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Fuzzy Number-Valued Fuzzy Graph

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Abstract. Graph theory has an important role in the area of applications of networks and clustering. In the case of dealing with uncertain data, we must utilize ambiguous data such as fuzzy value, fuzzy interval value or values of fuzzy number. In this study, values of fuzzy number were used. Initially, we utilized the fuzzy number value fuzzy relation and then proposed fuzzy number-value fuzzy graph on nodes and arcs. In this study, some properties of the graph on fuzzy number-value fuzzy graph were examined. First, we define the Cartesian product, composition, union and join operators on fuzzy number-value fuzzy graphs and then prove some of their properties and and give some examples for every one of definitions. We also introduced the notion of homomorphism, weak isomorphism,weak co-isomorphism, isomorphism, complete, weak complete and compliment on the fuzzy number fuzzy graphs and prove some of their properties and also present some examples for every one of them.

Keywords. Fuzzy numbers, Relation, Fuzzy relation, Graph, Fuzzy graph.
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1 Introduction

In mathematics, fuzzy sets are sets whose elements have degrees of membership. Fuzzy sets were introduced by Lotfi A. Zadeh in 1965 [19] as an extension of the classical notion of set. The usefulness of the introduced notion of fuzzy set theory was realized and applied in studies in almost all branches of science and technology by many researchers. ([4, 10, 11, 12, 13]).
The fuzzy relation theory as a generalization of Euler's graph theory was first introduced by Rosenfeld [14] in 1975. Some researchers work in graph by uncertain data on nodes and arcs such as fuzzy graph [15, 16], bipolar fuzzy graph [2, 4, 18] and fuzzy interval graph [8, 3]. A fuzzy number is a quantity whose value is imprecise, rather than exact as the case with "ordinary" (single-valued) numbers or interval numbers. M. Adabitabar firozja and S. Firouzian [1] define the fuzzy number valued fuzzy relation.
The remaining part of the paper is organized as follows: In section 2, a background of fuzzy concepts and fuzzy relations and also fuzzy numbers and some properties of fuzzy numbers are presented. We will introduce the fuzzy number-valued fuzzy graph and prove some properties of graph on fuzzy number-valued fuzzy graphs with examples in Section 3. Finally, conclusions are presented in Section 4.

2 Background

A fuzzy subset of X is a mapping $\mu : X \to [0,1]$ where μ as assigning to each element $x \in X$ a degree of membership, $0 \leq \mu(x) \leq 1$.
If S be a set and μ and ν be fuzzy subsets of S then following properties exist:
1. A fuzzy subset $\mu \subseteq \nu$ if and only if $\mu(x) \leq \nu(x)$ for all $x \in S$,
2. $(\mu \cup \nu)(x) = \mu(x) \bigvee \nu(x)$ for all $x \in S$,
3. $(\mu \cap \nu)(x) = \mu(x) \bigwedge \nu(x)$ for all $x \in S$,
where, max and min are shown with $\bigvee$ and $\bigwedge$ respectively.

Definition 1. (Rosenfeld, [14]) Let S and T be two sets and μ and ν be fuzzy subsets of S and T, respectively. A fuzzy relation ρ from the fuzzy subset μ into the fuzzy subset ν is a fuzzy subset ρ of $S \times T$ such that

$$\rho(x,y) \leq \mu(x) \bigwedge \nu(y), \forall x \in S, \forall y \in T$$

.
Definition 2. (Rosenfeld, [14]) Let $\rho : S \times T \to [0,1]$ be a fuzzy relation from a fuzzy subset μ of S into a fuzzy subset ν of T and $\omega : T \times U \to [0,1]$ be a fuzzy relation from a fuzzy subset ν of T into a fuzzy subset ξ of U. Define the composition ρ of ω and denote by $\rho o \omega : S \times U \to [0,1]$ where for all $x \in S$ and $z \in U$

$$\rho o \omega(x,z) = \bigvee \{\rho(x,y) \bigwedge \omega(y,z) | y \in T\}. \tag{1}$$

Notation ρ^2 to denote the composition $\rho o \rho$, ρ^k to denote the composition $\rho^{k-1} o \rho; k > 1$. Define $\rho^\infty(x,y) = \bigvee \{\rho^k(x,y) | k = 1,2,...\}$ for all $x, y \in S$.

Definition 3. (Coroianu, [7]) The set of all fuzzy numbers is denoted by FN and for fuzzy number $A \in FN$, we show the membership function by $A(x)$ which is given by

$$A(x) = \begin{cases} 0 & x \le a_1, \\ l_A(x) & a_1 \le x \le a_2, \\ 1 & a_2 \le x \le a_3, \\ r_A(x) & a_3 \le x \le a_4, \\ 0 & a_4 \le x \end{cases} \tag{2}$$

where $a_1, a_2, a_3, a_4 \in R$ and $l_A(.)$ is nondecreasing and $r_A(.)$ is non-increasing and $l_A(a_1) = 0$, $l_A(a_2) = 1$, $r_A(a_3) = 1$ and $r_A(a_4) = 0$. For any $r \in (0, 1]$, $r-$cut of fuzzy number A is a crisp interval as

$$[A]^r = \{x \in R : A(x) \ge r\} = [A_l(r), A_u(r)]. \tag{3}$$

Let A be a fuzzy subset of X; the support of A, denoted $supp(A)$ whose

$$supp(A) = \{x \in X | A(x) > 0\}. \tag{4}$$

Definition 4. [9] let $[A]^r = [A_l(r), A_u(r)]$ and $[B]^r = [B_l(r), B_u(r)]$ be two fuzzy numbers. We get:

$$A \bigvee B = [A_l(r) \bigvee B_l(r), A_u(r) \bigvee B_u(r)],$$

and

$$A \bigwedge B = [A_l(r) \bigwedge B_l(r), A_u(r) \bigwedge B_u(r)],$$

$$A + B = [A_l(r) + B_l(r), A_u(r) + B_u(r)],$$

$$k[A]^r = \begin{cases} [kA_l(r), kA_u(r)] & k \ge 0 \\ [kA_l(r), kA_u(r)] & k < 0, \end{cases}$$

and we used of the following ranking method

$$A \preceq B \Leftrightarrow \begin{cases} A_l(r) \le B_l(r) \\ and \\ A_u(r) \le B_u(r) \end{cases} \quad \forall\ r \tag{5}$$

$$A = B \Leftrightarrow \begin{cases} A_l(r) = B_l(r) \\ and \\ A_u(r) = B_u(r). \end{cases} \quad \forall\ r \tag{6}$$

Proposition 1. With the above ranking $A \bigwedge B \preceq A$ and $A \bigwedge B \preceq B$.

Definition 5. (Stefanini;[17], Bede-Stefanini;[5])The generalized difference (g-difference for short) of two fuzzy numbers A, B is given by its level sets as

$$[A \ominus_g B]^\alpha = Cl \bigcup_{\beta \ge \alpha} ([A]^\beta \ominus_{gH} [B]^\beta); \quad \forall \alpha \in [0, 1] \tag{7}$$

where

$$[A]^\beta \ominus_{gH} [B]^\beta = [C]^\beta \Leftrightarrow \begin{cases} [A]^\beta = [B]^\beta + [C]^\beta \\ or \\ [B]^\beta = [A]^\beta + (-1)[C]^\beta \end{cases} \tag{8}$$

Proposition 2. (Bede-Stefanini;[5]) The g-difference is given by the expression

$$[A \ominus_g B]^\alpha = [inf_{\beta \geq \alpha} min\ F(\beta), sup_{\beta \geq \alpha} max\ F(\beta)] \quad (9)$$

where $F(\beta) = \{A_l(\beta) - B_l(\beta), A_u(\beta) - B_u(\beta)\}$.
Proposition 3.(Proposition 5.20 ;[6]) For any fuzzy numbers A, B the g-difference $A \ominus_g B$ exists and it is a fuzzy number.

3 Fuzzy number-valued fuzzy graph (FN-VFG)

We show the set of all fuzzy numbers with support subset of $[0, 1]$ by FIN.
Definition 6. The fuzzy number valued fuzzy set A in V defined by

$$A : V \to FIN$$

where

$$A = \{(x, A(x))|x \in V\} = \{(x, [A(x)]^r)|x \in V, \quad r \in [0, 1]\} \quad (10)$$

Where $[A(x)]^r = [\underline{A(x)}(r), \overline{A(x)}(r)]$ is $r-$level.
For any two fuzzy number valued fuzzy sets A and B in V we define:

$$\begin{aligned} A \cup B &= \{(x, A(x) \vee B(x))|x \in V\} \\ A \cap B &= \{(x, A(x) \wedge B(x))|x \in V\} \end{aligned} \quad (11)$$

M. Adabitabar firozja and S. Firozian [1] define the fuzzy number valued fuzzy relation as follows:
Definition 7. Let S and T be two sets and $\mu : S \to FIN$ and $\nu : T \to FIN$ be two fuzzy number valued subsets of S and T, respectively. A FN-VFR ρ from μ into ν is a fuzzy number valued subset $\rho : S \times T \to FIN$ such that $\rho(x, y) \preceq \mu(x) \bigwedge \nu(y), \forall x \in S$ and $\forall y \in T$.

Definition 8. If $G^* = (V, E)$ is a graph and A is fuzzy number valued fuzzy set in V defined by definition 6, then by fuzzy number valued fuzzy relation B on set E by definition 7, we define $G = (A, B)$ is FN-VFG where
$A = \{(x, A(x))|x \in V, A(x) \in FIN\}$ and
$B = \{(xy, B(xy))|xy \in E, B(xy) \preceq \{A(x) \wedge A(y)\} \in FIN\}$.

Throughout this paper, G^* is a crisp graph and G is an fuzzy number valued fuzzy graph.
Example 1. Let $G^* = (V, E)$ is a graph such that $V = \{x, y, z\}$ and $E = \{xy, yz, zx\}$. Let A be an fuzzy number valued fuzzy set of V and B be an fuzzy number valued fuzzy set of $E \subseteq V \times V$ defined by:

$$A = \{(x, [0.2 + 0.1r, 0.4 - 0.1r]), (y, [0.3 + 0.1r, 0.5 - 0.1r]), (z, [0.4 + 0.05r, 0.5 - 0.05r])\}$$

$$B = \{(xy, [0.1 + 0.1r, 0.3 - 0.1r]), (yz, [0.2 + 0.1r, 0.4 - 0.1r]), (zx, [0.1 + 0.2r, 0.4 - 0.1r])\}$$

Then $G = (A, B)$ is FN-VFG.
Definition 9. The Cartesian product $G_1 \times G_2$ of two FN-VFGs, $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is defined as a pair $(A_1 \times A_2, B_1 \times B_2)$ such that

$$\{A_1 \times A_2\}(x_1, x_2) = A_1(x_1) \wedge A_2(x_2); \quad (x_1, x_2) \in V_1 \times V_2 \tag{12}$$

$$\{B_1 \times B_2\}((x, x_2)(x, y_2)) = A_1(x) \wedge B_2(x_2 y_2); \quad x \in V_1, \ x_2 y_2 \in E_2 \tag{13}$$

$$\{B_1 \times B_2\}((x_1, z)(y_1, z)) = B_1(x_1 y_1) \wedge A_2(z); \quad x_1 y_1 \in E_1, \ z \in V_2, \tag{14}$$

Example 2. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be graphs where $V_1 = \{a, b\}$, $V_2 = \{c, d\}$, $E_1 = \{ab\}$ and $E_2 = \{cd\}$. If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ two FN-VFGs where

$$A_1 = \{(a, [0.2 + 0.1r, 0.4 - 0.1r]), (b, [0.3 + 0.1r, 0.5 - 0.1r])\} \tag{15}$$

$$B_1 = \{(ab, [0.1 + 0.1r, 0.4 - 0.2r])\} \tag{16}$$

$$A_2 = \{(c, [0.1 + 0.2r, 0.4 - 0.1r]), (d, [0.2 + 0.1r, 0.6 - 0.2r])\} \tag{17}$$

$$B_2 = \{(cd, [0.1r, 0.4 - 0.2r])\} \tag{18}$$

then we have

$$\{A_1 \times A_2\}(a, c) = [0.1 + 0.2r, 0.4 - 0.1r]$$

$$\{A_1 \times A_2\}(a, d) = [0.2 + 0.1r, 0.4 - 0.1r]$$

$$\{A_1 \times A_2\}(b, c) = [0.1 + 0.2r, 0.4 - 0.1r]$$

$$\{A_1 \times A_2\}(b, d) = [0.2 + 0.1r, 0.5 - 0.1r]$$

$$\{B_1 \times B_2\}((a, c)(a, d)) = [0.1r, 0.4 - 0.2r]$$

$$\{B_1 \times B_2\}((b, c)(b, d)) = [0.1r, 0.4 - 0.2r]$$

$$\{B_1 \times B_2\}((a, c)(b, c)) = [0.1 + 0.1r, 0.4 - 0.2r]$$

$$\{B_1 \times B_2\}((a, d)(b, d)) = [0.1 + 0.1r, 0.4 - 0.2r]$$

Proposition 4. The Cartesian product $G_1 \times G_2 = (A_1 \times A_2, B_1 \times B_2)$ of two FN-VFGs of the graphs of G_1^* and G_2^* is a FN-VFG of $G_1^* \times G_2^*$.
Proof. Let $x \in V_1$, $x_2 y_2 \in E_2$. Then
$\{B_1 \times B_2\}((x, x_2)(x, y_2)) = A_1(x) \wedge B_2(x_2 y_2) \preceq A_1(x) \wedge (A_2(x_2) \wedge A_2(y_2))$
$= (A_1(x) \wedge A_2(x_2)) \wedge (A_1(x) \wedge A_2(y_2)) = A_1 \times A_2(x, x_2) \wedge A_1 \times A_2(x, y_2)$

Similarly, if $z \in V_2$, $x_1 y_1 \in E_1$ we have
$\{B_1 \times B_2\}((x_1, z)(y_1, z)) = B_1(x_1 y_1) \wedge A_2(z) \preceq (A_1(x_1) \wedge A_1(y_1)) \wedge A_2(z)$
$= (A_1(x_1) \wedge A_2(z)) \wedge (A_1(y_1) \wedge A_2(z)) = A_1 \times A_2(x_1, z) \wedge A_1 \times A_2(y_1, z)$

Definition 10. The composition $G_1[G_2] = (A_1 \circ A_2, B_1 \circ B_2))$ of two FN-VFGs G_1 and G_2 of the graphs G_1^* and G_2^* is defined as follows:

$$\begin{gathered} \{A_1 \circ A_2\}(x_1, x_2) = A_1(x_1) \wedge A_2(x_2) \quad \forall (x_1, x_2) \in V \\ \{B_1 \circ B_2\}((x, x_2)(x, y_2)) = A_1(x) \wedge B_2(x_2 y_2) \quad \forall x \in V_1 \ , x_2 y_2 \in E_2, \\ \{B_1 \circ B_2\}((x_1, z)(y_1, z)) = B_1(x_1 y_1) \wedge A_2(z) \quad \forall z \in V_2 \ , x_1 y_1 \in E_1, \\ \{B_1 \circ B_2\}((x_1, x_2)(y_1, y_2)) = A_2(x_2) \wedge A_2(y_2) \wedge B_1(x_1 y_1) \quad \forall (x_1, x_2)(y_1, y_2) \in E^0 - E \end{gathered} \tag{19}$$

Example 3. Let G_1^* and G_2^* be as in the previous example. Consider two FN-VFGs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ defined by

$$A_1 = \{(a, [0.2 + 0.1r, 0.4 - 0.1r]), (b, [0.3 + 0.1r, 0.5 - 0.1r])\} \tag{20}$$

$$B_1 = \{(ab, [0.1 + 0.1r, 0.4 - 0.2r])\} \tag{21}$$

$$A_2 = \{(c, [0.1 + 0.2r, 0.4 - 0.1r]), (d, [0.2 + 0.1r, 0.6 - 0.2r])\} \tag{22}$$

$$B_2 = \{(cd, [0.1r, 0.4 - 0.2r])\} \tag{23}$$

then we have

$$\{B_1 \circ B_2\}((a,c)(a,d)) = A_1(a) \wedge B_2(cd) = [0.1r, 0.4 - 0.2r] \tag{24}$$

$$\{B_1 \circ B_2\}((b,c)(b,d)) = A_1(b) \wedge B_2(cd) = [0.1r, 0.4 - 0.2r] \tag{25}$$

$$\{B_1 \circ B_2\}((a,c)(b,c)) = B_1(ab) \wedge A_2(c) = [0.1 + 0.1r, 0.4 - 0.2r] \tag{26}$$

$$\{B_1 \circ B_2\}((a,d)(b,d)) = B_1(ab) \wedge A_2(d) = [0.1 + 0.1r, 0.4 - 0.2r] \tag{27}$$

$$\{B_1 \circ B_2\}((b,c)(a,d)) = A_2(c) \wedge A_2(d) \wedge B_1(ab) = [0.1 + 0.1r, 0.4 - 0.2r] \tag{28}$$

Proposition 5. The composition $G_1[G_2]$ of FN-VFGs G_1 and G_2 of G_1^* and G_2^* is an FN-VFG of $G_1^*[G_2^*]$.
Proof. We proved one of cases and sufficient that show

$$\{B_1 \circ B_2\}((x,y)(x,z)) \preceq \{A_1 \circ A_2\}(x,y) \wedge \{A_1 \circ A_2\}(x,z) \quad ;\forall \ \ x \in A_1, \ yz \in E_1 \tag{29}$$

$\{B_1 \circ B_2\}((x,y)(x,z)) = A_1(x) \wedge B_2(yz) \preceq A_1(x) \wedge (A_2(y) \wedge A_2(z)) =$
$(A_1(x) \wedge A_2(y)) \wedge (A_1(x) \wedge A_2(z)) = \{A_1 \circ A_2\}(x,y) \wedge \{A_1 \circ A_2\}(x,z)$.
Definition 11. The union $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2))$ of two FN-VFGs G_1 and G_2 of the graphs G_1^* and G_2^* is defined as follows:

$$\begin{gathered}\{A_1 \cup A_2\}(x) = A_1(x) \vee A_2(x) \\ \{B_1 \cup B_2\}(xy) = B_1(xy) \vee B_2(xy)\end{gathered} \tag{30}$$

Example 4. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be graphs such that $V_1 = \{a, b, c, d, e\}$, $E_1 = \{ab, bc, be, ce, ad, ed\}$, $V_2 = \{a, b, c, d, f\}$ and $E_2 = \{ab, bc, cf, bf, bd\}$. If two FN-VFGs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ defined by

$$\begin{gathered}A_1 = \{(a, [0.1 + 0.3r, 0.7 - 0.1r]), (b, [0.2 + 0.2r, 0.6 - 0.2r]), \\ (c, [0.5 + 0.2r, 0.9 - 0.2r]), (d, [0.4 + 0.1r, 0.7 - 0.2r]), \\ (e, [0.3 + 0.2r, 0.8 - 0.1r])\}\end{gathered} \tag{31}$$

$$\begin{gathered}B_1 = \{(ab, [0.1 + 0.3r, 0.6 - 0.2r]), (bc, [0.2 + 0.2r, 0.6 - 0.2r]), \\ (ce, [0.3 + 0.2r, 0.8 - 0.1r]), (be, [0.2 + 0.2r, 0.6 - 0.2r]), \\ (ad, [0.1 + 0.3r, 0.7 - 0.2r]), (de, [0.3 + 0.2r, 0.7 - 0.2r])\}\end{gathered} \tag{32}$$

$$\begin{gathered}A_2 = \{(a, [0.2 + 0.1r, 0.6 - 0.2r]), (b, [0.5 + 0.2r, 0.8 - 0.1r]), \\ (c, [0.3 + 0.2r, 0.7 - 0.1r]), (d, [0.1 + 0.2r, 0.5 - 0.1r]), (f, [0.3 + 0.1r, 0.6 - 0.1r])\}\end{gathered} \tag{33}$$

$$\begin{gathered}B_2 = \{(ab, [0.2 + 0.1r, 0.6 - 0.2r]), (bc, [0.3 + 0.2r, 0.7 - 0.1r]), \\ (cf, [0.3 + 0.1r, 0.6 - 0.1r]), (bf, [0.3 + 0.1r, 0.6 - 0.1r]), \\ (bd, [0.1 + 0.2r, 0.5 - 0.1r])\}\end{gathered} \tag{34}$$

then we have

$$\{A_1 \cup A_2\}(a) = \begin{cases} [0.2 + 0.1r, 0.7 - 0.1r] & r \in [0, 0.5], \\ [0.1 + 0.3r, 0.7 - 0.1r] & r \in [0.5, 1] \end{cases} \tag{35}$$

$$\{A_1 \cup A_2\}(b) = [0.5 + 0.2r, 0.8 - 0.1r] \quad , \quad A_1 \cup A_2(c) = [0.5 + 0.2r, 0.9 - 0.2r] \tag{36}$$

$$\{A_1 \cup A_2\}(d) = [0.4 + 0.1r, 0.7 - 0.2r] \quad , \quad A_1 \cup A_2(e) = [0.3 + 0.2r, 0.8 - 0.1r] \tag{37}$$

$$\{A_1 \cup A_2\}(f) = [0.3 + 0.1r, 0.6 - 0.1r] \tag{38}$$

$$\{B_1 \cup B_2\}(ab) = \begin{cases} [0.2+0.1r, 0.6-0.2r] & r \in [0,0.5], \\ [0.1+0.3r, 0.6-0.2r] & r \in [0.5,1] \end{cases} \tag{39}$$

$$\{B_1 \cup B_2\}(bc) = [0.3+0.2r, 0.7-0.1r] \quad , \quad B_1 \cup B_2(ce) = [0.3+0.2r, 0.8-0.1r] \tag{40}$$

$$\{B_1 \cup B_2\}(be) = [0.2+0.2r, 0.6-0.2r] \quad , \quad B_1 \cup B_2(ad) = [0.1+0.3r, 0.7-0.2r] \tag{41}$$

$$\{B_1 \cup B_2\}(de) = [0.3+0.2r, 0.7-0.2r] \quad , \quad B_1 \cup B_2(bd) = [0.1+0.2r, 0.5-0.1r] \tag{42}$$

$$\{B_1 \cup B_2\}(bf) = [0.3+0.1r, 0.6-0.1r] \tag{43}$$

Proposition 6. The union of two FN-VFGs G_1 and G_2 of G_1^* and G_2^* is an FN-VFG.
Proof.

$$\begin{aligned} \{B_1 \cup B_2\}(xy) = B_1(xy) \vee B_2(xy) \preceq \{A_1(x) \wedge A_1(y)\} \vee \{A_2(x) \wedge A_2(y)\} \\ = \{A_1(x) \vee A_2(x)\} \wedge \{A_1(y) \vee A_2(y)\} = \{A_1 \cup A_2\}(x) \wedge \{A_1 \cup A_2\}(y) \end{aligned} \tag{44}$$

Definition 12. The join $G_1 + G_2 = (A_1 + A_2, B_1 + B_2))$ of two FN-VFGs G_1 and G_2 of the graphs G_1^* and G_2^* is defined as follows:

$$\begin{aligned} \{A_1 + A_2\}(x) &= A_1(x) \vee A_2(x) \\ \{B_1 + B_2\}(xy) &= B_1(xy) \vee B_2(xy); \quad if\ xy \in E_1 \cap E_2 \\ \{B_1 + B_2\}(xy) &= A_1(x) \wedge A_2(y); \qquad if\ xy \in E^{'} \end{aligned} \tag{45}$$

Where $E^{'}$ is the set of all edges joining the nodes of V_1 and V_2.
Proposition 7. The join of FN-VFGs is a FN-VFG.
Proof. If $xy \in E_1 \cap E_2$ then

$$\begin{aligned} \{B_1 + B_2\}(xy) = B_1(xy) \vee B_2(xy) \preceq \{A_1(x) \wedge A_1(y)\} \vee \{A_2(x) \wedge A_2(y)\} \\ = \{A_1(x) \vee A_2(x)\} \wedge \{A_1(y) \vee A_2(y)\} = \{A_1 + A_2\}(x) \wedge \{A_1 + A_2\}(y) \end{aligned} \tag{46}$$

If $xy \in E^{'}$ then

$$\begin{aligned} \{B_1 + B_2\}(xy) = A_1(x) \wedge A_2(y) \preceq \{A_1(x) \vee A_2(x)\} \wedge \{A_1(y) \vee A_2(y)\} \\ = \{A_1 + A_2\}(x) \wedge \{A_1 + A_2\}(y) \end{aligned} \tag{47}$$

Definition 13. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two FN-VFGs. A homomorphism $f : G_1 \to G_2$ is a mapping $f : V_1 \to V_2$ such that

$$\begin{aligned} &A_1(x) \preceq A_2(f(x)); && \forall x \in V_1 \\ &B_1(xy) \preceq B_2(f(x)f(y)); && \forall x \in V_1, xy \in E_1 \end{aligned} \tag{48}$$

A bijective homomorphism with the property $A_1(x) = A_2(f(x))$ is called a weak isomorphism.
A bijective homomorphism with the property $B_1(xy) = B_2(f(x)f(y))$ is called a weak co-isomorphism.
A bijective mapping is called an isomorphism if weak isomorphism and weak co-isomorphism.

Example 5. Consider graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that $V_1 = \{a_1, b_1\}$, $V_2 = \{a_2, b_2\}$, $E_1 = \{a_1b_1\}$ and $E_2 = \{a_2b_2\}$. If A_1, A_2, B_1 and B_2 be fuzzy number valued fuzzy subsets defined by:
(i)

$$A_1 = \{(a_1, [0.2+0.1r, 0.6-0.2r]), (b_1, [0.3+0.2r, 0.7-0.2r])\} \tag{49}$$

$$B_1 = \{(a_1b_1, [0.1+0.2r, 0.5-0.1r])\} \tag{50}$$

$$A_2 = \{(a_2, [0.3+0.2r, 0.8-0.2r]), (b_2, [0.2+0.2r, 0.6-0.1r])\} \tag{51}$$

$$B_2 = \{(a_2b_2, [0.2 + 0.2r, 0.6 - 0.2r])\} \tag{52}$$

Then, as is easy to see, the map $f : V_1 \to V_2$ defined by $f(a_1) = b_2$ and $f(b_1) = a_2$ is a homomorphism.

(ii)

$$A_1 = \{(a_1, [0.2 + 0.2r, 0.6 - 0.1r]), (b_1, [0.3 + 0.2r, 0.7 - 0.2r])\} \tag{53}$$

$$B_1 = \{(a_1b_1, [0.1 + 0.2r, 0.5 - 0.1r])\} \tag{54}$$

$$A_2 = \{(a_2, [0.3 + 0.2r, 0.7 - 0.2r]), (b_2, [0.2 + 0.2r, 0.6 - 0.1r])\} \tag{55}$$

$$B_2 = \{(a_2b_2, [0.2 + 0.1r, 0.5 - 0.1r])\} \tag{56}$$

Then, as is easy to see, the map $f : V_1 \to V_2$ defined by $f(a_1) = b_2$ and $f(b_1) = a_2$ is a weak isomorphism but it is not an isomorphism.

(iii)

$$A_1 = \{(a_1, [0.2 + 0.2r, 0.6 - 0.2r]), (b_1, [0.3 + 0.2r, 0.7 - 0.2r])\} \tag{57}$$

$$B_1 = \{(a_1b_1, [0.1 + 0.2r, 0.5 - 0.1r])\} \tag{58}$$

$$A_2 = \{(a_2, [0.3 + 0.2r, 0.8 - 0.2r]), (b_2, [0.2 + 0.3r, 0.6 - 0.1r])\} \tag{59}$$

$$B_2 = \{(a_2b_2, [0.1 + 0.2r, 0.5 - 0.1r])\} \tag{60}$$

Then, as is easy to see, the map $f : V_1 \to V_2$ defined by $f(a_1) = b_2$ and $f(b_1) = a_2$ is a weak co-isomorphism but it is an isomorphism.

(iv)

$$A_1 = \{(a_1, [0.2 + 0.2r, 0.6 - 0.1r]), (b_1, [0.3 + 0.2r, 0.7 - 0.2r])\} \tag{61}$$

$$B_1 = \{(a_1b_1, [0.1 + 0.2r, 0.5 - 0.1r])\} \tag{62}$$

$$A_2 = \{(a_2, [0.3 + 0.2r, 0.7 - 0.2r]), (b_2, [0.2 + 0.2r, 0.6 - 0.1r])\} \tag{63}$$

$$B_2 = \{(a_2b_2, [0.1 + 0.2r, 0.5 - 0.1r])\} \tag{64}$$

Then, as is easy to see, the map $f : V_1 \to V_2$ defined by $f(a_1) = b_2$ and $f(b_1) = a_2$ is an isomorphism.

Definition 14. A FN-VFG, $G = (A, B)$ is called complete if

$$B(xy) = A(x) \wedge A(y); \quad \forall xy \in E \tag{65}$$

Example 6. Consider a graph $G^* = (V, E)$ such that $V = \{x, y, z\}, E = \{xy, yz, zx\}$. If A and B are fuzzy number valued fuzzy subset defined by
$A = \{(x, [0.3 + 0.2r, 0.8 - 0.2r]), (y, [0.2 + 0.2r, 0.6 - 0.1r]), (z, [0.2 + 0.3r, 0.7 - 0.1r])\}$
$B = \{(xy, [0.2 + 0.2r, 0.6 - 0.1r]), (yz, [0.2 + 0.2r, 0.6 - 0.1r]), (xz, [0.2 + 0.3r, 0.7 - 0.1r])\}$
then $G = (A, B)$ is a complete FN-VFG of G^*.

Definition 15. The complement of a FN-VFG, $G = (A, B)$ of $G^* = (V, E)$ is a $\overline{G} = (\overline{A}, \overline{B})$ on $\overline{G^*} = (V, E)$, where $A = \overline{A}$ and $\overline{B}$ is defined by

$$\overline{B}(xy) = \{A(x) \wedge A(y)\} \ominus_g B(xy) \tag{66}$$

Proposition 8. The complement of FN-VFG is a FN-VFG.

Proof. If $\overline{G} = (A, \overline{B})$ is complement FN-VFG of $G = (A, B)$ then it is sufficient that show

$$\overline{B}(xy) \preceq A(x) \wedge A(y); \quad \forall xy \in E \tag{67}$$

where with Proposition 2. and Proposition 3. and Eq. (66) proof is evident.

Example 7. Consider a graph $G^* = (V, E)$ such that $V = \{x, y, z, t\}, E = \{xy, xt, yz, yt, zx, zt\}$. If $G = (A, B)$ is FN-VFG where

$A = \{(x, [0.3+0.2r, 0.8-0.2r]), (y, [0.2+0.2r, 0.6-0.1r]), (z, [0.2+0.3r, 0.7-0.1r]), (t, [0.4+0.2r, 0.9-0.2r])\}$
$B = \{(xy, [0.2+0.2r, 0.6-0.1r]), (yz, [0.1+0.2r, 0.6-0.1r]), (xz, [0.1+0.2r, 0.6-0.2r]), (xt, [0.1+0.1r, 0.3-0.1r])\}$
then $\overline{G} = (\overline{A}, \overline{B})$ is a complement of G where
$\overline{A} = A = \{(x, [0.3+0.2r, 0.8-0.2r]), (y, [0.2+0.2r, 0.6-0.1r]), (z, [0.2+0.3r, 0.7-0.1r]), (t, [0.4+0.2r, 0.9-0.2r])\}$
$\overline{B} = \{(tz, [0.2+0.3r, 0.7-0.1r]), (yz, [0, 0.1-0.1r]), (xz, [0.1+0.1r, 0.2]), (xt, [0.2+0.1r, 0.5-0.1r])\}$

Definition 16. A FN-VFG, $G = (A, B)$ is called weak complete if

$$0 < B(xy) \preceq A(x) \wedge A(y); \quad \forall xy \in E \tag{68}$$

Proposition 9. If $G = (A, B)$ is FN-VFG, then $G \cup \overline{G}$ is weak complete FN-VFG.
Proof. It sufficient that $B(xy) \vee \overline{B}(xy) > 0$ where with Equation (66)

4 Conclusions

It is well known that fuzzy number valued fuzzy sets constitute a generalization of the notion of fuzzy sets. The fuzzy number-valued fuzzy models give more precision, flexibility and compatibility to the system when compared to the classical and fuzzy models. Therefore, we have introduced fuzzy number-valued fuzzy graph and have presented several properties for this relation. The further study of fuzzy number-valued fuzzy graph may also be used for application on distribution networks such as water, electricity, gas etc. Moreover, in FN-VFG, other properties in graph can be reviewed.

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چکیده
نظریه گراف دارای نقش مهمی در زمینه کاربردهای شبکه‌ها و خوشه‌بندی است. وقتی با داده‌های مبهم مواجه می‌شویم بایستی داده‌های مبهم از قبیل مقادیر فازی، مقادیر بازه‌های فازی یا اعداد فازی استفاده کنیم. در این بررسی مقادیر اعداد فازی به کار رفته است. نخست، مقادیر اعداد فازی و روابط فازی را به کار برده و سپس گراف‌های با مقدار عدد فازی روی گره‌ها و کمان را ارائه می‌کنیم. در این تحقیق، برخی ویژگی‌های گراف مربوط به گراف‌های فازی با مقدار عدد فازی ارائه شد. نخست، ما حاصضرب دکارتی، ترکیب، اجتماع و پیوستگی را برای گراف‌های با مقادیر عدد فازی تعریف کرده و سپس برخی از خواص آنرا ثابت نموده و مثالهایی برای هریک از تعاریف ارائه می‌کنیم. همچنین مفاهیم هم‌ریختی، یکریختی ضعیف، هم‌یکریختی ضعیف، یکریختی، کامل، کامل ضعیف و متمم را برای این دسته از گراف‌ها معرفی و خواص مربوط به آنها را ثابت می‌کنیم و همچنین مثالهایی برای هر یک از آنها ارائه می‌کنیم.

کلمات کلیدی
عدد فازی، رابطه، رابطه فازی، گراف، گراف فازی.