Received: October 27, 2022 ; **Accepted:** June 20, 2023.
DOI. 10.30473/COAM.2023.65902.1218
Summer-Autumn (2023) Vol. 8, No. 2, (49-61)
Research Article

Open Access

Control and Optimization in Applied Mathematics - COAM

A New Weak Slater Constraint Qualification for Non-Smooth Multi-Objective Semi-Infinite Programming Problems

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Abstract. This paper addresses a non-smooth multi-objective semi-infinite programming problem that involves a feasible set defined by inequality constraints. Our focus is on introducing a new weak Slater constraint qualification and deriving the necessary and sufficient conditions for (weakly, properly) efficient solutions to the problem using (weak and strong) Karush-Kuhn-Tucker types. Additionally, we present two duals of the Mond-Weir type for the problem and provide (weak and strong) duality results for them. All of the results are given in terms of Clarke subdifferential.

Keywords. Semi-infinite programming, Multi-objective optimization, Constraint qualification, Optimality conditions.

MSC. 90C26.

How to Cite
Soroush, H., (2023). "A new weak Slater constraint qualification for non-smooth multi-objective semi-infinite programming problems", Control and Optimization in Applied Mathematics, 8(2): 49-61.

1 Introduction

In this paper, we consider the following multi-objective semi-infinite programming problem (MSIP for short):

$$\begin{array}{ll} \text{(P)} & f(x) := \inf \big(f_1(x), f_2(x), \dots, f_p(x)\big) \\ \text{s.t.} & g_t(x) \le 0 \quad t \in T, \\ & x \in \mathbb{R}^n, \end{array}$$

where f_i as $i \in I := \{1, 2, \dots, p\}$ and g_t as for $t \in T$ are locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$, The index set T is arbitrary and not necessarily finite, but non-empty. Such kinds of problems arise in various fields of engineering such as control systems design, resource allocation in decentralized systems, decision making under competition, multi-objective optimization, and filter design in signal processing ([7, 24]). There are many papers that have shown the necessary and sufficient optimality conditions of the Karush-Kuhn-Tucker (KKT) type MSIPs; see e.g., [7, 9, 21] for the linear case, [5, 11, 24] in differentiable case, [6, 8, 22] for the convex case, and [1, 2, 10, 13, 14, 15, 16, 17, 18, 19, 20, 25] for the non-smooth case. The so-called Slater constraint qualification plays an important role in the study of convex MSIPs. We recall from [6, 22] that the Slater constraint qualification is satisfied for (P) if

$$(\star) : \begin{cases} \text{(I): the } g_t \text{ functions are convex as } t \in T, \\ \\ \text{(II): } T \text{ is a compact subset of a metric space and the function} \\ \qquad (x,t) \to g_t(x) \text{ is continuous on } \mathbb{R}^n \times T\ , \\ \\ \text{(III): there exists a } x_* \in \mathbb{R}^n \text{ such that } g_t(x_*) < 0 \text{ for all } t \in T. \end{cases}$$

The Slater constraint qualification would not be useful without limiting assumptions (I) and (II), see e.g., [6, 10]). The goal of this paper is to modify and weaken conditions (I) and (II), and generalize the Slater constraint qualification for non-convex MSIPs. Since we do not assume that the data of (P) are differentiable, we replace the derivative appearing in the classical results with a known generalized derivative, named Clarke subdifferential.

In the next section, we provide preliminary results to be used throughout the remainder of the paper. We examine the Slater constraint qualification for problem (P), as well as investigate the weak and strong KKT type and FJ type optimality conditions in Section 3. Section 4 encompasses the weak and strong duality results for the two dual problems in the Mond-Weir type.

2 Preliminaries

This section provides a brief overview of the key concepts and preliminary information related to convex analysis and non-smooth analysis, which are extensively utilized in formulating and proving the main results presented in this paper. For a more comprehensive understanding, further discussion, and applications, readers are referred to references [3, 12].

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two given points in $\mathbb{R}^n$. We write

- $x \leqq y$ when $x_i \leq y_i$ for all $i = 1, \dots, n$.
- $x \leq y$ when $x_i \leq y_i$ for all $i = 1, \dots, n$ and $x_j < y_j$ for some $j = 1, \dots, n$.
- $x < y$ when $x_i < y_i$ for all $i = 1, \dots, n$.

Given a nonempty set $A \subseteq \mathbb{R}^n$, we denote the closure and the convex hull generated by $\overline{A}$ as $\overline{A}$ and conv(A), respectively. It is worth noting, as observed in [12] that if $\{A_\beta\}_{\beta\in\Lambda}$ is any class of convex subsets of $\mathbb{R}^n$, then

$$\text{conc}\Big(\bigcup_{\beta\in\Lambda} A_\beta\Big) = \Big\{\sum_{i=1}^{k} \lambda_{\beta_i} a_{\beta_i} \mid k \in \mathbb{N},\ a_{\beta_i} \in A_{\beta_i},\ \lambda_{\beta_i} \geq 0,\ \sum_{i=1}^{k} \lambda_{\beta_i} = 1\Big\}. \tag{1}$$

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Clarke directional derivative of ψ at $x_0 \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$, and the Clarke subdifferential of ψ at x_0 are respectively defined as follows:

$$\psi^0(x_0; d) := \limsup_{x\to x_0,\ \nu\downarrow 0} \frac{\psi(x+\nu d) - \psi(x)}{\nu},$$

and

$$\partial_c\psi(x_0) := \big\{\xi \in \mathbb{R}^n \mid \partial a\langle \xi, d\rangle \leq \psi^0(x_0; d) \quad \text{for all } d \in \mathbb{R}^n\big\},$$

where $\langle \xi, d\rangle$ denotes the standard inner product of ξ and d in $\mathbb{R}^n$. The zero vector of $\mathbb{R}^n$ is denoted by 0_n.

In the following theorem,we provide a concise summary of the key properties of the Clarke subdifferential from [3] that which are extensively utilized in what follows.

Theorem 1. Suppose that ψ_1 and ψ_2 are locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$, and $x_0 \in \mathbb{R}^n$ is given. Then,

(i) $\partial_c(\psi_1 + \psi_2)(x_0) \subseteq \partial_c\psi_1(x_0) + \partial_c\psi_2(x_0)$.

(ii) $\partial_c\big(\lambda\psi_1\big)(x_0) = \lambda\partial_c\psi_1(x_0), \quad \forall\lambda \in \mathbb{R}$.

(iii) $\partial_c\big(\max\{\psi_1, \psi_2\}\big)(x_0) \subseteq \text{conv}\big(\partial_c\psi_1(x_0) \cup \partial_c\psi_2(x_0)\big)$.

(iv) $\partial_c\psi_1(x_0)$ is a nonempty, convex, and compact subset of $\mathbb{R}^n$.

(v) $0_n \in \partial_c\psi_1(x_0)$ if ψ_1 attains its minimum at x_0.

3 Optimality Conditions

The feasible set of (P) is denoted by S, such that

$$S := \{x \in \mathbb{R}^n \mid g_t(x) \leq 0, \quad \forall t \in T\}.$$

Weak efficiency, efficiency, and proper efficiency are essential concepts in studying multi-objective optimization problems. There are various definitions for these efficiencies in the literature; refer to [4] for a comparison among these notions.

Definition 1. A feasible point $\widehat{x} \in S$ is called a

- weakly efficient solution of (P) when there is no $x \in S$ such that $f(x) < f(\widehat{x})$.
- efficient solution of (P) when there is no $x \in S$ such that $f(x) \leq f(\widehat{x})$.
- properly efficient solution of (P) when there exists a $\eta > 0_p$ such that

$$\langle \eta, f(\widehat{x}) \rangle \leq \langle \eta, f(x) \rangle, \quad \forall x \in S.$$

The following first-order optimality conditions are standard in study of non-smooth multi-objective (semi-infinite) optimization; see, e.g., [4, 6, 16, 20, 19].

Definition 2. Let $\widehat{x} \in S$ be a feasible point for (P).

- We say that $\widehat{x}$ is a Fritz-John (FJ) point for (P) if there exist some finite index set $T_1 \subseteq T(\widehat{x})$, some $(\alpha_1, \ldots, \alpha_p) \geqq 0_p$ and $\beta_t \geq 0$, for $t \in T_1$, such that not all of α_is and β_ts equal to zero and

$$0_n \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(\widehat{x}) + \sum_{t \in T_1} \beta_t \partial_c g_t(\widehat{x}). \tag{2}$$

- We say that $\widehat{x}$ is a weak Karush-Kuhn-Tuker (WKKT) point for (P) if there exist some finite index set $T_1 \subseteq T(\widehat{x})$, some $(\alpha_1, \ldots, \alpha_p) \geq 0_p$ and $\beta_t \geq 0$, for $t \in T_1$, satisfying (2).
- We say that $\widehat{x}$ is a strong Karush-Kuhn-Tuker (SKKT) point for (P) if there exist some finite index set $T_1 \subseteq T(\widehat{x})$, some $(\alpha_1, \ldots, \alpha_p) > 0_p$ and $\beta_t \geq 0$, for $t \in T_1$, satisfying (2).

Recall from [20, 19] that, if $\widehat{x} \in S$ is a weakly efficient solution of (P) and condition (II) in $(\star)$ holds at $\widehat{x}$, then $\left(\bigcup_{i \in I} \partial_c f_i(\widehat{x})\right) \cup \left(\bigcup_{t \in T(\widehat{x})} \partial_c g_t(\widehat{x})\right)$ is compact and

$$0_n \in \operatorname{conv}\Big(\Big(\bigcup_{i \in I} \partial_c f_i(\widehat{x})\Big) \cup \Big(\bigcup_{t \in T(\widehat{x})} \partial_c g_t(\widehat{x})\Big)\Big),$$

and hence, $\widehat{x}$ is a FJ point for (P) by (1). For extension of this result to problem (P) without assumption (II) in $(\star)$, we recall the following definition from [13, 14].

Definition 3. We say that (P) has the Pshenichnyi-Levin-Valadier (PLV) property at $\widehat{x} \in S$, if $\vartheta(\cdot)$ is finite-valued Lipschitz around $\widehat{x}$, and

$$\partial_c \vartheta(\widehat{x}) \subseteq conv\Big(\bigcup_{t \in T(\widehat{x})} \partial_c g_t(\widehat{x})\Big),$$

where, $\vartheta(\cdot)$ is defined as

$$\vartheta(x) := \sup_{t \in T} g_t(x), \quad \forall x \in \mathbb{R}^n.$$

Remark 1. It should be observed from [6, 13] that the PLV property is strictly weaker than condition (II) in $(\star)$.

Now, we can extend the FJ necessary condition at weakly efficient solutions of (P) as follows.

Theorem 2. (FJ necessary condition) Let $\widehat{x}$ be a weakly efficient solution of (P) and the PLV property holds at $\widehat{x}$. Then, $\widehat{x}$ is a FJ point for (P).

Proof. As the first, we define the function $f : \mathbb{R}^n \to \mathbb{R}$ as

$$\varphi(x) := \max\left\{f_1(x) - f_1(\widehat{x}), f_2(x) - f_2(\widehat{x}), \dots, f_p(x) - f_p(\widehat{x})\right\}, \quad \forall x \in \mathbb{R}^n.$$

Since $\widehat{x}$ is a weakly efficient solution for (P), we have

$$\varphi(x) \geq 0, \quad \forall x \in S. \tag{3}$$

Put

$$\Psi(x) := \max\{\vartheta(x), \varphi(x)\}, \quad \forall x \in \mathbb{R}^n,$$

where $\vartheta(x)$ is defined as Definition 3. If $x \in S$, we have $\vartheta(x) \leq 0$ and hence, $\Psi(x) \geq 0$ by (3). If $x \notin S$, then there exists a $t_0 \in T$ such that $g_{t_0}(x) > 0$, and so $\Psi(x) \geq \vartheta(x) \geq g_{t_0}(x) > 0$. Consequently,

$$\Psi(x) \geq 0 = \Psi(\widehat{x}), \quad \forall x \in \mathbb{R}^n.$$

This means $\widehat{x}$ is a global minimizer for $\Psi(\cdot)$, and so $0_n \in \partial_c\Psi(\widehat{x})$ by Theorem 1. From this, Theorem 1, and PLV property we get

$$0_n \in \text{conv}\big(\partial_c\vartheta(\widehat{x}) \cup \partial_c\varphi(\widehat{x})\big) \subseteq \text{conv}\Bigg(\Big(\bigcup_{i\in I}\partial_c f_i(\widehat{x})\Big) \cup \Big(\bigcup_{t\in T(\widehat{x})}\partial_c g_t(\widehat{x})\Big)\Bigg).$$

This inclusion and (1) imply that $\widehat{x}$ is a FJ point for (P), and the proof is complete. $\square$

Like classical multi-objective problems, additional conditions, called constraint qualifications, are necessary to obtain KKT type necessary conditions. weaken conditions (I) and (III) in $(\star)$ and achieve a weak Slater constraint qualification, it is necessary to introduce the following definitions, which are inspired by [2, 6, 10, 11, 25, 26].

Definition 4. The problem (P) is said satisfies in weak Slater condition at $x_0 \in S$ if for each finite index set $T_1 \subseteq T(x_0)$, there exists a point $x_1 \in S$ such that $g_t(x_1) < 0$ for all $t \in T_1$.

Example 1. Let $T = \mathbb{N} \cup \{0\}$, $g_t(x_1, x_2) = (x_1 - \frac{1}{t})x_2^2$, for $t \in \mathbb{N}$, and $g_0(x_1, x_2) = -x_1$. Since there is not any $x_* \in S = \{0\} \times \mathbb{R}$ with $g_t(x_*) < 0$, for all $t \in T$, the condition (iii) in $(*)$ is not valid. Taking $x_0 = (0, 0) \in S$, we conclude that $T(x_0) = T$ and for each finite set $T_1 \subseteq T(x_0)$, we have $g_t(\widehat{x}) < 0, t \in T$, with $\widehat{x} = (\dfrac{1}{1 + max(T_1)}, 1)$. Thus the weak Slater condition holds at x_0.

Definition 5. Suppose that the function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is given. A locally Lipschitz function $\hbar : \mathbb{R}^n \to \mathbb{R}$ is said to be generalized η-invex respect to $A \subseteq \mathbb{R}^n$ at $x_0 \in A$ if there exists a function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to (0, +\infty)$ such that for each $x \in A$ one has:

$$\rho(x, x_0)\big(\hbar(x) - \hbar(x_0)\big) \geq \langle \xi, \eta(x.x_0)\rangle, \qquad \forall \xi \in \partial_c\hbar(x_0).$$

Example 2. Suppose that the function $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given as $\eta(x, y) = \sin(x - y)$. The function $\hbar(x) = \sin x$ is generalized η−respect to $A = \mathbb{R}$ at $x_0 = 0$. In fact, if we take $\rho(x, y) = 1$ for all $x, y \in \mathbb{R}$, owing to $\nabla\hbar(x) = \cos x$, we have

$$1(\sin x - \sin 0) = \cos 0(\sin(x - 0)), \quad \forall x \in \mathbb{R}.$$

Remark 2. It is noteworthy that if $\rho(x, y) = 1$ and $\eta(x, y) = x - y$ for all $x, y \in \mathbb{R}^n$, generalized η-invexity of $\overline{h}(\cdot)$ reduces to convexity of $\overline{h}(\cdot)$. So, Definition 5 is strictly weaker than condition (I) in $(\star)$. Also, it is clear that Definition 4 is strictly weaker than condition (III) in $(\star)$.

Now, we can introduce a new constraint qualification for (P) as follows.

Definition 6. Suppose that $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is given. We say that (P) satisfies the η-weak Slater constraint qualification (η-WSCQ) at $\widehat{x} \in S$ if the PLV property holds at $\widehat{x}$, the weak Slater condition is satisfied at $\widehat{x}$, and for each $t \in T(\widehat{x})$, the g_t function is generalized η-invex respect to S at $\widehat{x}$.

According to Remarks 1 and 2, we understand that the above definition is strictly weaker that Slater constraint qualification, defined in $(\star)$. The following theorem guaranties that η-WSCQ is actually a constraint qualification.

Theorem 3. (WKKT necessary condition) Assume that $\widehat{x} \in S$ is a weakly efficient solution of (P) and η-WSCQ is satisfied at $\widehat{x}$. Then, $\widehat{x}$ is a WKKT point for (P).

Proof. Employing Theorem 2, we can find some $\alpha = (\alpha_1, \ldots, \alpha_p) \geqq 0_p$ and $\beta_t \geq 0$ for $t \in T_1 \subseteq T(\widehat{x})$ with $|T_1| < \infty$, not all of them equal to zero, as well as some $(\xi_1, \ldots, \xi_p) \in \prod_{i=1}^{p} \partial_c f_i(\widehat{x})$ and $\zeta_t \in \partial_c g_t(\widehat{x})$ for $t \in T_1$, such that

$$\sum_{i=1}^{p} \alpha_i \xi_i + \sum_{t \in T_1} \beta_t \zeta_t = 0_n.$$

If $\alpha = 0_p$, then $\sum_{t \in T_1} \beta_t \zeta_t = 0_n$. So, according to generalized η-invexity of g_t functions respect to S at $\widehat{x}$, for each $x \in S$ we obtain

$$0 = \langle \sum_{t \in T_1} \beta_t \zeta_t, \eta(x, \widehat{x}) \rangle = \sum_{t \in T_1} \beta_t \langle \zeta_t, \eta(x, \widehat{x}) \rangle \leq \sum_{t \in T_1} \beta_t \rho_t(x, \widehat{x}) \big(g_t(x) - \underbrace{g_t(\widehat{x})}_{=0} \big).$$

On the other hand, since T_1 is a finite subset of $T(\widehat{x})$, there exists a $x_1 \in S$ such that $g_t(x_1) < 0$ for all $t \in T_1$. Consequently, the above inequality implies the following contradiction:

$$0 \leq \sum_{t \in T_1} \beta_t \rho_t(x_1, \widehat{x}) g_t(x_1) < 0,$$

where the last inequality holds by $\beta_t \neq 0$ for some $t \in T_1$ and by $\rho_t(x_1, \widehat{x}) \geq 0$ for all $t \in T_1$. This contradiction shows that $\alpha \neq 0_p$, and the proof is complete. □

Corollary 1. If in Theorems 2 and 3 the "weakly efficient" is replaced by "efficient", the results are also true.

Proof. Since each efficient solution is a weakly efficient solution, the corollary is clearly true. □

The following theorem shows the role of generalized η-invexity in WKKT sufficient condition. It is noteworthy that the following theorem is a generalization of sufficient KKT conditions that are presented in [6, 17, 13, 18].

Theorem 4. (WKKT sufficient condition) Suppose that $\widehat{x} \in S$ is a WKKT point for (P). If f_i and g_t functions, for all $i \in I$ and $t \in T(\widehat{x})$, are generalized η-invex at $\widehat{x}$, then $\widehat{x}$ is a weakly efficient solution for (P).

Proof. Since $\widehat{x}$ is a WKKT point for (P), there exist some $(\alpha_1, \ldots, \alpha_p) \geq 0_p$ and $\beta_t \geq 0$ for $t \in T_1 \subseteq T(\widehat{x})$ with $|T_1| < \infty$, as well as some $(\xi_1, \ldots, \xi_p) \in \prod_{i=1}^{p} \partial_c f_i(\widehat{x})$ and $\zeta_t \in \partial_c g_t(\widehat{x})$ for $t \in T_1$, such that

$$\sum_{i=1}^{p} \alpha_i \xi_i + \sum_{t \in T_1} \beta_t \zeta_t = 0_n. \tag{4}$$

Suppose on the contrary that $\widehat{x}$ is not a weakly efficient solution for (P), then there is some $x_* \in S$ such that $f_i(x_*) < f_i(\widehat{x})$ for $i \in I$. From this inequality and strictly positivity of some α_is and all $\rho_i(x_*, \widehat{x})$s, we have

$$0 > \sum_{i=1}^{p} \rho_i(x_*, \widehat{x})\big(\alpha_i f_i(x_*) - \alpha_i f_i(\widehat{x})\big) \geq \sum_{i=1}^{p} \alpha_i \langle \xi_i, \eta(x_*, \widehat{x})\rangle. \tag{5}$$

On the other hand, for all $t \in T_1$ we obtain that

$$\sum_{t \in T_1} \beta_t \langle \zeta_t, \eta(x_*, \widehat{x})\rangle \leq \sum_{t \in T_1} \rho_t(x_*, \widehat{x})\big(\beta_t \underbrace{g_t(x_*)}_{\leq 0} - \beta_t \underbrace{g_t(\widehat{x})}_{=0}\big) \leq 0.$$

Adding this inequality with (5), we get

$$\begin{aligned} 0 &> \sum_{i=1}^{p} \alpha_i \langle \xi_i, \eta(x_*, \widehat{x})\rangle + \sum_{t \in T_1} \beta_t \langle \zeta_t, \eta(x_*, \widehat{x})\rangle \\ &= \Big\langle \sum_{i=1}^{p} \alpha_i \xi_i + \sum_{t \in T_1} \beta_t \zeta_t \ , \ \eta(x_*, \widehat{x}) \Big\rangle = 0, \end{aligned}$$

where the last equality holds by (4). This contradiction shows that $\widehat{x}$ is a weakly efficient solution for (P).

Example 3. Taking $f_1(x_1, x_2) = x_1$, $f_2(x_1, x_2) = x_2$, $T = [\pi, \frac{3\pi}{2}]$, $\widehat{x} = (0. -1)$ and $g_t(x_1, x_2) = (\cos t)x_1 + (\sin t)x_2 - 1$, for all $t \in T$. we have

$$\begin{aligned} T(\widehat{x}) &= \Big\{ t \in [\pi, \frac{3\pi}{2}] \quad | - \sin t - 1 = 0 \Big\} = \{\frac{3\pi}{2}\}, \\ \partial f_1(\widehat{x}) &= (1, 0), \quad \partial f_2(\widehat{x}) = (0, 1), \\ \partial g_t(\widehat{x}) &= (\cos \frac{3\pi}{2}, sin \frac{3\pi}{2}) = (0, -1). \end{aligned}$$

Thus, the following KKT equality holds:

$$\alpha_1 \{(1, 0)\} + \alpha_2 \{(0, 1)\} + \beta_1 \{(0, -1)\} = \{(0, 0)\},$$

by $\alpha_1 = 0$ and $\alpha_2 = \beta_1 = \frac{1}{2}$. Therefore, $\widehat{x}$ is a weak KKT point for the considered problem and it is a weakly effieicent point by Theorem 4.

□

Theorem 5. (SKKT necessary condition) Assume that $\widehat{x} \in S$ is a properly efficient solution of (P) and η-WSCQ is satisfied at $\widehat{x}$. Then, $\widehat{x}$ is a SKKT point for (P).

Proof. Since $\widehat{x}$ is a properly efficient solution for (P), there exist some scalars $\mu_i > 0$ (for $i \in I$) such that

$$\sum_{i=1}^{p} \mu_i f_i(\widehat{x}) \leq \sum_{i=1}^{p} \mu_i f_i(x), \quad \forall x \in S.$$

This means that $\widehat{x}$ is a minimizer (i.e., weakly efficient solution) for the following semi-infinite optimization problem:

$$\min \sum_{i=1}^{p} \mu_i f_i(x) \quad \text{subject to} \quad g_t(x) \leq 0, \quad t \in T.$$

Employing Theorem 3, we find some $T_1 \subseteq T(\widehat{x})$ with $|T_1| < \infty$, and $\beta_t \geq 0$ as $t \in T_1$, and $\tau > 0$ such that

$$0_n \in \tau \partial_c \Big(\sum_{i=1}^{p} \mu_i f_i(\cdot) \Big)(\widehat{x}) + \sum_{t \in T_1} \beta_t \partial_c g_t(\widehat{x}).$$

This inclusion and the fact that $\partial_c \Big(\sum_{i=1}^{p} \mu_i f_i(\cdot) \Big)(\widehat{x}) \subseteq \sum_{i=1}^{p} \mu_i \partial_c f_i(\widehat{x})$ by Theorem 1, imply that

$$0_n \in \sum_{i=1}^{p} \tau \mu_i \partial_c f_i(\widehat{x}) + \sum_{t \in T_1} \beta_t \partial_c g_t(\widehat{x}).$$

Taking $\alpha_i := \tau \mu_i > 0$ for $i \in I$ in above inclusion, we get (2), and the proof is complete. $\square$

Example 4. Let us consider the following problem: $\min(f_1(x), f_2(x))$ subject to

$$g_t(x) = x_1^2 + x_2^2 - 2x_1 + 2(t-3)x_2 + (\frac{5}{9}t^2 - 2t + 1) \leq 0, \quad t \in [0, 3],$$

with

$$f_1(x) = f_2(x) = \begin{cases} x_2^2 + 2x_2, & x_2 \geq 0, \\ \\ 2x_2, & x_2 < 0. \end{cases}$$

The point $\widehat{x} = (1, 0)$ is a weakly efficient for the problem and $T(\widehat{x}) = \{0\}$. Thus, $g_t(1, \frac{1}{2}) < 0$, for all $t \in T(\widehat{x})$, and the $\eta-$WSCQ is satisfied at $\widehat{x}$, and $\widehat{x}$ is a KKT point for the problem by theorem 5.

Now, we can state the sufficient condition for properly efficiency of (P) as follows.

Theorem 6. (SKKT sufficient condition) Suppose that $\widehat{x} \in S$ is a SKKT point for (P). If f_i and g_t functions as $i \in I$ and $t \in T(\widehat{x})$ are generalized η-invex at $\widehat{x}$, then $\widehat{x}$ is a properly efficient solution for (P).

Proof. Since $\widehat{x}$ is a SKKT point for (P), there exist some $(\alpha_1, \dots, \alpha_p) > 0_p$ and $\beta_t \geq 0$ for $t \in T_1 \subseteq T(\widehat{x})$ with $|T_1| < \infty$, as well as some $(\xi_1, \dots, \xi_p) \in \prod_{i=1}^{p} \partial_c f_i(\widehat{x})$ and $\zeta_t \in \partial_c g_t(\widehat{x})$ for $t \in T_1$, such that (4) holds. So, for all $x \in S$ we have

$$\sum_{i=1}^{p} \alpha_i \langle \xi_i, \eta(x, \widehat{x}) \rangle + \sum_{t \in T_1} \beta_t \langle \zeta_t, \eta(x, \widehat{x}) \rangle = \Big\langle \sum_{i=1}^{p} \alpha_i \xi_i + \sum_{t \in T_1} \beta_t \zeta_t \ , \ \eta(x, \widehat{x}) \Big\rangle = 0_n \tag{6}$$

On the other hand, owing to the generalized η-invexity of f_i and g_t functions at $\widehat{x}$ respect to S for all $i \in I$ and $t \in T_1$, we have

$$\begin{cases} \sum_{i=1}^{p} \alpha_i \Big(f_i(x) - f_i(\widehat{x}) \Big) \geq \sum_{i=1}^{p} \alpha_i \langle \xi_i, \eta(x,\widehat{x}) \rangle, \\ \\ \sum_{t\in T_1} \beta_t \Big(\underbrace{g_t(x)}_{\leq 0} - \underbrace{g_t(\widehat{x})}_{=0} \Big) \geq \sum_{t\in T_1} \beta_t \langle \zeta_t, \eta(x,\widehat{x}) \rangle, \end{cases} \quad \text{for all } \ x \in S.$$

This deduces that

$$\sum_{i=1}^{p} \alpha_i \Big(f_i(x) - f_i(\widehat{x}) \Big) + \underbrace{\sum_{t\in T_1} \beta_t \Big(g_t(x) - g_t(\widehat{x}) \Big)}_{\leq 0} = 0, \quad \text{for all } \ x \in S$$

and hence,

$$\sum_{i=1}^{p} \alpha_i \Big(f_i(x) - f_i(\widehat{x}) \Big) \geq 0, \quad \text{for all } \ x \in S.$$

Consequently,

$$\sum_{i=1}^{p} \alpha_i f_i(\widehat{x}) \leq \sum_{i=1}^{p} \alpha_i f_i(x), \quad \text{for all } \ x \in S,$$

and the result is proved. □

4 Duality Results

As applications of the optimality conditions presented in the previous section, we introduce two dual problems in the Mond-Weir [23] type for (P) that are connected to weakly and properly efficient solutions.

For $y \in \mathbb{R}^n$, $T_1 \subseteq T$ with $|T_1| < \infty$, and $\beta := (\beta_t)_{t\in T_1} \geqq 0_{|T_1|}$, put

$$\Upsilon(y,\beta,T_1) := \Big(f_1(y) + \sum_{t\in T_1} \beta_t g_t(y), \ldots, f_p(y) + \sum_{t\in T_1} \beta_t g_t(y) \Big).$$

Consider the following two dual problems:

$$(MW_1): \quad \max \Big\{ \Upsilon(y,\beta,T_1) \mid \exists \alpha := (\alpha_1,\ldots,\alpha_p), \ \sum_{i=1}^{p} \alpha_i = 1, \ (\alpha,y,\beta,T_1) \in S_1 \Big\},$$

and

$$(MW_2): \quad \max \Big\{ \Upsilon(y,\beta,T_1) \mid \exists \alpha := (\alpha_1,\ldots,\alpha_p), \ \sum_{i=1}^{p} \alpha_i = 1, \ (\alpha,y,\beta,T_1) \in S_2 \Big\},$$

where the feasible sets S_1 and S_2 are defined by

$$S_1 := \Big\{ (\alpha,y,\beta,T_1) \mid y \in \mathbb{R}^n, \ T_1 \subseteq T, \ |T_1| < \infty, \ (\beta_t)_{t\in T_1} \geqq 0_{|T_1|},$$
$$\alpha \geq 0_p, \ \ 0_n \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(y) + \sum_{t\in T_1} \beta_t \partial_c g_t(y) \Big\},$$

and

$$S_2 := \Big\{(\alpha, y, \beta, T_1) \mid y \in \mathbb{R}^n,\ T_1 \subseteq T,\ |T_1| < \infty,\ (\beta_t)_{t\in T_1} \geqq 0_{|T_1|},\\ \alpha > 0_p,\ \ 0_n \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(y) + \sum_{t\in T_1} \beta_t \partial_c g_t(y)\Big\}.$$

It is important to note that the difference between problems (MW_1) and (MW_2) lies in the freedom of α in S_1 compared to S_2. The following two theorems describe the (weak and strong) duality relations between the primal problem (P) and the dual problem (MW_1).

Theorem 7. (weak duality for MW_1) Suppose that $x \in S$ and $(\alpha, y, \beta, T_1) \in S_1$ are given, and that the f_i and g_t functions are η-invex at y for $i \in I$ and $t \in T_1$. Then, $f(x) \nless \Upsilon(y, \beta, T_1)$.

Proof. The proof is by contradiction. If $f(x) < \Upsilon(y, \beta, T_1)$, then

$$f_i(x) < f_i(y) + \sum_{t\in T_1} \beta_t g_t(y) \implies f_i(x) - f_i(y) < \sum_{t\in T_1} \beta_t g_t(y), \quad \forall i \in I. \tag{7}$$

Owing to $(\alpha, y, \beta, T_1) \in S_1$, we can find some $\xi_i^f \in \partial_c f_i(y)$ and $\xi_t^g \in \partial_c g_t(y)$ as $i \in I$ and $t \in T_1$ such that

$$\sum_{i=1}^{p} \alpha_i \xi_i^f + \sum_{t\in T_1} \beta_t \xi_t^g = 0_n,$$

and hence,

$$\begin{aligned} 0 &= \Big\langle \sum_{i=1}^{p} \alpha_i \xi_i^f + \sum_{t\in T_1} \beta_t \xi_t^g\ ,\ \eta(x,y)\Big\rangle \\ &= \sum_{i=1}^{p} \alpha_i \big\langle \xi_i^f, \eta(x,y)\big\rangle + \sum_{t\in T_1} \beta_t \big\langle \xi_t^g, \eta(x,y)\big\rangle \\ &\le \sum_{i=1}^{p} \alpha_i \rho(x,y)\big(f_i(x) - f_i(y)\big) + \sum_{t\in T_1} \beta_t \rho(x,y)\big(\underbrace{g_t(x)}_{\le 0} - g_t(y)\big) \\ &\le \sum_{i=1}^{p} \alpha_i \rho(x,y)\big(f_i(x) - f_i(y)\big) - \sum_{t\in T_1} \beta_t \rho(x,y) g_t(y) \\ &< \sum_{i=1}^{p} \alpha_i \rho(x,y) \sum_{t\in T_1} \beta_t g_t(y) - \sum_{t\in T_1} \beta_t \rho(x,y) g_t(y) = 0, \end{aligned} \tag{8}$$

where the final line holds by (7), $\alpha \geq 0_p$ and $\sum_{i=1}^{p} \alpha_i = 1$. This contradiction shows that $f(x) \nless \Upsilon(y, \beta, T_1)$, as required. □

Theorem 8. (strong duality for (MW_1)) Suppose that $\widehat{x}$ is a weakly efficient solution for (P) and that η-WSCQ is satisfied at $\widehat{x}$. If the f_i functions as $i \in I$ are η-invex at $\widehat{x}$, then there exist $\alpha \geq 0_p$ and $T_1 \subseteq T(\widehat{x})$ with $|T_1| < \infty$ and $\beta \geq 0_{|T_1|}$ such that $(\alpha, \widehat{x}, \beta, T_1) \in S_1$ and $f(\widehat{x}) = \Upsilon(\widehat{x}, \beta, T_1)$. Furthermore, $(\alpha, \widehat{x}, \beta, T_1)$ is a weakly efficient solution for dual problem (MW_1).

Proof. Employing Theorem 3, we can find some $\alpha := (\alpha_1, \ldots, \alpha_p) \geq 0_p$ and $T_1 \subseteq T(\widehat{x})$ with $|T_1| < \infty$, as well as $\beta_t \geq 0$ for $t \in T_1$ satisfying (2). So, $(\alpha, \widehat{x}, \beta, T_1) \in S_1$, and by $T_1 \subseteq T(\widehat{x})$ we have

$$\Upsilon(\widehat{x}, \beta, T_1) = \Big(f_1(\widehat{x}) + \sum_{t\in T_1} \beta_t g_t(\widehat{x}), \ldots, f_p(\widehat{x}) + \sum_{t\in T_1} \beta_t g_t(\widehat{x})\Big) = f(\widehat{x}).$$

Owing to the above equality and Theorem 7, we get $\Upsilon(\widehat{x}, \beta, T_1) \not< \Upsilon(y, \gamma, T_2)$ for any $(\alpha, y, \gamma, T_2) \in S_1$, and hence $(\widehat{x}, \beta, T_1)$ is a weakly efficient solution for the dual problem (MW_1). □

The next two theorems describe (weak and strong) duality relations between the primal problem (P) and the dual problem (MW_2).

Theorem 9. (weak duality for (MW_2)) Suppose that $x \in S$ and $(\alpha, y, \beta, T_1) \in S_1$ are given, and that the f_i and g_t functions are η-invex at y for $i \in I$ and $t \in T_1$. Then, $f(x) \nleq \Upsilon(y, \beta, T_1)$.

Proof. Assume on the contrary that $f(x) \leq \Upsilon(y, \beta, T_1)$. Thus, there exists an index $k \in I$ such that

$$\begin{cases} f_i(x) \leq f_i(y) + \sum_{t\in T_1} \beta_t g_t(y), & \forall i \in I \setminus \{k\}, \\ f_k(x) < f_k(y) + \sum_{t\in T_1} \beta_t g_t(y). \end{cases}$$

Similar proof of Theorem 7, the above relation and $\alpha > 0_p$ concludes (8), and this contradition finishes the proof. □

Theorem 10. (strong duality for (MW_2)) Suppose that $\widehat{x}$ is a properly efficient solution for (P) and that η-WSCQ is satisfied at $\widehat{x}$. If the f_i functions as $i \in I$ are η-invex at $\widehat{x}$, then there exist $\alpha > 0_p$ and $T_1 \subseteq T(\widehat{x})$ with $|T_1| < \infty$ and $\beta \geq 0_{|T_1|}$ such that $(\alpha, \widehat{x}, \beta, T_1) \in S_2$ and $f(\widehat{x}) = \Upsilon(\widehat{x}, \beta, T_1)$. Furthermore, $(\alpha, \widehat{x}, \beta, T_1)$ is an efficient solution for dual problem (MW_2).

Proof. Based on Theorem 9, the result is proved similar Theorem 8. □

Remark 3. It should be noted that our strong duality result, as presented in Theorem 10 is not typical in the sense that the solution of the dual problem is not guaranteedto be properly efficient, only efficient, while the solution to the primal problem is properly efficient.

5 Conclusion

This paper addressed the issue of non-smooth multi-objective semi-infinite programming problems which are characterized by a feasible set defined by inequality constraints. To tackle this problem, we proposed a new weak Slater constraint qualification based on the concept of a generalized η-inven function. By utilizing this new CQ, we can present both weak and strong KKT-type optimality conditions for the weakly and properly efficient solutions of the problem. Furthermore, we derived two dual problems and establish weak and strong duality results for them.

Declarations

Availability of supporting data

All data generated or analyzed during this study are included in this published paper.

Funding

No funds, grants, or other support were received for conducting this study.

Competing interests

The authors have no competing interests to declare that are relevant to the content of this paper.

Authors' contributions

The main manuscript text is written collectively by the authors.

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