\begin{aligned}
&\tilde{A} + \tilde{B} = \bigcup_{\alpha} \left(\tilde{A} + \tilde{B}\right)_{\alpha},\\
&\left(\tilde{A} + \tilde{B}\right)_{\alpha} = \left[\frac{\phi + \varphi}{2} + \left(\frac{\underline{A}_{\alpha} + \underline{B}_{\alpha}}{2}\right), \frac{\phi + \varphi}{2} + \left(\frac{\overline{A}_{\alpha} + \overline{B}_{\alpha}}{2}\right)\right],
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\tilde{A} - \tilde{B} = \bigcup_{\alpha} \left(\tilde{A} - \tilde{B}\right)_{\alpha},\\
&\left(\tilde{A} - \tilde{B}\right)_{\alpha} = \left[\frac{\phi - 3\varphi}{2} + \left(\frac{\underline{A}_{\alpha} + \underline{B}_{\alpha}}{2}\right), \frac{\phi - 3\varphi}{2} + \left(\frac{\overline{A}_{\alpha} + \overline{B}_{\alpha}}{2}\right)\right],
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\tilde{A}.\tilde{B} = \bigcup_{\alpha} \left(\tilde{A}.\tilde{B}\right)_{\alpha},\\
&\left(\tilde{A}.\tilde{B}\right)_{\alpha} = \begin{cases}
\left[\left(\frac{\varphi}{2}\right)\underline{A}_{\alpha} + \left(\frac{\phi}{2}\right)\underline{B}_{\alpha} + \left(\frac{\varphi}{2}\right)\overline{A}_{\alpha} + \left(\frac{\phi}{2}\right)\overline{B}_{\alpha}\right], & \phi > 0, \varphi > 0,\\
\left[\left(\frac{\varphi}{2}\right)\overline{A}_{\alpha} + \left(\frac{\phi}{2}\right)\underline{B}_{\alpha} + \left(\frac{\varphi}{2}\right)\underline{A}_{\alpha} + \left(\frac{\phi}{2}\right)\overline{B}_{\alpha}\right], & \phi > 0, \varphi < 0,\\
\left[\left(\frac{\varphi}{2}\right)\overline{A}_{\alpha} + \left(\frac{\phi}{2}\right)\overline{B}_{\alpha} + \left(\frac{\varphi}{2}\right)\underline{A}_{\alpha} + \left(\frac{\phi}{2}\right)\underline{B}_{\alpha}\right], & \phi < 0, \varphi < 0,\\
\left[\left(\frac{\varphi}{2}\right)\underline{A}_{\alpha} + \left(\frac{\phi}{2}\right)\overline{B}_{\alpha} + \left(\frac{\varphi}{2}\right)\overline{A}_{\alpha} + \left(\frac{\phi}{2}\right)\underline{B}_{\alpha}\right], & \phi < 0, \varphi > 0,
\end{cases}
\end{aligned} \tag{9}$$

$$\tilde{A}^{-1} = \bigcup_{\alpha} \left(\tilde{A}^{-1}\right)_{\alpha}, \left(\tilde{A}^{-1}\right)_{\alpha} = \left[\frac{1}{\phi^2}\underline{A}_{\alpha}, \frac{1}{\phi^2}\overline{A}_{\alpha}\right], \tag{10}$$

$$\tilde{A}.\tilde{B}^{-1} = \bigcup_{\alpha} \left(\tilde{A}.\tilde{B}^{-1}\right)_{\alpha},$$

$$\left(\tilde{A}.\tilde{B}^{-1}\right)_{\alpha} = \begin{cases} \left[\left(\frac{1}{2\varphi}\right)\underline{A}_{\alpha} + \left(\frac{\phi}{2\varphi^2}\right)\underline{B}_{\alpha} + \left(\frac{1}{2\varphi}\right)\overline{A}_{\alpha} + \left(\frac{\phi}{2\varphi^2}\right)\overline{B}_{\alpha}\right], & \phi > 0, \varphi > 0, \\ \left[\left(\frac{1}{2\varphi}\right)\overline{A}_{\alpha} + \left(\frac{\phi}{2\varphi^2}\right)\underline{B}_{\alpha} + \left(\frac{1}{2\varphi}\right)\underline{A}_{\alpha} + \left(\frac{\phi}{2\varphi^2}\right)\overline{B}_{\alpha}\right], & \phi > 0, \varphi < 0, \\ \left[\left(\frac{1}{2\varphi}\right)\overline{A}_{\alpha} + \left(\frac{\phi}{2\varphi^2}\right)\overline{B}_{\alpha} + \left(\frac{1}{2\varphi}\right)\underline{A}_{\alpha} + \left(\frac{\phi}{2\varphi^2}\right)\underline{B}_{\alpha}\right], & \phi < 0, \varphi < 0, \\ \left[\left(\frac{1}{2\varphi}\right)\underline{A}_{\alpha} + \left(\frac{\phi}{2\varphi^2}\right)\overline{B}_{\alpha} + \left(\frac{1}{2\varphi}\right)\overline{A}_{\alpha} + \left(\frac{\phi}{2\varphi^2}\right)\underline{B}_{\alpha}\right], & \phi < 0, \varphi > 0. \end{cases} \tag{11}$$

Definition 7. [4] Let $F_C(R)$ be a set of pseudo-geometric fuzzy numbers defined on a set of real numbers. Then, for each $\tilde{A}$ there exists $0_{\tilde{A}}$ such that

$$\tilde{A} + 0_{\tilde{A}} = 0_{\tilde{A}} + \tilde{A} = \tilde{A}, \quad \tilde{A} - \tilde{A} = 0_{\tilde{A}},$$

for $\tilde{A} = (\underline{a}, a, \overline{a}, l_{\tilde{A}}(x), r_{\tilde{A}}(x))$, we have:

$$0_{\tilde{A}} = (\underline{a} - a, 0, \overline{a} - a, l_{\tilde{A}}(x + a), r_{\tilde{A}}(x + a)), \tag{12}$$

and for $\tilde{A} = (\underline{a}, a_1, a_2, \overline{a}, l_{\tilde{A}}(x), r_{\tilde{A}}(x))$, we have:

$$0_{\tilde{A}} = \left(\underline{a} - \phi, \frac{a_1 - a_2}{2}, \frac{a_2 - a_1}{2}, \overline{a} - \phi, l_{\tilde{A}}(x + \phi), r_{\tilde{A}}(x + \phi)\right). \tag{13}$$

Definition 8. [4] Let $\tilde{A}$ and $\tilde{B}$ be two NCC fuzzy sets. Then,

$$\tilde{A} \cong \tilde{B} \quad \textit{if and only if} \quad ac\left(\tilde{A}\right) = ac\left(\tilde{B}\right).$$

3 The Proposed Method

Definition 9. Let $\tilde{A} = \left[\tilde{a}_{ij}\right]$ and $\tilde{B} = \left[\tilde{b}_{ij}\right]$, $1 \leqslant j,\ j \leqslant n$ be fuzzy matrices. It is said that $\tilde{A} \cong \tilde{B}$, if:

$$\forall\ 1 \leqslant j,\ j \leqslant n,\ ac\left(\tilde{a}_{ij}\right) = ac\left(\tilde{b}_{ij}\right).$$

Definition 10. Let $\tilde{A} = \left[\tilde{a}_{ij}\right]$, $1 \leqslant j,\ j \leqslant n$ be a fuzzy matrix. The corresponding zero matrix is shown by $O_{\tilde{A}}$ and can be defined as follows:

$$O_{\tilde{A}} = \begin{pmatrix} 0_{\tilde{a}_{11}} & 0_{\tilde{a}_{12}} & \cdots & 0_{\tilde{a}_{1n}} \\ 0_{\tilde{a}_{21}} & 0_{\tilde{a}_{22}} & \cdots & 0_{\tilde{a}_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{\tilde{a}_{n1}} & 0_{\tilde{a}_{n2}} & \cdots & 0_{\tilde{a}_{nn}} \end{pmatrix},$$

where $0_{\tilde{a}_{ij}}$, $1 \leqslant i,\ j \leqslant n$ is determined based on (12) and (13).

Lemma 1. If $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are fuzzy matrices, then:

i. $\tilde{A}-\tilde{A} \cong O_{\tilde{A}}$,

ii. $\tilde{A}+O_{\tilde{A}} \cong \tilde{A}$,

iii. $\tilde{A}.\tilde{B}=\tilde{B}.\tilde{A}$,

iv. $\tilde{A}.\tilde{B}+\tilde{A}+\tilde{C} \cong \tilde{A}.\left(\tilde{B}+\tilde{C}\right)$.

Definition 11. The determinant of a 2×2 fuzzy matrix $\tilde{A}=\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix}$ is shown by $|\tilde{A}|$ and is defined by

$$|\tilde{A}| = (\tilde{a}_{11}.\tilde{a}_{22}) - (\tilde{a}_{12}.\tilde{a}_{21}). \tag{14}$$

Definition 12. Let $\tilde{A}=\left[\tilde{a}_{ij}\right]_{n\times n}$. The (i,j)-minor of $\tilde{A}$, which is the determinant of the matrix of $\tilde{A}$ formed by deleting the i-th row and j-th column of $\tilde{A}$, is denoted by $\tilde{M}_{ij}$.

Example 1. Consider the following fuzzy matrix:

$$\tilde{A}=\begin{pmatrix} (-4,1,2,-,-) & (5,6,7,-,-) & \left(1,2,4,\left(1-(x-2)^2\right)^{\frac{1}{2}},\left(1-\frac{1}{4}(x-2)^2\right)^{\frac{1}{2}}\right) \\ (1,2,3,-,-) & (6,6,7,-,-) & (1,2,3,-,-) \\ (-4,1,2,-,-) & (4,5,7,-,-) & (-7,2,3,-,-) \end{pmatrix}.$$

The $(3,1)$-minor of $\tilde{A}$ is obtained as:

$$\tilde{M}_{31}=\begin{pmatrix} (5,6,7,-,-) & \left(1,2,4,\left(1,(x-2)^2\right)^{\frac{1}{2}},\left(1-\frac{1}{4}(x-2)^2\right)^{\frac{1}{2}}\right) \\ (6,6,7,-,-) & (1,2,3,-,-) \end{pmatrix}.$$

Definition 13. Consider the fuzzy matrix

$$\tilde{A}=\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{nn} \end{pmatrix}.$$

The (i,j) element of the cofactor matrix of $\tilde{A}$ is shown by $\tilde{A}_{ij}$ and is defined as follows:

$$\tilde{A}_{ij}=(-1)^{i+j}|\tilde{M}_{ij}|.$$

Example 2. Consider the matrix $\tilde{A}$ of Example 1. Using Definitions 11 to 13 and the TA-based arithmetic operations (2) to (6), it is obtained:

$$\tilde{A}_{13}=\left(-\frac{21}{4},4,8,-,-\right).$$

Definition 14. (Expansion method for calculating the determinant of an $n \times n$ fuzzy matrix). Consider the fuzzy matrix

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{nn} \end{pmatrix}.$$

The determinant of the matrix can be evaluated by expanding each row or column of the matrix. For example, by expanding on the first row, we have:

$$|\tilde{A}| = \tilde{a}_{11}.\tilde{A}_{11} + \tilde{a}_{12}.\tilde{A}_{12} + \cdots + \tilde{a}_{1n}.\tilde{A}_{1n}. \tag{15}$$

Example 3. Consider the matrix $\tilde{A}$ of Example 1. From (15), and (2) to (6), it can be achieved that:

$$|\tilde{A}| = \bigcup_{\alpha} \left[-\frac{60}{16} - \frac{1}{2}\sqrt{1-\alpha^2} + \frac{28}{16}\alpha, \frac{29}{16} - \frac{61}{16}\alpha + \frac{17}{32}\sqrt{1-\alpha^2} \right].$$

Definition 15. The fuzzy matrix $\tilde{A}$ is called singular, if $|\tilde{A}| \cong 0_{|\tilde{A}|}$ and is called non-singular, if $ac\left(|\tilde{A}|\right) \neq 0$.

Definition 16. The following system

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{pmatrix}, \tag{16}$$

is called a fully fuzzy vector system and is denoted by $\tilde{A}\tilde{X} = \tilde{B}$, where $\tilde{A} = \left[\tilde{a}_{ij}\right]$, $1 \leqslant i$, $j \leqslant n$ is a known $n \times n$ fuzzy matrix, $\tilde{B} = \left[\tilde{b}_i\right]$ is a known $n \times 1$ fuzzy vector, and $\tilde{X} = [\tilde{x}_i]$ is an unknown $n \times 1$ fuzzy vector.

Properties of the fuzzy determinant

- If two rows or two columns of a fuzzy matrix $\tilde{A}$ are equal, then $|\tilde{A}| \cong 0_{|\tilde{A}|}$.
- In a fuzzy matrix $\tilde{A}$, if for ith row and jth column, $j = 1, 2, \ldots, n$, there is $\tilde{a}_{ij} \cong 0_{\tilde{a}_{ij}}$, then $|\tilde{A}| \cong 0_{|\tilde{A}|}$.
- For any fuzzy square matrix $\tilde{A}$, we have $|\tilde{A}| \cong |\tilde{A}^T|$.
- If two rows or two columns of a fuzzy matrix $\tilde{A}$ are switched and the obtained (or resulting) matrix is called $\tilde{B}$, then $|\tilde{B}| \cong |\tilde{A}|$.

- In a fuzzy matrix $\tilde{A} = \left[\tilde{a}_{ij}\right]$, if we have $\tilde{a}_{ij} \cong 0_{\tilde{a}_{ij}}$ for each $i, j = 1, 2, \ldots, n$, then $|\tilde{A}| = [0, 0]$.

Example 4. Consider the fuzzy matrix

$$\tilde{A} = \begin{pmatrix} \left(0,4,6,-,1-\frac{1}{4}(x-4)^2\right) & \left(0,4,6,-,1-\frac{1}{4}(x-4)^2\right) \\ (-3,-2,-1,-,-) & (-3,-2,-1,-,-) \end{pmatrix},$$

in which the first and second columns are equal. From (14), it can be found that

$$|A| = \bigcup_{\alpha} \left[-2 + 2\alpha - 2\sqrt{1-\alpha}, 6 - 6\alpha\right],$$

and as a result, $|\tilde{A}| \cong 0_{|\tilde{A}|}$.

The Fuzzy Cramer method: The solution to the fuzzy system (16) obtained using the fuzzy Cramer method is achieved as follows:

$$\tilde{x}_j = \frac{|\tilde{A}_j|}{|A|}, \qquad j = 1, 2, \ldots, n, \tag{17}$$

in which $\tilde{A}_j$ is determined by substituting $\tilde{B}$ in the jth column of $\tilde{A}$.

Theorem 1. If a fuzzy matrix $\tilde{A}$ is non-singular, then the Cramer method always has a fuzzy solution for the fuzzy system (16).

Definition 17. The following system

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix} + \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{pmatrix} = \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1n} \\ \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{c}_{n1} & \tilde{c}_{n2} & \cdots & \tilde{c}_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{x}_{n1} & \tilde{x}_{n2} & \cdots & \tilde{x}_{nn} \end{pmatrix} + \begin{pmatrix} \tilde{d}_1 \\ \tilde{d}_2 \\ \vdots \\ \tilde{d}_n \end{pmatrix}, \tag{18}$$

is called the dual fully fuzzy system and can be shown as $\tilde{A}\tilde{X}+\tilde{B} = \tilde{C}\tilde{X}+\tilde{D}$ by considering $\tilde{A} = \left[\tilde{a}_{ij}\right]_{n\times n}$, $\tilde{B} = \left[\tilde{b}_i\right]_{n\times 1}$, $\tilde{C} = \left[\tilde{c}_{ij}\right]_{n\times n}$ and $\tilde{D} = \left[\tilde{d}_i\right]_{n\times 1}$. It is assumed that the matrix $A - C = \left[a_{ij}\right] - \left[c_{ij}\right]$ is non-singular.

Now, the goal is to solve $\tilde{A}\tilde{X} + \tilde{B} = \tilde{C}\tilde{X} + \tilde{D}$. Therefore, we have:

$$\tilde{A}\tilde{X} + \tilde{B} - \tilde{B} = \tilde{C}\tilde{X} + \tilde{D} - \tilde{B}.$$

Using Lemma 1, (i) and (ii), we have:

$$\tilde{A}\tilde{X} \cong \tilde{C}\tilde{X} + \tilde{D} - \tilde{B}.$$

Adding $-\tilde{C}\tilde{X}$ to both sides of the above equation and using Lemma 1, (i) and (ii), it is obtained:

$$\tilde{A}\tilde{X} + \left(-\tilde{C}\right)\tilde{X} \cong \tilde{D} - \tilde{B}.$$

Moreover, using Lemma 1,(iv), we achieve:

$$\left(\tilde{A}-\tilde{C}\right)\tilde{X} \cong \tilde{D}-\tilde{B}. \tag{19}$$

Finally, the solution for the fuzzy system (19) is obtained as follows using the fuzzy Cramer method:

$$\tilde{x}_j = \frac{\left|\left(\tilde{A}-\tilde{C}\right)_j\right|}{\left|\left(\tilde{A}-\tilde{C}\right)\right|} \qquad \tilde{x}_j = \bigcup_{\alpha}\left(\tilde{x}_j\right)_{\alpha} \tag{20}$$

in which $\left(\tilde{A}-\tilde{C}\right)_j$ is determined by substituting the elements of $\tilde{D}-\tilde{B}$ in the j-th column of $\tilde{A}-\tilde{C}$.

Theorem 2. If the matrix $\left(\tilde{A}-\tilde{C}\right)$ is non-singular, the dual fully fuzzy system (20) has a fuzzy solution.

4 Numerical Examples

Example 5. Consider the following system of linear equations.

$$\begin{pmatrix} (-4,1,2,-,-) & (4,7,8,-,-) \\ (2,4,6,-,-) & (6,6,7,-,-) \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} (1,5,7,-,-) \\ (1,2,3,-,-) \end{pmatrix}. \tag{21}$$

Using the arithmetic operations (2) to (6), the fuzzy Cramer method (17) and determinant definition (14), we obtain:

$$\tilde{x} = \frac{\left|\begin{pmatrix} (1,5,7,-,-) & (4,7,8,-,-) \\ (1,2,3,-,-) & (6,6,7,-,-) \end{pmatrix}\right|}{\left|\begin{pmatrix} (-4,1,2,-,-) & (4,7,8,-,-) \\ (2,4,6,-,-) & (6,6,7,-,-) \end{pmatrix}\right|} = \left(-\frac{2142}{1936}, -\frac{16}{22}, -\frac{801}{1936}, -, -\right),$$

$$\tilde{y} = \frac{\left|\begin{pmatrix} (-4,1,2,-,-) & (1,5,7,-,-) \\ (2,4,6,-,-) & (1,2,3,-,-) \end{pmatrix}\right|}{\left|\begin{pmatrix} (-4,1,2,-,-) & (4,7,8,-,-) \\ (2,4,6,-,-) & (6,6,7,-,-) \end{pmatrix}\right|} = \left(\frac{1128}{1936}, \frac{18}{22}, \frac{27445}{21296}, -, -\right),$$

which is shown in Figure 1.

Example 6. Consider the following dual fuzzy system:

$$\begin{aligned} &\begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 5 \\ -2 & 4 & 1 \end{pmatrix}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} + \begin{pmatrix} (-2,0,1,1,-,-) \\ \left(1,2,4,6,x-1,\left(1-\frac{1}{4}(x,-4)^2\right)^{\frac{1}{2}}\right) \\ (-2,0,2,4,-,-) \end{pmatrix} = \\ &\begin{pmatrix} 2 & 0 & -3 \\ 1 & -2 & 0 \\ 6 & 1 & -1 \end{pmatrix}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} + \begin{pmatrix} \left(1,2,7,9,\left(1-(x-2)^2\right)^{\frac{1}{2}},\left(1-\frac{1}{4}(x-7)^2\right)^{\frac{1}{2}}\right) \\ (-3,-2,1,3,-,-) \\ (-2,0,1,1,-,-) \end{pmatrix}. \end{aligned} \tag{22}$$

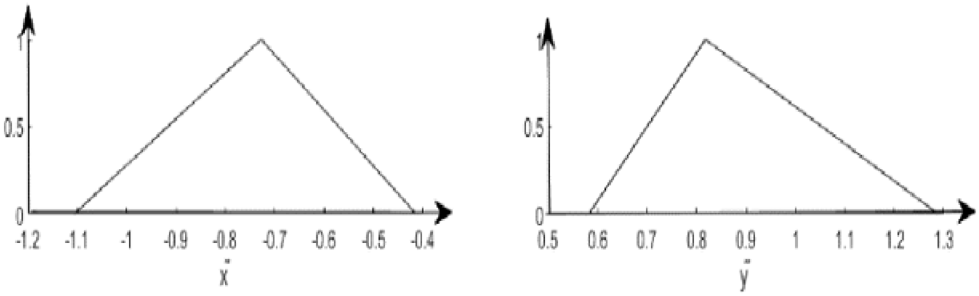

Figure 1: The fuzzy solution to Example 5.

Using (19) and the difference (8), we obtain:

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 5 \\ -8 & 3 & 2 \end{pmatrix}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{3}_3 \end{pmatrix} \cong \begin{pmatrix} \left(\frac{5}{2}, 3, 7, 8, \left(1-(6-2x)^2\right)^{\frac{1}{2}}\right) \\ \left(-5, -\frac{9}{1}, -\frac{3}{2}, -\frac{1}{2}, 2x+10, \left(1-\left(x+\frac{3}{2}\right)^2\right)^{\frac{1}{2}}\right) \\ \left(-\frac{13}{4}, -\frac{5}{4}, \frac{1}{4}, \frac{5}{4}, \frac{x}{2}+\frac{13}{8}, \frac{5}{4}-x\right) \end{pmatrix}.$$

Finally, using the fuzzy Cramer method (17), fuzzy determinant (15), and arithmetic operations (7) to (11), the solution to the dual fuzzy system (22) is obtained as follows:

$$\tilde{x}_1 = \frac{\begin{vmatrix} \left(\frac{5}{2}, 3, 7, 8, \left(1-(6-2x)^2\right)^{\frac{1}{2}}, \left(1-(x-7)^2\right)^{\frac{1}{2}}\right) & 2 & 2 \\ \left(-5, -\frac{9}{2}, -\frac{3}{2}, -\frac{1}{2}, 2x+10, \left(1-\left(x+\frac{3}{2}\right)^2\right)^{\frac{1}{2}}\right) & 2 & 5 \\ \left(-\frac{13}{4}, -\frac{5}{4}, \frac{1}{4}, \frac{5}{4}, \frac{x}{2}+\frac{13}{8}, \frac{5}{4}-x\right) & 3 & 2 \end{vmatrix}}{\begin{vmatrix} -1 & 2 & 2 \\ 2 & 2 & 5 \\ -8 & 3 & 2 \end{vmatrix}}$$

$$= \bigcup_{\alpha}\left[\frac{7523}{4224} + \frac{36}{4224}\alpha + \frac{12}{4224}\sqrt{1-\alpha^2}, \frac{8551}{4224} - \frac{88}{4224}\alpha + \frac{88}{4224}\sqrt{1-\alpha^2}\right],$$

$$\tilde{x}_2 = \frac{\begin{vmatrix} -1 & \left(\frac{5}{2}, 3, 7, 8, \left(1-(6-2x)^2\right)^{\frac{1}{2}}, \left(1-(x-7)^2\right)^{\frac{1}{2}}\right) & 2 \\ 2 & \left(-5, -\frac{9}{2}, -\frac{3}{2}, -\frac{1}{2}, 2x+10, \left(1-\left(x+\frac{3}{2}\right)^2\right)^{\frac{1}{2}}\right) & 5 \\ -8 & \left(-\frac{13}{4}, -\frac{5}{4}, \frac{1}{4}, \frac{5}{4}, \frac{x}{2}+\frac{13}{8}, \frac{5}{4}-x\right) & 2 \end{vmatrix}}{\begin{vmatrix} -1 & 2 & 2 \\ 2 & 2 & 5 \\ -8 & 3 & 2 \end{vmatrix}}$$

$$= \bigcup_{\alpha}\Big[-\frac{8069}{8448}+\frac{35}{2112}\alpha-\frac{11}{132}\sqrt{1-\alpha^2}, -\frac{4595}{8448}-\frac{21}{2112}\alpha+\frac{61}{1056}\sqrt{1-\alpha^2}\Big],$$

$$\tilde{x}_3 = \frac{\begin{vmatrix} -1 & 2 & \left(\frac{5}{2},3,7,8,\left(1-(6-2x)^2\right)^{\frac{1}{2}},\left(1-(x-7)^2\right)^{\frac{1}{2}}\right) \\ 2 & 2 & \left(-5,-\frac{9}{2},-\frac{3}{2},-\frac{1}{2},2x+10,\left(1-\left(x+\frac{3}{2}\right)^2\right)^{\frac{1}{2}}\right) \\ -8 & 3 & \left(-\frac{13}{4},-\frac{5}{4},\frac{1}{4},\frac{5}{4},\frac{x}{2}+\frac{13}{8},\frac{5}{4}-x\right) \end{vmatrix}}{\begin{vmatrix} -1 & 2 & 2 \\ 2 & 2 & 5 \\ -8 & 3 & 2 \end{vmatrix}}$$
$$= \bigcup_{\alpha}\Big[-\frac{14367}{4224}+\frac{35}{4224}\alpha-\frac{64}{4224}\sqrt{1-\alpha^2}, -\frac{13936}{4224}-\frac{20}{4224}\alpha+\frac{70}{4224}\sqrt{1-\alpha^2}\Big],$$

which is shown in Figure 2.

Example 7. Consider the following fuzzy system:

$$\begin{pmatrix} (-1,1,3,-,-) & \left(1,2,4,\left(1-(x-2)^2\right)^{\frac{1}{2}},\left(1-\frac{1}{4}(x-2)^2\right)^{\frac{1}{2}}\right) & (1,3,6,-,-) \\ (-2,-2,2,-,-) & \left(3,4,5,\left(1-(x-4)^2\right)^{\frac{1}{2}},5-x\right) & \left(4,7,9,\frac{1}{3}(x-4),\left(1-\frac{1}{4}(x-7)^2\right)^{\frac{1}{2}}\right) \\ (2,3,5,-,-) & (2,6,8,-,-) & \left(8,9,10,x-8,\left(1-(x-9)^2\right)^{\frac{1}{2}}\right) \end{pmatrix}\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \tilde{x}_3\end{pmatrix} \tag{23}$$
$$= \begin{pmatrix} (2,4,7,-,-) \\ \left(1,2,6,\left(1-(x-2)^2\right)^{\frac{1}{2}},\left(1-\frac{1}{16}(x-2)^2\right)^{\frac{1}{2}}\right) \\ (3,6,8,-,-) \end{pmatrix}.$$

Assuming:

$$\tilde{A} = \begin{pmatrix} (-1,1,3,-,-) & \left(1,2,4,\left(1-(x-2)^2\right)^{\frac{1}{2}},\left(1-\frac{1}{4}(x-2)^2\right)^{\frac{1}{2}}\right) & (1,3,6,-,-) \\ (-2,-2,2,-,-) & \left(3,4,5,\left(1-(x-4)^2\right)^{\frac{1}{2}},5-x\right) & \left(4,7,9,\frac{1}{3}(x-4),\left(1-\frac{1}{4}(x-7)^2\right)\right)^{\frac{1}{2}} \\ (2,3,5,-,-) & (2,6,8,-,-) & \left(8,9,10,x-8,\left(1,(x-9)^2\right)^{\frac{1}{2}}\right) \end{pmatrix},$$

we have $ac\left(|\tilde{A}|\right) = 0$ (because $\begin{vmatrix} 1 & 2 & 3 \\ -2 & 4 & 7 \\ 3 & 6 & 9 \end{vmatrix} = 0$), that is, the coefficient matrix of the system is singular, and the system (23) does not have a fuzzy solution.

Example 8. Suppose you want to calculate the approximate prices of pistachios and almonds in 1390, while you know that one of your colleagues bought about 1 kg ($\tilde{1} = (0,1,2,-,-)$) of pistachios and about 2 kg ($\tilde{2} = (1,2,4,-,-)$) of almonds this year at a price of about 10 tomans ($\tilde{10} = (8,10,11,-,-)$), and your other colleague bought about 3 kg ($\tilde{3} = (2,3,5,-,-)$) of pistachios and about 4 kg ($\tilde{4} = (3,4,7,-,-)$) of almonds

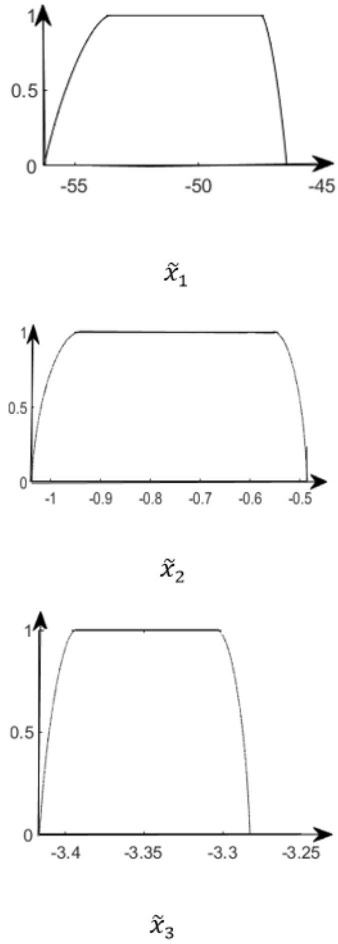

Figure 2: The fuzzy solution to Example 6.

this year at a price of about 24 tomans ($\tilde{24} = (22, 24, 25, -, -)$). (Shopping malls had incentive packages for shoppers.) Now, considering the approximate price of pistachios as $\tilde{x}$ and the approximate price of almonds as $\tilde{y}$, we form the following fuzzy system:

$$\begin{cases} \tilde{1}.\tilde{x} + \tilde{2}.\tilde{y} = \tilde{10}, \\ \tilde{3}.x + \tilde{4}.\tilde{y} = \tilde{24}. \end{cases} \tag{24}$$

Using the fuzzy Cramer method (17), fuzzy determinant (14), and TA-based arithmetic operations (2) to (6), the solution to the fuzzy system (24) is obtained:

$$\tilde{x} = \frac{\begin{vmatrix} (8, 10, 11, -, -) & (1, 2, 4, -, -) \\ (22, 24, 25, -, -) & (3, 4, 7, -, -) \end{vmatrix}}{\begin{vmatrix} (0, 1, 2, -, -) & (1, 2, 4, -, -) \\ (2, 3, 5, -, -) & (3, 4, 7, -, -) \end{vmatrix}} = \left(-\frac{37}{8}, 4, \frac{75}{8}, -, -\right),$$

$$\tilde{y} = \frac{\begin{vmatrix} (0,1,2,-,-) & (8,10,11,-,-) \\ (2,3,5,-,-) & (22,24,25,-,-) \end{vmatrix}}{\begin{vmatrix} (0,1,2,-,-) & (1,2,4,-,-) \\ (2,3,5,-,-) & (3,4,7,-,-) \end{vmatrix}} = \left(-3,3,\frac{15}{2},-,-\right).$$

That is, the price of pistachios was about 4 tomans ($\tilde{4} = \left(-\frac{37}{8},4,\frac{75}{8},-,-\right)$), and the price of almonds was about 3 tomans ($\tilde{3} = \left(-3,3,\frac{15}{2},-,-\right)$).

5 Conclusion

An analytical method for solving a system of fuzzy linear equations is the Cramer method. We can find some limitations in the methods used in the literature. The methods based on arithmetic operations using the extension principle and α-cuts have problems in subtraction and division operations, as well as problems in attaining membership functions for the operators and also the dependence effect in the fuzzy arithmetic operations. Therefore, in this paper, using TA-based fuzzy arithmetic, which is more realistic than other arithmetic operations, we solved a fuzzy system by a Cramer method, which does not have the limitations of the other methods presented by e.g., Allahviranloo et al. or Radhakrishnan et al. In other words, the proposed method was used for all fuzzy systems such as the fully fuzzy and the dual fuzzy systems with all numbers such as quasi-triangular and quasi-trapezoidal numbers as inputs and calculates all the solutions of the fuzzy systems, including non-negative and non-positive solutions. Finally, using the proposed method and assuming that the 1-cut coefficient matrix of the fuzzy system is non-singular, the fuzzy system always contains a fuzzy solution.

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