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Dynamical Behaviour of Fractional Order *SEIR* Mathematical Model for Infectious Disease Transmission

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Abstract. This paper presents an extension of the $SEIR$ mathematical model for infectious disease transmission to a fractional-order model. The model is formulated using the Caputo derivative of order $\alpha \in (0, 1]$. We study the stability of equilibrium points, including the disease-free equilibrium (E_f), and the infected steady-state equilibrium (E_e) using the stability theorem of Fractional Differential Equations. The model is also analyzed under certain conditions, and it is shown that the disease-free equilibrium is locally asymptotically stable. Additionally, the extended Barbalat's lemma is applied to the fractional-order system, and a suitable Lyapunov functional is constructed to demonstrate the global asymptotic stability of the infected steady-state equilibrium. To validate the theoretical results, a numerical simulation of the problem is conducted.

Keywords. Fractional calculus, Caputo derivatives, $SEIR$ model, Lyapunov function, Stability.

MSC. 97M60; 34D20.

1 Introduction

Fractional calculus has found applications in various fields, including engineering, sciences, applied mathematics, and economics [8, 10, 11, 12, 21, 23, 24, 26]. Fractional differential equations $(FDEs)$, are particularly useful in epidemiological studies, as fractional derivatives can increase the stability region of the system. Due to their ability to model and describe the abnormal dynamics of real-world processes with memory and hereditary properties, many mathematicians have used fractional calculus to model natural processes [22]. Fractional constitutive models have been developed to model many real-world problems in various fields.

To obtain a good mathematical model of real-world problems, it is essential to identify the main parameters. Mathematical models have been developed to study more realistic aspects of disease spreading.

There are several mathematical models for disease dynamics, such as the SIS, SIR, $SIRS$, and $SEIR$ models, which represent the stages of the disease for each individual in a population. Some researchers have extended classical epidemic models to fractional-order epidemic systems with the Caputo derivative and discussed the local stability of the disease-free (non-negative boundary) equilibrium and endemic (positive) equilibrium [5, 8, 9].

Stability is a powerful tool for analyzing the qualitative properties of epidemiological mathematical models. The Lyapunov's direct method, also called the second Lyapunov's method, is a method for studying the stability of mathematical models in biology. The Lyapunov direct method is based on constructing an appropriate functional, called the Lyapunov functional, which investigates the stability of an equilibrium point without explicitly solving for an integer-order nonlinear system. The Lyapunov direct method is a sufficient condition to show the stability of systems, which means the system may still be stable, even if one cannot find a Lyapunov function candidate to conclude the system stability property [18].

In [4, 7, 15, 17], the authors introduced the fractional Lyapunov's second method. In recent years, different Lyapunov functions have been proposed to study the stability analysis of Caputo differential equations of fractional order [29]. The stability of nonlinear fractional differential systems with Caputo derivatives using the comparison method is studied also [8].

The remainder of this paper is structured as follows. In Section 2, we first provide some definitions and preliminaries about fractional calculus. In Section 3, we present the classical $SEIR$ mathematical model for infectious disease transmission and extend it to the context of fractional calculus. In Section 4, we study the existence, as well as uniqueness, of the equilibrium points and their local and global asymptotic stability. In Section 5, we present numerical results with a numerical example. Finally, in Section 6, we conclude the paper with some final remarks.

2 Essential Concepts

In this section, we provide definitions of fractional-order integration and derivative.

Definition 1. [25]. Let $\alpha \geq 0$ and $L_1[t_0, t_f]$ be a *Lebesgue space*. The operator $\mathcal{I}_a^{\alpha}$, which is defined on $L_1[t_0, t_f]$ by

$$\mathcal{I}_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s) ds \quad \alpha > 0,$$
$$\mathcal{I}_{t_0}^{0} f(t) = f(t), \tag{1}$$

for $t_0 \leq t \leq t_f$, is called the *Riemann-Liouville fractional integral* operator of order α, where $\Gamma(.)$ is the Gamma function.

Definition 2. The *Caputo fractional derivatives* is defined by

$${}^{C}\mathcal{D}_{0^+}^{\alpha} f(t) = (\mathcal{I}_{0^+}^{n-\alpha} f^{(n)})(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \tag{2}$$

where $n-1 \leq \alpha < n$, $n \in \mathbb{N}$, $\mathcal{D} = \frac{d}{dt}$.

For the Caputo derivative, we have

$${}^{C}\mathcal{D}_{0^+}^{\alpha} [\beta f(x) + \gamma g(x)] = \beta \, {}^{C}\mathcal{D}_{0^+}^{\alpha} f(x) + \gamma \, {}^{C}\mathcal{D}_{0^+}^{\alpha} g(x),$$
$${}^{C}\mathcal{D}_{0^+}^{\alpha} \beta = 0,$$

where β and γ are constants.

The Caputo derivative of x^{β} is given by

$${}^{C}\mathcal{D}_{0^+}^{\alpha} x^{\beta} = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \alpha \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geq \lceil \alpha \rceil, \text{or } \beta > \lfloor \alpha \rfloor. \end{cases} \tag{3}$$

where we use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α and the floor function $\lfloor \alpha \rfloor$ to denote the largest integer less than or equal to α and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

3 The Mathematical Model of Infectious Disease Transmission

Let us introduce the classical $SEIR$ mathematical model for infectious disease transmission, which has been extensively studied in the literature [2, 3, 16, 28]. The $SEIR$ model is a theoretical study that divides the total host population, $N(t)$, into four classes: Susceptible $(S(t))$, Exposed $(E(t))$, Infected $(I(t))$, and Recovered $(R(t))$. We assume that once an individual recovers from the disease, they acquire lifelong immunity. Additionally, the birth and death rates are assumed to be, constant in time, equal, and denoted by ν. Thus, it follows that the population size $N(t)$ remains constant for a long time. Specifically, we have $N(t) = S(t) + E(t) + I(t) + R(t)$.

The model of the infection disease transmission is given by the following system of differential equations:

$$\begin{cases} \dot{S}(t) = \nu N - \rho S(t) I(t) - (\nu + \mu) S(t), \\ \dot{E}(t) = \rho S(t) I(t) - (\delta + \nu) E(t), \\ \dot{I}(t) = \delta E(t) - (\nu + \lambda + \beta + \gamma) I(t), \\ \dot{R}(t) = (\beta + \gamma) I(t) + \mu S(t) - \nu R(t). \end{cases} \tag{4}$$

The initial conditions now are $(S(0), E(0), I(0), R(0)) \in \mathbb{R}_+^4$. In these equations, all the parameters are non-negative. The main parameters of the model (4) are defined in Table 1.

Table 1: Parameters used for system (4).

Symbol	Description	Values range	Reference
ν	Rate of birth population	0.0121	[14]
ρ	Transmission coefficient	0.125	[1]
δ	Rate of moving from E to I	0.02	[2]
β	Rate of moving from I to R without treatment	0.025	[1]
γ	Rate of treatment per year	$0 \leq \gamma \leq 1$	-
μ	Rate of vaccination per year	$0 \leq \mu \leq 1$	-
λ	Rate of death in population by infectious disease	0.0008	[1]

For simplicity, we normalize the population size to 1. Then, $s = \frac{S}{N}$, $e = \frac{E}{N}$, $i = \frac{I}{N}$ and $r = \frac{R}{N}$ represent the fractions of susceptible, exposed, infective, and recovered individuals in the population, respectively, and satisfy the constraint $s + e + i + r = 1$ [14]. Thus, model (4) becomes:

$$\begin{cases} \dot{s}(t) = \nu - \rho s(t)i(t) - (\nu + \mu)s(t), \\ \dot{e}(t) = \rho s(t)i(t) - (\delta + \nu)e(t), \\ \dot{i}(t) = \delta e(t) - (\nu + \lambda + \beta + \gamma)i(t), \\ \dot{r}(t) = (\beta + \gamma)i(t) + \mu s(t) - \nu r(t). \end{cases} \tag{5}$$

The initial conditions are $(s(0), e(0), i(0), r(0)) \in \mathbb{R}^4_+$.

3.1 Fractional-order $SEIR$ model

In this section, we present the fractional-order $SEIR$ model using system (5). First, we replace each ordinary derivative in the system with fractional derivative of order α in the sense of Caputo, where α is an arbitrary real number belonging to the interval $(0, 1)$. Therefore, the proposed fractional $SEIR$ model is given as follows:

$$\begin{cases} {}^C\mathcal{D}^{\alpha}_{0+} s(t) = \nu - \rho s(t)i(t) - (\nu + \mu)s(t), \\ {}^C\mathcal{D}^{\alpha}_{0+} e(t) = \rho s(t)i(t) - (\delta + \nu)e(t), \\ {}^C\mathcal{D}^{\alpha}_{0+} i(t) = \delta e(t) - (\nu + \lambda + \beta + \gamma)i(t), \\ {}^C\mathcal{D}^{\alpha}_{0+} r(t) = (\beta + \gamma)i(t) + \mu s(t) - \nu r(t). \end{cases} \tag{6}$$

The initial conditions are $(s(0), e(0), i(0), r(0)) \in \mathbb{R}^4_+$. We assume that $s(t)$, $e(t)$, $i(t)$, $r(t)$, and their Caputo fractional derivatives are continuous functions at $t \geq 0$. Since $r(t)$ does not appear in the first three equations of system (6), without loss of generality, we consider the following system:

$$\begin{cases} {}^C\mathcal{D}^{\alpha}_{0+} s(t) = \nu - \rho s(t)i(t) - (\nu + \mu)s(t), \\ {}^C\mathcal{D}^{\alpha}_{0+} e(t) = \rho s(t)i(t) - (\delta + \nu)e(t), \\ {}^C\mathcal{D}^{\alpha}_{0+} i(t) = \delta e(t) - (\nu + \lambda + \beta + \gamma)i(t). \end{cases} \tag{7}$$

with initial conditions $(s(0), e(0), i(0)) \in \mathbb{R}^3_+$. Moreover, since the population is constant, we have $r(t) = 1 - s(t) - e(t) - i(t)$ at any time t. In the next section, we investigate the existence and stability conditions of both equilibrium of system (7) by constructing an appropriate Lyapunov functional.

4 Dynamical Behavior of Fractional Order Model

To investigate the existence, uniqueness, non-negativity of the solution, and stability of the equilibrium points of the system (7), we consider the following initial value problems for fractional differential equations in the form of

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{0+} X(t) = f(t, X(t)), \\ X(t_0) = X_0, \end{cases} \tag{8}$$

where, $f(t, X(t)) : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a vector field.

Theorem 1. [19]. Let $\mathcal{J} = [t_0 - a, t_0 + a]$, $\mathcal{B} = \{x \in \mathbb{R}^n | \|x - x_0\| \leq b\}$, and $\mathcal{A} = \{(t, X) \in \mathbb{R} \times \mathbb{R}^n | t \in \mathcal{J}, x \in \mathcal{B}\}$. Assume that the function $f : \mathcal{A} \longrightarrow \mathbb{R}^n$ satisfies the following conditions:

a. $f(t, X)$ is Lebesgue measurable with respect to t on $\mathcal{J}$.

b. $f(t, X)$ is continuous with respect to x on $\mathcal{B}$.

c. There exists a real-valued function $m(t) \in L^2(\mathcal{J})$ such that

$$\|f(t, X)\| \leq m(t),$$

for almost every $t \in \mathcal{J}$ and all $x \in \mathcal{B}$.

Then, for $\alpha > \frac{1}{2}$, there exists at least one solution of the initial value problem (8) on the interval $[t_0 - h, t_0 + h]$ for some positive number h.

Theorem 2. [19]. Assume that the vector field $f(t, X)$ satisfies the first two conditions of Theorem 1 globally and

$$\|f(t, X)\| \leq \omega + \theta \|X\|,$$

for almost every $t \in \mathbb{R}$ and all $X \in \mathbb{R}^n$. Here ω and θ, are two positive constants. Then, there exists a function $X(t)$ on $(-\infty, +\infty)$ that solves the problem (8).

Lemma 1. [13]. Suppose that $f(t) \in C[a, b]$ and ${}^{C}\mathcal{D}^{\alpha}_{0+} f(t) \in C[a, b]$ for $0 < \alpha \leq 1$. If ${}^{C}\mathcal{D}^{\alpha}_{0+} f(t) \geq 0$, $for\ all\ t \in [a, b]$, then $f(t)$ is non-decreasing for each $t \in [a, b]$. If ${}^{C}\mathcal{D}^{\alpha}_{0+} f(t) \leq 0$, $for\ all\ t \in (a, b)$, then $f(t)$ is non-increasing for each $t \in [a, b]$.

Theorem 3. The system (7) has a unique non-negative solution.

Proof. The existence and uniqueness of the solution follow from Theorem 1 and Remark 3.2 of [19], respectively. To show the non-negativity of the solution, we consider the second equation of system (7), and obtain

$${}^{C}\mathcal{D}^{\alpha}_{0+} e(t) = \rho s(t) i(t) - (\delta + \nu) e(t) \geq -(\delta + \nu) e(t),$$

which leads to

$$e(t) \geq E_{\alpha,\alpha+1}(-(\delta+\nu)t^{\alpha})e(0) \geq 0. \tag{9}$$

Similarly, from the third equation of system (7), we have

$${}^{C}\mathcal{D}^{\alpha}_{0^+} i(t) = \delta e(t) - (\nu+\lambda+\beta+\gamma)i(t) \geq -(\nu+\lambda+\beta+\gamma)i(t),$$

which implies

$$e(t) \geq E_{\alpha,\alpha+1}(-(\nu+\lambda+\beta+\gamma)t^{\alpha})i(0) \geq 0. \tag{10}$$

By using to the systems (7), (9) and (10), we get

$$\begin{aligned}
{}^{C}\mathcal{D}^{\alpha}_{0^+} s(t)|_{s=0} &= \nu \geq 0,\\
{}^{C}\mathcal{D}^{\alpha}_{0^+} e(t)|_{e=0} &= \rho s(t) i(t) \geq 0,\\
{}^{C}\mathcal{D}^{\alpha}_{0^+} i(t)|_{i=0} &= \delta e(t) \geq 0.
\end{aligned}$$

Since $(s(0), e(0), i(0)) \in \mathbb{R}^3_+$ and applying Lemma 1, we conclude that $s(t), e(t)$, and $i(t)$ are non-negative for any $t \geq 0$. Therefore, the solution of system (7) lies in $\mathbb{R}^3_+$. Next, we show that the solution of system (7) is bounded. Define a function

$$M(t) = s(t) + e(t) + i(t),$$

then

$${}^{C}\mathcal{D}^{\alpha}_{0^+} M(t) = {}^{C}\mathcal{D}^{\alpha}_{0^+} s(t) + {}^{C}\mathcal{D}^{\alpha}_{0^+} e(t) + {}^{C}\mathcal{D}^{\alpha}_{0^+} i(t).$$

Adding all equations of system (7), we obtain

$${}^{C}\mathcal{D}^{\alpha}_{0^+} M(t) \leq \nu - \nu M(t).$$

By applying the Laplace transform in the previous inequality, we get

$$S^{\alpha} L(M) - S^{\alpha-1} M(0) \leq \nu S^{-1} + S^{\alpha-1} M(0),$$

which implies

$$M(t) \leq \nu t^{\alpha} E_{\alpha,\alpha+1}(-\nu t^{\alpha}) + E_{\alpha,1}(-\nu t^{\alpha} M(0),$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function. Let $K = max\{1, M(0)\}$, then

$$M(t) \leq K\left[\nu t^{\alpha} E_{\alpha,\alpha+1}(-\nu t^{\alpha}) + E_{\alpha,1}(-\nu t^{\alpha}\right] = K\frac{1}{\Gamma(1)} = K.$$

Therefore, the solution of system (7) is bounded. This completes the proof. □

Hence, we conclude that the feasible region of system (7) is defined by

$$\Omega = \left\{(s,e,i) \in \mathbb{R}^3 \mid s > 0,\ e \geq 0,\ i \geq 0 \text{ and } s+e+i \leq K\right\}.$$

We state the following theorem from [25] to analyze the stability of equilibrium points for a commensurate fractional-order system.

Theorem 4. [25]. Consider the following commensurate fractional-order system:

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{0+}X(t) = f(t, X(t)), \\ X(t_0) = X_0, \end{cases} \tag{11}$$

where,$\alpha \in (0, 1]$ and $f(t, X(t)) : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a vector field. The equilibrium points of this system are locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix $\frac{\partial f(t,X)}{\partial X}$ evaluated at the equilibrium points satisfy the following condition:

$$|arg\lambda_i| > \alpha\frac{\pi}{2}.$$

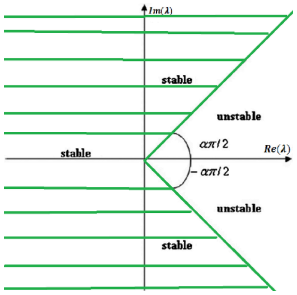

Figure 1: Stability region of the fractional-order system.

Definition 3. [20]. A point X^* is an equilibrium point of the problem (11) if and only if $f(t, X^*) = 0$.

Remark 1. [6, 8]. When $0 < \alpha < 1$, problem (11) has the same equilibrium points as the integer order system.

Theorem 5. The model (7) has at most two equilibrium points:

a. A disease free equilibrium $E_f = (\frac{\nu}{\nu+\mu}, 0, 0)$.

b. A infected steady-state equilibrium: $E_e = (s^*, e^*, i^*)$, where

$$\begin{aligned} s^* &= \frac{\nu + \lambda + \beta + \gamma}{\rho\delta}, \\ e^* &= \frac{(\rho\delta\nu) - (\nu + \lambda + \beta + \gamma)(\nu + \mu)}{\rho\delta}, \\ i^* &= \frac{(\rho\delta\nu) - (\nu + \lambda + \beta + \gamma)(\nu + \mu)(\nu + \delta)}{\rho(\nu + \lambda + \beta + \gamma)(\nu + \delta)}). \end{aligned}$$

Lemma 2. [3]. Let M be a 3×3 real matrix. If $tr(M)$, det(M), and det$(M^{[2]})$ are all negative, then all of the eigenvalues of M have a negative real part.

Theorem 6. The equilibrium E_f of the system (7) is locally asymptotically stable if

$$\frac{(\rho\delta\nu)}{(\nu + \lambda + \beta + \gamma)(\nu + \mu)(\nu + \delta)} < 1.$$

Proof. We determine the Jacobian matrix of the fractional system (7) at E_f:

$$J = \begin{bmatrix} -(\nu+\mu) & 0 & -\frac{\rho\nu}{\nu+\mu} \\ 0 & -(\delta+\nu) & \frac{\rho\nu}{\nu+\mu} \\ 0 & \delta & -(\nu+\lambda+\beta+\gamma) \end{bmatrix}.$$

Then,

$$\text{tr}(J) = -\big[(\nu+\mu)+(\delta+\nu)+(\nu+\lambda+\beta+\gamma)\big] < 0,$$

and

$$\det(J) = -(\nu+\mu)\big[(\delta+\nu)(nu+\lambda+\beta+\gamma) - \frac{(\rho\delta\nu)}{(\nu+\mu)}\big] < 0.$$

The second compound [16] of the Jacobian matrix is

$$J^{[2]} = \begin{bmatrix} -(2\nu+\mu+\delta) & \frac{\rho\nu}{\nu+\mu} & \frac{\rho\nu}{\nu+\mu} \\ \delta & -(2\nu+\mu+\lambda+\beta+\gamma) & 0 \\ 0 & 0 & -(\delta+2\nu+\lambda+\beta+\gamma) \end{bmatrix}.$$

The determinant of $J^{[2]}$ is

$$\det(J^{[2]}) = -(\delta+2\nu+\lambda+\beta+\gamma)\big[(2\nu+\mu+\delta)(2\nu+\mu+\lambda+\beta+\gamma) - \frac{\rho\nu\delta}{\nu+\mu}\big] < 0,$$

for

$$\frac{(\rho\delta\nu)}{(\nu+\lambda+\beta+\gamma)(\nu+\mu)(\nu+\delta)} < 1.$$

Thus, $\text{tr}(J)$, $\det(J)$ and $\det(J^{[2]})$ are all negative. Therefore, by Lemma 2, the eigenvalues of J all have negative real parts. Hence, according to Theorem 4, the disease-free equilibrium at E_f of the model (7), is locally asymptotically stable if

$$\frac{(\rho\delta\nu)}{(\nu+\lambda+\beta+\gamma)(\nu+\mu)(\nu+\delta)} < 1.$$

□

To establish the global stability of infected steady-state E_e using the Lyapunov functional, we need to introduce the following lemma for fractional-order systems.

Lemma 3. [6]. Let $X(t) \in \mathbb{R}^+$ be a continuous function. Then for any time $t \geq t_0$

$${}^{C}\mathcal{D}^{\alpha}_{0+}\big[X(t) - X^* - X^* \ln\frac{X(t)}{X^*}\big] \leq \big(1 - \frac{X^*}{X(t)}\big)\,{}^{C}\mathcal{D}^{\alpha}_{0+}X(t),$$
$$X^* \in \mathbb{R}^+ \quad \text{for all } \alpha \in (0,1).$$

Using this lemma, we can prove the following theorem:

Theorem 7. For the system (7), the equilibrium E_e is globally asymptotically stable if

$$\frac{(\rho\delta\nu)}{(\nu+\lambda+\beta+\gamma)(\nu+\mu)(\nu+\delta)} > 1.$$

Proof. For the equilibrium E_e, we define the following Lyapunov function

$$V(t) = s^* g(\frac{s(t)}{s^*}) + e^* g(\frac{e(t)}{e^*}) + \frac{\delta+\nu}{\delta} i^* g(\frac{i(t)}{i^*}),$$

where

$$g(x) = x - 1 - \ln x.$$

For all $s(t) > 0$, $e(t) > 0$, and $i(t) > 0$, V is well-defined, continuous, and positive definite. According to Lemma 3, we get

$$\begin{aligned}
{}^C\mathcal{D}^{\alpha}_{0+} V(t) &\leq (1 - \frac{s^*}{s(t)}) \, {}^C\mathcal{D}^{\alpha}_{0+}(s) + (1 - \frac{e^*}{e(t)}) \, {}^C\mathcal{D}^{\alpha}_{0+}(e) + \frac{\delta+\nu}{\delta}(1 - \frac{i^*}{i(t)}) \, {}^C\mathcal{D}^{\alpha}_{0+}(i) \\
&= (1 - \frac{s^*}{s(t)})(\rho s^* i^* + (\nu+\mu)s^* - \rho s(t) i(t) - (\nu+\mu)s(t)) + (1 - \frac{e^*}{e(t)})(\rho s(t) i(t) \\
&\quad - (\delta+\nu)e(t)) + \frac{\delta+\nu}{\delta}(1 - \frac{i^*}{i(t)})(\delta e(t) - (\nu+\lambda+\beta+\gamma)i(t)) \\
&= -(\nu+\mu)\frac{(s-s*)^2}{s} + \rho s^* i^* (1 - \frac{s^*}{s(t)})(1 - \frac{s(t)i(t)}{s^* i^*}) \\
&\quad + \rho s^* i^* (1 - \frac{e^*}{e(t)})(1 - \frac{s(t)i(t)}{s^* i^*} - \frac{e(t)}{e^*}) + (1 - \frac{i^*}{i(t)})(\delta+\nu)e(t) \\
&\quad - \frac{\delta+\nu}{\delta}(\nu+\lambda+\beta+\gamma)i(t)) \\
&= -(\nu+\mu)\frac{(s-s*)^2}{s} + \rho s^* i^* (1 - \frac{s^*}{s(t)})(-\frac{s(t)i(t)}{s^* i^*}) \\
&\quad \rho s^* i^* (1 - \frac{e^*}{e(t)})(-\frac{s(t)i(t)}{s^* i^*} - \frac{e(t)}{e^*}) + \rho s^* i^* (1 - \frac{i^*}{i(t)})(\frac{e(t)}{e^*} - \frac{i(t)}{i^*}) \\
&= -(\nu+\mu)\frac{(s-s*)^2}{s} + \rho s^* i^* \mathcal{F},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F} &= (1 - \frac{s^*}{s(t)})(-\frac{s(t)i(t)}{s^* i^*}) + (1 - \frac{e^*}{e(t)})(-\frac{s(t)i(t)}{s^* i^*} - \frac{e(t)}{e^*}) \\
&\quad + (1 - \frac{i^*}{i(t)})(\frac{e(t)}{e^*} - \frac{i(t)}{i^*}) = 3 - \frac{s^*}{s(t)}) - \frac{s(t)i(t)e^*}{s^* i^* e(t)} - \frac{e(t)i^*}{e^* i(t)} \\
&= -(\frac{s^*}{s(t)} + \frac{s(t)i(t)e^*}{s^* i^* e(t)} + \frac{e(t)i^*}{e^* i(t)} - 3) \\
&= -[g(\frac{s^*}{s(t)}) + g(\frac{s(t)i(t)e^*}{s^* i^* e(t)} + g(\frac{e(t)i^*}{e^* i(t)})].
\end{aligned}$$

Thus, if

$$\frac{(\rho\delta\nu)}{(\nu+\lambda+\beta+\gamma)(\nu+\mu)(\nu+\delta)} > 1,$$

then $\mathcal{F} < 0$, it follows that ${}^C\mathcal{D}^{\alpha}_{0+} V(t)$ is negative definite. Therefore, the infected steady-state E_e is globally asymptotically stable. This completes the proof. □

5 Numerical Results and Discussion

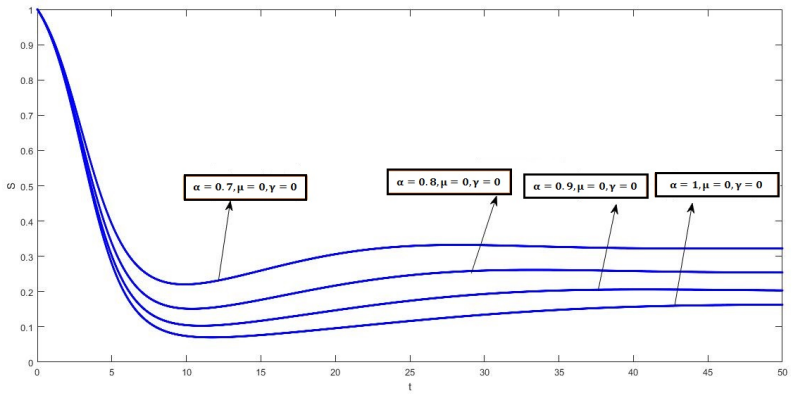

Figure 2: The behavior of S for different values of α.

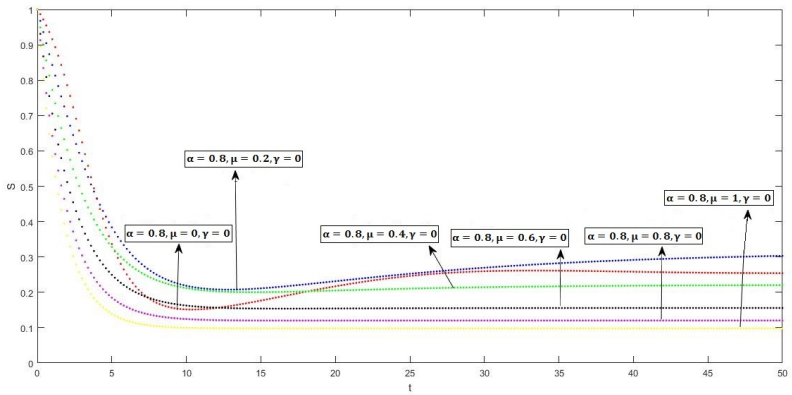

Figure 3: The behavior of S both with control and without control.

Numerical simulations of the problem (6) with the parameter values is presented in Table 1. We investigate two scenarios, with and without control, and report the following results:

Figures 2 and 3 show that the susceptible population decreases as the control and vaccination increase, leading to a reduction in the infected population. With the continuation of vaccination, horizontal transmission is dramatically reduced. The infected population increases if the disease is not treated, and the disease can turn into an epidemic, increasing vertical transmission in society. However, the infected population decreases with an increase in control and drug therapy. With the continuation of drug therapy, vertical transmission decreases, and the disease eventually disappears. Figure 5 indicates that drug therapy alone leads to an increase in the number of recovered individuals in a certain period. Vaccination alone also produces an increase in the number of recovered individuals if it is continued at the social level. If both vaccination and drug therapy are used simultaneously to control and prevent the incidence of the disease at the social level, the rate of increase in the number of recovered individuals will be significantly higher compared to the previous two cases. This method of controlling infectious diseases at the social level yields good results.

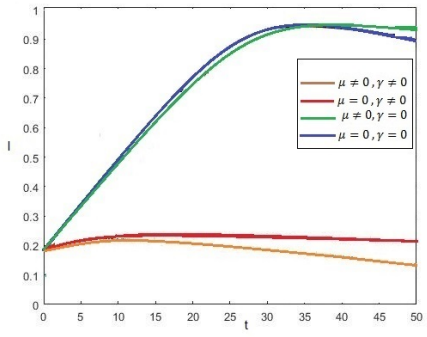

Figure 4: The behavior of I both with control and without control.

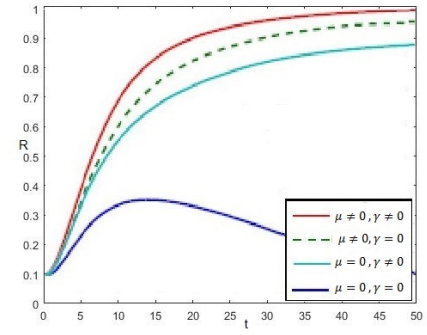

Figure 5: The behavior of R both with control and without control.

6 Conclusions

In this paper, we extended the classical $SEIR$ mathematical model for infectious disease transmission to a system of fractional ordinary differential equations. The fractional order system was considered because it is a generalization of integer order differential equations. We proved that our fractional model has a unique non-negative solution. The stability of equilibrium points was studied, and under certain conditions, the local asymptotic stability at the disease-free equilibrium was analyzed. In this paper, Barbalat's Lemma is extended to the fractional case, and a suitable Lyapunov functional is constructed to prove the global asymptotic stability of the infected steady-state. Finally, under certain conditions, it is proved that the infected steady-state is globally asymptotically stability. Finally, a numerical simulation is conducted to demonstrate our theoretical results. The results suggest that the simultaneous use of both controllers is the best way to control disease in society.

Declarations

Availability of supporting data

All data generated or analyzed during this study are included in this published paper.

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Competing interests

The authors have no competing interests to declare that are relevant to the content of this paper.

Authors' contributions

The main manuscript text is written collectively by the authors.

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