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Research Article

The Smallest Number of Colors Needed for a Coloring of the Square of the Cartesian Product of Certain Graphs

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Abstract. Given any graph G, its square graph G^2 has the same vertex set as G, with two vertices adjacent in G^2 whenever they are at distance 1 or 2 in G. The Cartesian product of graphs G and H is denoted by $G \square H$. One of the most studied NP-hard problems is the graph coloring problem. A method such as Genetic Algorithm (GA) is highly preferred to solve the Graph Coloring problem by researchers for many years. In this paper, we use the graph product approach to this problem. In fact, we prove that $\chi((D(m', n') \square D(m, n))^2) \leqslant 10$ for $m, n \geqslant 3$, where $D(m, n)$ is the graph obtained by joining a vertex of the cycle C_m to a vertex of degree one of the paths P_n and $\chi(G)$ is the chromatic number of the graph G.

Keywords. 2-Distance coloring; Chromatic number; Cartesian product; Dragon graph.

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1 Introduction

Let $G = (V, E)$ be a finite and simple graph. For any graph G, we denote the vertex-set and the edge-set of G by $V(G)$ and $E(G)$, respectively. A proper vertex k-coloring of a graph G is a mapping $c : V(G) \to \{1, \dots, k\}$, with the property that $c(u) \neq c(v)$ whenever $uv \in E(G)$. The smallest k for which there exists a k-coloring of G, called the chromatic number of G, is denoted by $\chi(G)$. Graph coloring has numerous applications in scheduling and other practical problems. The square of a graph G, denoted by G^2, is a graph with $V(G) = V(G^2)$, in which two vertices are adjacent if their distance in G is at most two. A 2-distance coloring of G is a vertex coloring of G such that any two distinct vertices at distance less than or equal to 2 are assigned different colors. The 2-distance chromatic number of a graph G is the minimum number of colors necessary to have a 2-distance coloring of G, which is denoted by $\chi_2(G)$. Hence $\chi_2(G)$ is equal to $\chi(G^2)$. The 2-distance coloring of graphs was introduced by Wegner in [15]. Wegner conjectured that if G is a planar graph with maximum degree $\Delta(G) \geqslant 8$, then $\chi(G^2) \leqslant \lceil 3/2\Delta(G) \rceil + 1$. The best known upper bound is $\lceil 5/3\Delta(G) \rceil + 78$ for all Δ in [9]. The problem of determining the chromatic number of the square of particular graphs has attracted a lot of attention, with a particular focus on the square of planar graphs (see, for example [3, 5, 14, 7, 9, 4]). The Cartesian product of graphs G and H is the graph $G \square H$ with the vertex set $V(G) \times V(H)$ and $(x_1, x_2)(y_1, y_2) \in E(G \square H)$ whenever $x_1 y_1 \in E(G)$ and $x_2 = y_2$ or $x_2 y_2 \in E(H)$ and $x_1 = y_1$, see [13]. The subgraph of $G \square H$ induced by $u \times V(H)$ is isomorphic to H. It is called an $H-$fiber and denoted by H^u. An (m, n)-dragon graph denoted by $D(m, n)$ is the graph obtained by joining a vertex of the cycle C_m to a vertex of degree one of the path P_n introduced in [8]. It has $(m + n)$ vertices and $(m + n)$ edges. The Dragon graph $D(6, 3)$ is shown in Figure 1.

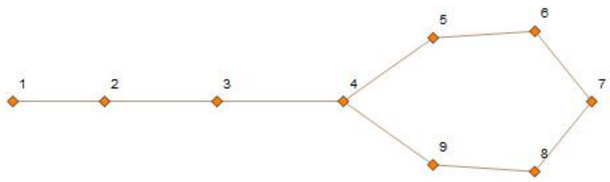

Figure 1: Dragon graph D(6,3).

In [6], Jamison et al. established acyclic colorings of products of trees. Chiang and Yan studied the chromatic number of the square of Cartesian products of paths and cycles and proved the following result.

Theorem 1 (Chiang and Yan [2])**.** If $G = C_m \square P_n$ with $m \geqslant 3$ and $n \geqslant 2$, then

$$\chi(G^2) = \begin{cases} 4, & if \ \ n = 2 \text{ and } m \equiv 0 (mod\ \ 4), \\ 6, & if \ \ n = 2 \text{ and } m \in \{3, 6\}, \\ 6, & if \ \ n \geqslant 3 \text{ and } m \not\equiv 0 (mod 5), \\ 5, & \text{Otherwise.} \end{cases} \tag{1}$$

Theorem 2 ([11])**.** If $T_{m,n} = C_m \square C_n$ with $3 \geqslant m \geqslant n$, then $\chi(T^2_{m,n}) \leqslant 7$ except $\chi(T^2_{3,3}) = 9$ and $\chi(T^2_{3,5}) = \chi(T^2_{4,4}) = 8$.

Shao and Vesel in [10] worked on the chromatic number square of Cartesian product of two cycles and proved the following theorem.

Theorem 3. If $m,n \geqslant 40$, then $\chi(T^2_{m,n}) \leqslant 6$.

In [1], Chegini et al. studied the square chromatic number of the torus and proved the following.

Theorem 4. If $m,n \in S(5,6)$, then $\chi(T^2_{m,n}) \leqslant 6$.

In this paper, the 2-distance chromatic numbers of Cartesian products of square two dragon graphs are investigated. In particular, we extend Theorem 4. In fact we establish the 2-distance chromatic number for $(D(m',n')\Box D(m,n))$ and we find an upper bound for $\chi_2(D(m',n')\Box D(m,n))$ for all $n,m,m',n' \geq 3$.

2 Main Results

Given two integers x and y, let $S(x,y)$ denote the set of all non-negative integer combinations of x and y defined as follows.

$$S(x,y) = \{\alpha x + \beta y : \alpha, \beta \ \textit{non-negative integers}\}.$$

To prove the main theorems, we need the following auxiliary lemmas. The following two lemmas are necessary and useful in proving the main theorems.

Lemma 1 ([12])**.** If x and y are relatively prime integers greater than 1, then $n \in S(x,y)$ for all $n \geqslant (x-1)(y-1)$.

Lemma 2. Let $m,m',m'' \geq 3$, $n,n',n'' \geq 3$, $s,r \geq 1$, $m'' \leq m'$, and $k \geq 9$ be integers and let f be a k-coloring of $(D(m',n')\Box D(m,n))^2$. Consider g as the restriction of f to $(V(D^i_{(m',n')}),\dots,V(D^{i+m''+n''-1}_{(m',n')}))$. If g is a k-coloring of $(D(m'',n'')\Box D(m,n))^2$, then

$$\chi((D(m'+(s-1)m'',n'+(r-1)n'')\Box D(m,n))^2) \leq k.$$

Proof. Define a function h from $V((D(m'+(s-1)m'',n'+(r-1)n'')\Box D(m,n))^2)$ onto the set $\{1,2,\dots,k\}$ by

$$h(i,j) = \begin{cases} f(i,j) & if \ i < m', \\ f(i-m' \equiv (mod \ m'),j) & if \ m' \leq i \leq m'+(s-1)m'', \\ g(i,j) & if \ m'+(s-1)m'' < i \leq m'+(s-1)m''+n'-1, \\ t_{i,j} \in A & if \ i = m'+(s-1)m''+n', \\ g(i-3,j) & if \ i > m'+(s-1)m''+n', \end{cases}$$

where

$$A = \Big\{t_{i,j} \in \{1,2,\dots,k\} \ | t_{i,m-1} \notin \{t_{i,m-2}, t_{i,m-3}, t_{i,1}, g(i-1,m-2), g(i-1,m-1), g(i-1,m),$$
$$g(i-2,m-2), g(i-2,m-1), g(i-2,m)\}, t_{i,m} \notin \{t_{i,m-1}, t_{i,m-2}, t_{i,1},$$
$$t_{i,2}, g(i-1,1), g(i-1,m-1), g(i-1,m), g(i-1,m+1), g(i-2,m)\}, t_{i,j} \notin \{t_{i,j-1}, t_{i,j-2},$$

$g(i-1,j-1), g(i-1,j), g(i-1,j+1), g(i-2,j-1), g(i-2,j), g(i-2,j+1)\}, g(i,0)$
$\overset{m}{\equiv} g(i,m)$ *and* $t_{i,-1}, t_{i,0} \in \emptyset\Big\}$.

In order to see that h is a k-coloring of $V(D(m'+(s-1)m'', n'+(r-1)n'')\Box D(m,n))^2$, consider first the vertex (m',j). This vertex is adjacent to $(m'-2,j)$, $(m'-1,j)$, $(m'-1,j-1)$, and $(m'-1,j+1)$. Note that $h(m',j) = f(0,j)$, since $(0,j)$ is also adjacent to $(m'-2,j)$, $(m'-1,j)$, $(m'-1,j-1)$, and $(m'-1,j+1)$ in $(D(m',n')\Box D(m,n))^2$. Moreover, f is a k-coloring of $(D(m',n')\Box D(m,n))^2$. Thus the case where $(m',j) \in V(D(m'+(s-1)m'', n'+(r-1)n'')\Box D(m,n)), j = 1,2,...,m+n$ is proved.
If $i = m'+(s-1)m''+n'$, according to the definition of $t_{i,j}$, it is obvious that $h(i,j)$ is a k-coloring of $\{(i,j)|(i,j) \in V(D(m'+(s-1)m'', n'+(r-1)n'')\Box D(m,n)), i \le m'+(s-1)m''+n'$ *and* $j = 1,2,...,m+n\}$.
The vertex $(m'+(s-1)m''+n'+1,j)$ is adjacent to $(m'+(s-1)m''+n'+1,j-2)$, $(m'+(s-1)m''+n'+1,j-1)$, $(m'+(s-1)m''+1,j+1)$, $(m'+(s-1)m''+n'+1,j+2)$, $(m'+(s-1)m''+n',j-1)$, $(m'+(s-1)m''+n',j)$, $(m'+(s-1)m''+n',j+1)$ and $(m'+(s-1)m''+n'-1,j)$. Note that $h(i,j) = g(i-3,j)$ for $i > m'+(s-1)m''+n'$, since g is a k-coloring of $(D(m'',n'')\Box D(m,n))^2$. Therefore, g assigns distinct colors of the vertex $(m'+(s-1)m''+n'+1,j)$ to the vertices $(m'+(s-1)m''+n'+1,j-2)$, $(m'+(s-1)m''+n'+1,j-1)$, $(m'+(s-1)m''+1,j+1)$ and $(m'+(s-1)m''+n'+1,j-2)$.
Moreover, according to the definition of $t_{i,j}$, we have $h(m'+(s-1)m''+n'+1,j)$ such that $h(m'+(s-1)m''+n',j)$, $h(m'+(s-1)m''+n',j-1)$, $h(m'+(s-1)m''+n',j+1)$ and $h(m'+(s-1)m''+n'-1,j)$ are distinct. The proof for the other vertices is similar. Therefore, $h(i,j)$ is a k-coloring of $\{(i,j)|(i,j) \in V(D(m'+(s-1)m'', n'+(r-1)n'')\Box D(m,n)), i > m'+(s-1)m''+n'$ *and* $j = 1,2,...,m+n\}$. This completes the proof. □

Theorem 5. If $m', m \in S(6,5)$ and $n', n \in S(4,3)$, then

$$\chi((D(m',n')\Box D(m,n))^2) \le 10.$$

Proof. Let $m', m \in S(6,5)$ and let $n', n \in S(4,3)$. For convenience, a 10-coloring of $(D(m',n')\Box D(m,n))^2$ will be represented as $(m'+n')$-by-$(m+n)$ patterns where the entry on the ith row, jth column will be the coloring of vertex (i,j). We shall construct explicit colorings using combinations of patterns given in matrix form. Each pattern can be thought of as a proper coloring of the square of the dragon graph of the same size. The following patterns A_1, A_2, A_3 and A_4, depicted in Figure 2 provide in an obvious way a proper 10-coloring of the following graphs respectively.

$$(D(5,3)\Box D(5,4))^2, (D(6,4)\Box D(5,4))^2, (D(5,3)\Box D(6,3))^2, (D(6,4)\Box(6,3))^2.$$

Applying Lemmas 1 and 2 and combinations of A_1 and A_2, we obtain an $((m'+n')\times 9)$ pattern A_5 for graph $(D(m',n')\Box D(5,4))^2$, and using combinations of A_3 and A_4, we get an $((m'+n')\times 9)$ pattern A_6 for the graph $(D(m',n')\Box D(6,3))^2$. Moreover, using combinations of A_5 and A_6, we obtain an $((m'+n')\times(m+n))$ pattern for $(D(m',n')\Box D(m,n))^2$. This provides a proper 10-coloring of $(D(m',n')\Box D(m,n))^2$. □

By Theorem 5, we only need to find the value of $\chi((D(m',n')\Box D(m,n))^2)$ for $m' \in \{3,4,7,8,9,13,14,19\}$ and $n' = 5$.

$$A_1=\begin{array}{|ccccccccc|}\hline 1&2&7&4&1&2&5&7&3\\ 3&4&8&10&5&3&4&1&6\\ 2&5&6&9&4&1&6&3&2\\ 1&3&2&7&6&2&7&4&5\\ 5&6&9&10&3&4&8&2&1\\ 2&10&1&8&7&9&3&6&4\\ 1&5&10&9&5&6&2&1&3\\ 7&6&3&7&8&3&4&5&2\\ 3&4&5&10&1&2&7&6&4\\ \hline\end{array}\quad A_2=\begin{array}{|ccccccccc|}\hline 1&2&7&4&1&2&5&7&3\\ 3&4&8&10&5&3&4&1&6\\ 2&5&6&9&4&1&6&3&2\\ 1&3&2&7&6&2&7&4&5\\ 5&6&9&10&3&4&8&2&1\\ 2&10&1&8&7&9&3&6&4\\ 1&5&10&9&5&6&2&1&3\\ 7&6&3&7&8&3&4&5&2\\ 3&4&5&10&1&2&7&6&4\\ 2&1&9&2&7&6&1&3&8\\ \hline\end{array}$$

$$A_3=\begin{array}{|ccccccccc|}\hline 1&2&7&4&1&2&5&7&3\\ 3&4&8&10&5&3&4&1&6\\ 2&5&6&9&4&1&6&3&2\\ 1&3&2&7&6&2&7&4&5\\ 5&6&9&10&3&4&8&2&1\\ 2&10&1&8&7&9&3&6&4\\ 1&5&10&9&5&6&2&1&3\\ 7&6&3&7&8&3&4&5&2\\ 3&4&5&10&1&2&7&6&4\\ \hline\end{array}\quad A_4=\begin{array}{|ccccccccc|}\hline 1&2&7&4&1&2&5&7&3\\ 3&4&8&10&5&3&4&1&6\\ 2&5&6&9&4&1&6&3&2\\ 1&3&2&7&6&2&7&4&5\\ 5&6&9&10&3&4&8&2&1\\ 2&10&1&8&7&9&3&6&4\\ 1&5&10&9&5&6&2&1&3\\ 7&6&3&7&8&3&4&5&2\\ 3&4&5&10&1&2&7&6&4\\ 2&1&9&2&7&6&1&3&8\\ \hline\end{array}$$

Figure 2: Patterns for Theorem 5.

Theorem 6. If $m \notin \{3,4,7,8,9,13,14,19\}$ and $n \neq 5$, then

$$\chi((D(3,n')\square D(m,n))^2) \leqslant 10.$$

Proof. Consider the following patterns B_1, B_2, B_3, and B_4 for graphs

$$(D(3,3)\square D(5,4))^2, (D(3,4)\square D(5,4))^2, (D(3,3)\square D(6,3))^2 \text{ and } (D(3,4)\square D(6,3))^2,$$

respectively.

$$B_1=\begin{array}{|ccccccccc|}\hline 1&2&9&10&1&2&5&7&3\\ 3&4&1&8&5&3&4&1&6\\ 7&5&6&7&4&8&6&9&2\\ 4&3&2&9&6&10&7&4&5\\ 5&6&4&10&3&4&5&2&1\\ 3&2&1&5&2&1&3&6&4\\ \hline\end{array}\quad B_2=\begin{array}{|ccccccccc|}\hline 1&2&9&10&1&2&5&7&3\\ 3&4&1&8&5&3&4&1&6\\ 7&5&6&7&4&8&6&9&2\\ 4&3&2&9&6&10&7&4&5\\ 5&6&4&10&3&4&5&2&1\\ 3&2&1&5&2&1&3&6&4\\ 4&7&3&8&7&6&8&9&10\\ \hline\end{array}$$

$$B_3=\begin{array}{|ccccccccc|}\hline 1&2&9&10&1&2&5&7&3\\ 3&4&1&8&5&3&4&1&6\\ 7&5&6&7&4&8&6&9&2\\ 4&3&2&9&6&10&7&4&5\\ 5&6&4&10&3&4&5&2&1\\ 3&2&1&5&2&1&3&6&4\\ \hline\end{array}\quad B_4=\begin{array}{|ccccccccc|}\hline 1&2&9&10&1&2&5&7&3\\ 3&4&1&8&5&3&4&1&6\\ 7&5&6&7&4&8&6&9&2\\ 4&3&2&9&6&10&7&4&5\\ 5&6&4&10&3&4&5&2&1\\ 3&2&1&5&2&1&3&6&4\\ 4&7&3&8&7&6&8&9&10\\ \hline\end{array}$$

By combinations of B_1 and B_2, we get a $((3+n')\times 9)$ pattern B_5 for the graph $(D(3,n')\square D(5,4))^2$, and using combinations of B_3 and B_4, we obtain a $((3+n')\times 9)$ pattern B_6 for the graph $(D(3,n')\square D(6,3))^2$. Moreover, using combinations of B_5 and B_6, we get a $((3+n')\times(m+n))$ pattern $(D(3,n')\square D(m,n))^2$. This provides a proper 10-coloring of $(D(3,n')\square D(m,n))^2$. $\square$

Corollary 1. If $m \notin \{3,4,7,8,9,13,14,19\}$ and $n \neq 5$, then

$$\chi((D(3k,n')\square D(m,n))^2) \leqslant 10,$$

where k is any positive integer.

Proof. By using Theorem 6, it is obvious that $\chi((D(3k,n')\Box D(m,n))^2)\leqslant 10$. $\square$

Theorem 7. If $m\notin\{3,4,7,8,9,13,14,19\}$ and $n\neq 5$, then

$$\chi((D(4,n')\Box D(m,n))^2)\leqslant 10.$$

Proof. Consider the following patterns C_1, C_2, C_3, and C_4 for graphs

$$(D(4,3)\Box D(5,4))^2,(D(4,4)\Box D(5,4))^2,(D(4,3)\Box D(6,3))^2 \ \ and \ \ (D(4,4)\Box D(6,3))^2,$$

respectively.

$$C_1=\begin{array}{|ccccccccc|}\hline 1&2&7&9&1&2&5&7&3\\ 3&4&1&8&5&3&4&1&6\\ 7&5&6&4&7&8&6&9&2\\ 1&3&2&10&6&2&7&4&5\\ 5&6&4&9&3&4&5&2&1\\ 3&2&1&6&2&1&3&8&4\\ 6&4&7&3&5&8&2&7&9\\ \hline\end{array} \quad C_2=\begin{array}{|ccccccccc|}\hline 1&2&7&9&1&2&5&7&3\\ 3&4&1&8&5&3&4&1&6\\ 7&5&6&4&7&8&6&9&2\\ 1&3&2&10&6&2&7&4&5\\ 5&6&4&9&3&4&5&2&1\\ 3&2&1&6&2&1&3&8&4\\ 6&4&7&3&5&8&2&7&9\\ 2&1&10&2&7&6&1&3&8\\ \hline\end{array}$$

$$C_3=\begin{array}{|ccccccccc|}\hline 1&2&7&9&1&2&5&7&3\\ 3&4&1&8&5&3&4&1&6\\ 7&5&6&4&7&8&6&9&2\\ 1&3&2&10&6&2&7&4&5\\ 5&6&4&9&3&4&5&2&1\\ 3&2&1&6&2&1&3&8&4\\ 6&4&7&3&5&8&2&7&9\\ \hline\end{array} \quad C_4=\begin{array}{|ccccccccc|}\hline 1&2&7&9&1&2&5&7&3\\ 3&4&1&8&5&3&4&1&6\\ 7&5&6&4&7&8&6&9&2\\ 1&3&2&10&6&2&7&4&5\\ 5&6&4&9&3&4&5&2&1\\ 3&2&1&6&2&1&3&8&4\\ 6&4&7&3&5&8&2&7&9\\ 2&1&10&2&7&6&1&3&8\\ \hline\end{array}$$

By combining the above subpatterns, similar to the proofs of the above theorems, we find a proper coloring for the graph $(D(4,n')\Box D(m,n))^2$ with 10 colors. $\square$

Corollary 2. If $m\notin\{3,4,7,8,9,13,14,19\}$ and $n\neq 5$, then

$$\chi((D(4k,n')\Box D(m,n))^2)\leq 10 \qquad k=1,2,3,\ldots.$$

Theorem 8. If $m\notin\{3,4,7,8,9,13,14,19\}$ and $n\neq 5$, then

$$\chi((D(7,n')\Box D(m,n))^2)\leq 10.$$

Proof. Consider the following patterns D_1, D_2, D_3, and D_4 for graphs

$$(D(7,3)\Box D(5,4))^2,(D(7,4)\Box D(5,4))^2,(D(7,3)\Box D(6,3))^2 \ \ and \ \ (D(7,4)\Box D(6,3))^2,$$

respectively.

$$D_1=\begin{array}{|ccccccccc|}\hline 1&2&3&10&9&3&1&2&4\\ 3&4&5&6&8&5&6&3&7\\ 2&1&7&2&10&7&2&1&5\\ 4&3&6&4&9&6&4&7&8\\ 1&2&5&1&2&5&1&9&3\\ 3&4&7&6&4&3&6&8&5\\ 8&10&9&2&1&7&9&10&6\\ 4&5&6&4&10&5&4&3&8\\ 1&2&7&1&9&6&1&7&5\\ 3&6&4&8&7&3&8&6&2\\ \hline\end{array} \quad D_2=\begin{array}{|ccccccccc|}\hline 1&2&3&10&9&3&1&2&4\\ 3&4&5&6&8&5&6&3&7\\ 2&1&7&2&10&7&2&1&5\\ 4&3&6&4&9&6&4&7&8\\ 1&2&5&1&2&5&1&9&3\\ 3&4&7&6&4&3&6&8&5\\ 8&10&9&2&1&7&9&10&6\\ 4&5&6&4&10&5&4&3&8\\ 1&2&7&1&9&6&1&7&5\\ 3&6&4&8&7&3&8&6&2\\ 5&8&3&9&6&4&7&5&4\\ \hline\end{array}$$

$$D_3=\begin{array}{|ccccccccc|}\hline 1&2&3&10&9&3&1&2&4\\ 3&4&5&6&8&5&6&3&7\\ 2&1&7&2&10&7&2&1&5\\ 4&3&6&4&9&6&4&7&8\\ 1&2&5&1&2&5&1&9&3\\ 3&4&7&6&4&3&6&8&5\\ 8&10&9&2&1&7&9&10&6\\ 4&5&6&4&10&5&4&3&8\\ 1&2&7&1&9&6&1&7&5\\ 3&6&4&8&7&3&8&6&2\\ \hline\end{array}\qquad D_4=\begin{array}{|ccccccccc|}\hline 1&2&3&10&9&3&1&2&4\\ 3&4&5&6&8&5&6&3&7\\ 2&1&7&2&10&7&2&1&5\\ 4&3&6&4&9&6&4&7&8\\ 1&2&5&1&2&5&1&9&3\\ 3&4&7&6&4&3&6&8&5\\ 8&10&9&2&1&7&9&10&6\\ 4&5&6&4&10&5&4&3&8\\ 1&2&7&1&9&6&1&7&5\\ 3&6&4&8&7&3&8&6&2\\ 5&8&3&9&6&4&7&5&4\\ \hline\end{array}$$

By combining the above subpatterns, we find a proper coloring for the following graph with 10 colors.

$$(D(7,n')\square D(m,n))^2.$$

□

Corollary 3. If $m \notin \{3,4,7,8,9,13,14,19\}$ and $n \neq 5$, then

$$\chi((D(7k,n')\square D(m,n))^2) \leq 9,$$

where k is a positive integer.

Theorem 9. If $m \notin \{3,4,7,8,9,13,14,19\}$ and $n \neq 5$, then

$$\chi((D(13,n')\square D(m,n))^2) \leq 10.$$

Proof. Consider the following patterns E_1, E_2, E_3, and E_4 for graphs

$$(D(13,3)\square D(5,4))^2, (D(13,4)\square D(5,4))^2, (D(13,3)\square D(6,3))^2 \; and \; (D(13,4)\square D(6,3))^2,$$

respectively.

$$E_1=\begin{array}{|ccccccccc|}\hline 1&2&3&1&2&3&1&4&5\\ 3&4&5&6&4&5&2&3&7\\ 2&1&7&2&3&1&4&5&8\\ 4&3&6&1&5&2&3&9&4\\ 1&2&4&3&6&4&5&2&7\\ 3&5&1&2&8&1&6&3&9\\ 2&4&3&5&4&2&7&1&6\\ 1&6&2&1&3&5&4&2&7\\ 3&5&4&6&2&1&3&5&8\\ 2&1&3&5&7&4&2&1&9\\ 4&6&2&1&6&3&5&4&10\\ 1&3&4&7&5&1&6&3&8\\ 2&5&1&2&3&4&2&9&6\\ 3&4&6&5&1&6&3&4&10\\ 1&2&3&4&7&2&1&5&9\\ 4&5&1&6&5&3&4&7&8\\ \hline\end{array}\qquad E_2=\begin{array}{|ccccccccc|}\hline 1&2&3&1&2&3&1&4&5\\ 3&4&5&6&4&5&2&3&7\\ 2&1&7&2&3&1&4&5&8\\ 4&3&6&1&5&2&3&9&4\\ 1&2&4&3&6&4&5&2&7\\ 3&5&1&2&8&1&6&3&9\\ 2&4&3&5&4&2&7&1&6\\ 1&6&2&1&3&5&4&2&7\\ 3&5&4&6&2&1&3&5&8\\ 2&1&3&5&7&4&2&1&9\\ 4&6&2&1&6&3&5&4&10\\ 1&3&4&7&5&1&6&3&8\\ 2&5&1&2&3&4&2&9&6\\ 3&4&6&5&1&6&3&4&10\\ 1&2&3&4&7&2&1&5&9\\ 4&5&1&6&5&3&4&7&8\\ 8&6&7&9&2&7&8&6&1\\ \hline\end{array}$$

$$E_3=\begin{array}{|ccccccccc|}\hline
1&2&3&1&2&3&1&4&5\\
3&4&5&6&4&5&2&3&7\\
2&1&7&2&3&1&4&5&8\\
4&3&6&1&5&2&3&9&4\\
1&2&4&3&6&4&5&2&7\\
3&5&1&2&8&1&6&3&9\\
2&4&3&5&4&2&7&1&6\\
1&6&2&1&3&5&4&2&7\\
3&5&4&6&2&1&3&5&8\\
2&1&3&5&7&4&2&1&9\\
4&6&2&1&6&3&5&4&10\\
1&3&4&7&5&1&6&3&8\\
2&5&1&2&3&4&2&9&6\\
3&4&6&5&1&6&3&4&10\\
1&2&3&4&7&2&1&5&9\\
4&5&1&6&5&3&4&7&8\\ \hline
\end{array}
\qquad
E_4=\begin{array}{|ccccccccc|}\hline
1&2&3&1&2&3&1&4&5\\
3&4&5&6&4&5&2&3&7\\
2&1&7&2&3&1&4&5&8\\
4&3&6&1&5&2&3&9&4\\
1&2&4&3&6&4&5&2&7\\
3&5&1&2&8&1&6&3&9\\
2&4&3&5&4&2&7&1&6\\
1&6&2&1&3&5&4&2&7\\
3&5&4&6&2&1&3&5&8\\
2&1&3&5&7&4&2&1&9\\
4&6&2&1&6&3&5&4&10\\
1&3&4&7&5&1&6&3&8\\
2&5&1&2&3&4&2&9&6\\
3&4&6&5&1&6&3&4&10\\
1&2&3&4&7&2&1&5&9\\
4&5&1&6&5&3&4&7&8\\
8&6&7&9&2&7&8&6&1\\ \hline
\end{array}$$

By using combinations of E_1 and E_2, we get a proper 10-coloring of $(D(13,n)\square D(5,4))^2$ as E_5, and by combining patterns E_3 and E_4, we obtain a proper 10- coloring for $(D(13,n)\square D(6,3))^2$ as E_6. Finally by combinations of E_5 and E_6, we obtain a proper 10-coloring of $(D(13,n)\square D(m,n))^2$. $\square$

Theorem 10. If $m \notin \{3,4,7,8,9,13,14,19\}$ and $n \neq 5$, then

$$\chi((D(19,n')\square D(m,n))^2) \leq 10.$$

Proof. Consider the following patterns J_1, J_2, J_3, and J_4 for graphs

$$(D(19,4)\square D(5,4))^2, (D(19,3)\square D(5,4))^2, (D(19,4)\square D(6,3))^2 \textit{ and } (D(19,3)\square D(6,3))^2,$$

$$J_1=\begin{array}{|ccccccccccccccccccccccc|}\hline
1&2&3&4&1&2&5&3&4&1&2&5&3&4&1&2&5&3&4&1&2&4&3\\
3&4&1&2&5&3&4&1&2&5&3&4&1&2&5&3&4&1&2&5&3&1&6\\
2&5&6&3&4&1&2&5&3&4&1&2&9&3&4&1&2&9&3&4&6&5&2\\
1&3&4&1&9&5&3&4&9&2&5&3&7&6&2&5&3&7&6&1&2&4&7\\
4&8&2&5&3&4&1&2&5&3&4&9&2&5&3&9&6&1&2&5&8&1&6\\
2&1&3&4&1&9&5&3&4&1&2&5&3&4&6&1&2&3&7&4&6&5&2\\
3&4&6&2&5&3&4&1&2&5&3&4&6&1&2&3&7&4&5&1&2&4&9\\
5&2&1&3&4&1&2&5&3&4&1&2&5&3&4&5&1&8&2&6&5&1&8\\
6&7&9&8&7&6&8&7&6&8&7&6&1&9&8&7&4&5&9&3&7&9&3\\ \hline
\end{array}$$

$$J_2=\begin{array}{|cccccccccccccccccccccc|}\hline
2&3&4&1&2&5&3&4&1&2&5&3&4&1&2&5&3&4&1&2&4&3\\
4&1&2&5&3&4&1&2&5&3&4&1&2&5&3&4&1&2&5&3&1&6\\
5&6&3&4&1&2&5&3&4&1&2&9&3&4&1&2&9&3&4&6&5&2\\
3&4&1&9&5&3&4&9&2&5&3&7&6&2&5&3&7&6&1&2&4&7\\
8&2&5&3&4&1&2&5&3&4&9&2&5&3&9&6&1&2&5&8&1&6\\
1&3&4&1&9&5&3&4&1&2&5&3&4&6&1&2&3&7&4&6&5&2\\
4&6&2&5&3&4&1&2&5&3&4&6&1&2&3&7&4&5&1&2&4&9\\
2&1&3&4&1&2&5&3&4&1&2&5&3&4&5&1&8&2&6&5&1&8\\
7&9&8&7&6&8&7&6&8&7&6&1&9&8&7&4&5&9&3&7&9&3\\ \hline
\end{array}$$

$$J_3=\begin{array}{|ccccccccccccccccccccccc|}\hline
1&2&3&4&1&2&5&3&4&1&2&5&3&4&1&2&5&3&4&1&2&4&3\\
3&4&1&2&5&3&4&1&2&5&3&4&1&2&5&3&4&1&2&5&3&1&6\\
2&5&6&3&4&1&2&5&3&4&1&2&9&3&4&1&2&9&3&4&6&5&2\\
1&3&4&1&9&5&3&4&9&2&5&3&7&6&2&5&3&7&6&1&2&4&7\\
4&8&2&5&3&4&1&2&5&3&4&9&2&5&3&9&6&1&2&5&8&1&6\\
2&1&3&4&1&9&5&3&4&1&2&5&3&4&6&1&2&3&7&4&6&5&2\\
3&4&6&2&5&3&4&1&2&5&3&4&6&1&2&3&7&4&5&1&2&4&9\\
5&2&1&3&4&1&2&5&3&4&1&2&5&3&4&5&1&8&2&6&5&1&8\\
6&7&9&8&7&6&8&7&6&8&7&6&1&9&8&7&4&5&9&3&7&9&3\\ \hline
\end{array}$$

$J_4=$

2	3	4	1	2	5	3	4	1	2	5	3	4	1	2	5	3	4	1	2	4	3
4	1	2	5	3	4	1	2	5	3	4	1	2	5	3	4	1	2	5	3	1	6
5	6	3	4	1	2	5	3	4	1	2	9	3	4	1	2	9	3	4	6	5	2
3	4	1	9	5	3	4	9	2	5	3	7	6	2	5	3	7	6	1	2	4	7
8	2	5	3	4	1	2	5	3	4	9	2	5	3	9	6	1	2	5	8	1	6
1	3	4	1	9	5	3	4	1	2	5	3	4	6	1	2	3	7	4	6	5	2
4	6	2	5	3	4	1	2	5	3	4	6	1	2	3	7	4	5	1	2	4	9
2	1	3	4	1	2	5	3	4	1	2	5	3	4	5	1	8	2	6	5	1	8
7	9	8	7	6	8	7	6	8	7	6	1	9	8	7	4	5	9	3	7	9	3

Applying Lemmas 1 and 2 and combinations of J_1 and J_2, we get a proper 9-coloring of $(D(19,n)\Box D(5,4))^2$ as J_5, and by combining the patterns J_3 and J_4, we obtain a proper 9-coloring for $(D(19,n)\Box D(6,3))^2$ as J_6. Finally by combinations of J_5 and J_6, we obtain a proper 9-coloring of $(D(19,n)\Box D(m,n))^2$.

$\square$

Theorem 11. If $n \neq 5$, then $\chi((D(m',5)\Box D(m,n))^2) \leq 10$.

Proof. Consider the following patterns F_1, F_2, F_3, and F_4 for graphs

$$(D(3,5)\Box D(5,4))^2, (D(4,5)\Box D(5,4))^2, (D(3,5)\Box D(6,3))^2 \ \textit{and} \ \ (D(4,5)\Box D(6,3))^2,$$

respectively.

$F_1=$

1	2	3	1	8	3	1	2	4
3	4	5	6	9	5	6	3	1
2	1	7	2	10	7	2	4	8
4	3	6	4	3	8	1	5	9
1	2	5	10	7	4	3	6	1
3	4	7	8	5	6	2	4	3
2	1	6	2	4	7	5	1	9
5	7	3	5	1	3	4	8	6

$F_2=$

1	2	3	1	8	3	1	2	4
3	4	5	6	9	5	6	3	1
2	1	7	2	10	7	2	4	8
4	3	6	4	3	8	1	5	9
1	2	5	10	7	4	3	6	1
3	4	7	8	5	6	2	4	3
2	1	6	2	4	7	5	1	9
5	7	3	5	1	3	4	8	6
6	8	1	7	2	5	7	3	4

$F_3=$

1	2	3	1	8	3	1	2	4
3	4	5	6	9	5	6	3	1
2	1	7	2	10	7	2	4	8
4	3	6	4	3	8	1	5	9
1	2	5	10	7	4	3	6	1
3	4	7	8	5	6	2	4	3
2	1	6	2	4	7	5	1	9
5	7	3	5	1	3	4	8	6

$F_4=$

1	2	3	1	8	3	1	2	4
3	4	5	6	9	5	6	3	1
2	1	7	2	10	7	2	4	8
4	3	6	4	3	8	1	5	9
1	2	5	10	7	4	3	6	1
3	4	7	8	5	6	2	4	3
2	1	6	2	4	7	5	1	9
5	7	3	5	1	3	4	8	6
6	8	1	7	2	5	7	3	4

Similar to Theorem 9, by combining the above subpatterns, we find a proper coloring for the graph $(D(m',5)\Box D(m,n))^2$ with 10 colors. $\square$

Declarations

Availability of supporting data

All data generated or analyzed during this study are included in this published paper.

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Competing interests

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Authors' contributions

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