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Hesitant Fuzzy Equation

Fatemeh Babakordi ✉, Nemat Allah Taghi-Nezhad

Department of Mathematics, Faculty of Science, Gonbad Kavous University, Gonbad Kavous, Iran.

✉ **Correspondence:**
Fatemeh Babakordi
E-mail:
babakordif@yahoo.com

Abstract. This paper presents the introduction of two novel equation types: the partial hesitant fuzzy equation and the half hesitant fuzzy equation. Additionally, an efficient method is proposed to solve these equations by defining four solution categories: Controllable, Tolerable Solution Set (TSS), Controllable Solution Set (CSS), and Algebraic Solution Set (ASS). Furthermore, the paper establishes eight theorems that explore different types of solutions and lay out the conditions for the existence and non-existence of hesitant fuzzy solutions. The practicality of the proposed method is demonstrated through numerical examples.

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1 Introduction

A crucial and effective tool that is utilized in various fields of science, including engineering, human health, cancer treatment and so on is linear equation. In practical and complex problems, the data that are not deterministic are handled . Therefore, the parameters of these systems are considered non-deterministic. To model practical problems, these uncertainties must be considered. To model the existing ambiguities and uncertainties of natural problems, fuzzy sets are used . Currently, in various fields such as [4, 9, 15, 17, 19, 21], researchers use fuzzy set theory. First, researchers focused on the fuzzy linear equation, and initial studies has been done in this field [11]. Then, various techniques, for instance, Newton's method and some iterative methods to solve fuzzy equations and systems of fuzzy equations, were dealt with [1, 5, 12, 13, 18]. The ranking method, to solve dual fuzzy polynomial equations is one of the last works in this area. The concept of the hesitant fuzzy set, which allows membership degree to acquire different possible crisp values between zero and one, was introduced by Torra [22] and Torra and Narukawa [23]. Recently, the hesitant fuzzy set captured the interest of researchers, thereby prompting them to apply hesitant fuzzy set to multi-criteria decision-making (MCDM) problems [3, 26, 28, 30].

Interval-valued hesitant-valued fuzzy sets (IVHFS), are a complex and flexible decision information presentation tool that has practical value in multi-criteria decision-making. However, uncertainty measurement cannot distinguish different IVHFS in some fields. To address this issue, XU proposed that there should be two types of uncertain for an IVHFS: the vagueness of an IVHFS and the non-specificity of IVHFS. Because, the available indicators to measure the uncertainty of IVHF are all single indicators that cannot consider two aspects of one IVHFS. Hence, according to the uncertainty criteria of IVHFS Xu suggested a two-tuple index to measure it, including one index to measure the fuzziness of the hesitant fuzzy set and another index to measure its non-specificity. The proposed two-tuple index can compensate for the entropy measurement error of the hesitant fuzzy set with the existing interval value [27].

An essential benefit of using IVHFSs is that decision-makers can evaluate a variable using numerous interval numbers, making it a valuable tool for handling decision-making problems. However, they cannot obtain gray information. To improve this problem, Zhao et al. defined a new fuzzy set called interval grey hesitant fuzzy set by combining the interval-valued hesitant fuzzy set and the gray fuzzy set. They obtained more reasonable results and improved the description of the factual information. The possible degrees of gray numbers indicate the upper and lower limits of the distance number, and by defining the basic operating rules, score function, entropy method, and measurement distance, they developed a multi-criteria decision-making model [29].

Probabilistic hesitant elements (PHFEs) are a valuable enhancements to the Hesitant Fuzzy Element (HFE) that provide decision makers with more flexibility in expressing their biases. However, the existing processes for homogenization and sorting of components in PHFEs, lead to various disadvantages. To address this issue, a PHFE equalization method was introduced in [14], to avoid operational performance defects while maintaining the inherent characteristics of PHFE probabilities.

To simplify the solution of complex multi-criteria decision-making problems, the fermatean hesitant fuzzy set is used. In [16], the concept of the fermatean hesitant fuzzy set, operations related to this concept, and main features of aggregation operators based on the fermatean hesitant fuzzy sets were investigated. A new multi-criteria decision-making method was introduced with these operators to select the best alternative in practice.

In the field of visualization and evaluation of strategies and plans to design active communities, government policymakers face problems that need to be evaluated to study cooperation between ministries. To address this issue, a complex hesitant fuzzy graph was proposed in [2]. This graph showed the cooperation and its agents between the ministries using two variables that change the consistency and performance of hesitant fuzzy charts in displaying information. The chart was able to convert complex information into a more understandable form.

Babakardi discussed various types of hesitant fuzzy sets [6]. In [7], the hesitant fuzzy system was introduced and solved using the definition of norm. Fully hesitant fuzzy equations and double hesitant fuzzy equations were introduced next. A method for solving one-element hesitant fuzzy equations was proposed, and then, the proposed method was expanded to effectively solve n-element hesitant fuzzy equations. Its application in determining the equilibrium point of the market was investigated [8]. Recently, in [20], the fully hesitant parametric fuzzy equation was discussed and investigated by introducing hesitant parametric fuzzy sets and operations on them. However, in this paper, two new categories of hesitant fuzzy equations, namely partial hesitant fuzzy equations and half hesitant fuzzy equations, have been introduced and solved. These equations have not been discussed in any other research so far. The primary purposes of introducing these equations are to discuss how the product of a real number in a hesitant fuzzy set is checked (partial hesitant fuzzy equation), how the product of a fuzzy set in a hesitant fuzzy set can become a hesitant fuzzy set (Equation (4)), and how the product of two hesitant fuzzy sets can be a fuzzy set (Equation (5)).

This study is structured as follows. Section 2 provides the preliminaries. In Section 3, we define types of the hesitant fuzzy equation in the form of $ax = b$. Section 4 investigates different solutions of hesitant fuzzy equations and presents a method for solving such problems. Section 5 offers the conclusion.

2 Preliminaries

This section describes the notation, required concepts, and basic definitions used in this paper. It briefly recalls some basic definitions of hesitant fuzzy set, which is extensively used throughout the paper.

Definition 1 ([24])**.** Let U be a fixed set. Each element of fuzzy sets is mapped to the interval $[0, 1]$ by the membership function.

$$\mu_{\tilde{A}} : U \to [0, 1].$$

Definition 2 ([22, 23, 25])**.** Consider the fixed set U. An applied function on X which returns a subset of $[0, 1]$ is called a hesitant fuzzy set on U. Xia and Xu in [25] expressed the hesitant fuzzy set using the following mathematical symbol:

$$A = \{\langle x, h_A(x)\rangle | x \in U\}. \tag{1}$$

where $h_A(x)$ denotes the possible membership degrees of the element $x \in U$ to the set A and is a set of values in the interval $[0, 1]$. For convenience, $h = h_A(x)$ is called a hesitant fuzzy element. Let h, h_1, h_2 be three hesitant fuzzy elements, and $\lambda > 0$. Some operations on hesitant fuzzy elements can be described as follows:

$$\begin{aligned}
&H^c = \cup_{\delta \in H}\{1-\delta\},\\
&H^\rho = \cup_{\delta \in H}\{\gamma^\rho\},\\
&\rho H = \cup_{\delta \in H}\{1-(1-\delta)^\rho\},\\
&H_1 \cup H_2 = \cup_{\delta_1 \in H_1, \delta_2 \in H_2} \max\{\delta_1, \delta_2\},\\
&H_1 \cap H_2 = \cap_{\delta_1 \in H_1, \delta_2 \in H_2} \min\{\delta_1, \delta_2\},\\
&H_1 \oplus H_2 = \cup_{\delta_1 \in H_1, \delta_2 \in H_2}\{\delta_1 + \delta_2 - \delta_1\delta_2\},\\
&H_1 \otimes H_2 = \cup_{\delta_1 \in H_1, \delta_2 \in H_2}\{\delta_1\delta_2\}.
\end{aligned} \tag{2}$$

3 Hesitant Fuzzy Equations

In this section, we will define various types of hesitant fuzzy equations.

Remark 1. In this paper, for simplification, we consider the single-element fuzzy set. The single-element fuzzy set $\tilde{A}$ is represented by $\tilde{A} = (a, \mu_{\tilde{A}})$, where a is a real number, and $\mu_{\tilde{A}}$ is the degree of membership belonging to the interval $[0, 1]$.

Remark 2. In this paper, for simplicity, the single-element hesitant fuzzy set is considered. The single-element hesitant fuzzy set A is denoted as $A = \langle x, h_A(x) \rangle$.

Definition 3. The equation

$$aY = B, \tag{3}$$

is referred to as a partial hesitant fuzzy equation, where a is a known real number, $B = \langle b, h_B \rangle$ is a known hesitant fuzzy set, and $Y = \langle y, h_Y \rangle$ is an unknown hesitant fuzzy set.

Definition 4. Each of the following equations

$$\tilde{A}Y = B, \tag{4}$$

$$AY = \tilde{B}, \tag{5}$$

is called half hesitant fuzzy equations, in which one of the equation parameters ($\tilde{A}$ or $\tilde{B}$) is fuzzy set and the other is known hesitant fuzzy set, and Y is an unknown hesitant fuzzy set.

4 The Method

In this section, we will explain the four solution types proposed by interval analysis researchers [10] for solving a hesitant fuzzy equation. Additionally, we will describe the method for solving each type of hesitant fuzzy equation.

Definition 5. Let X be the algebraic solution for Equation (3) when aX is exactly equal to B. In Equation (4), assume $\tilde{A} = (a, \mu_{\tilde{A}})$, $B = \langle b, h_B \rangle$ and $X = \langle x, h_X \rangle$, where for all $i = 1, \ldots, n$, $\mu_{\tilde{A}}, h_i, h'_i \in [0, 1]$ and $h_B = \{h_1, h_2, \ldots, h_n\}$ and $h_X = \{h'_1, h'_2, \ldots, h'_n\}$. Then, for each $i = 1, \ldots, n$, the four groups of the solutions of this equation are defined as follows:

Definition 6. The United Solution Set (USS) of the hesitant fuzzy equation of (4) is a Hesitant fuzzy set $X = \langle x, h_X \rangle$ that satisfies the condition:

$$\left\{\mu_{\tilde{A}} h'_i | h'_i \in h_X \in [0, 1]\right\} \cap \{h_i | h_i \in h_B\} \neq \emptyset.$$

Definition 7. Hesitant fuzzy set $X = \langle x, h_X \rangle$ is called The Tolerable Solution Set (TSS) of the hesitant fuzzy equation of (4) is a hesitant fuzzy set $X = \langle x, h_X \rangle$ that satisfies the condition:

$$\left\{\mu_{\tilde{A}} h'_i | h'_i \in h_X \in [0, 1]\right\} \subset \{h_i | h_i \in h_B\}.$$

Definition 8. A Controllable Solution Set (CSS) of the hesitant fuzzy equation of (4) is a hesitant fuzzy set $X = \langle x, h_X \rangle$ that satisfies the condition:

$$\left\{\mu_{\tilde{A}} h'_i | h'_i \in h_X \in [0, 1]\right\} \supset \{h_i | h_i \in h_B\}.$$

Definition 9. The Algebraic Solution Set (ASS) of the hesitant fuzzy equation of (4) is a hesitant fuzzy set $X = \langle x, h_X \rangle$ that satisfies the condition:

$$\left\{\mu_{\tilde{A}} h'_i | h'_i \in h_X \in [0, 1]\right\} = \{h_i | h_i \in h_B\}.$$

In Equation (5), let's assume $A = \langle a, h_A \rangle$, $\tilde{B} = (b, \mu_{\tilde{B}})$ and $X = \langle x, h_X \rangle$. For all $i = 1, \ldots, n$, we have $\mu_{\tilde{B}}, h_i, h'_i \in [0, 1]$ and $h_A = \{h_1, h_2, \ldots, h_n\}$ and $h_X = \{h'_1, h'_2, \ldots, h'_n\}$. Then, for each $i = 1, \ldots, n$ we can define four groups for this equation as follows:

Definition 10. The United Solution Set (USS) of the hesitant fuzzy equation in (5) is a hesitant fuzzy set $X = \langle x, h_X \rangle$ that satisfies the condition:

$$\left\{h_i h'_i | h'_i \in h_X \in [0, 1], h_i \in h_A\right\} \cap \left\{\mu_{\tilde{B}}\right\} \neq \emptyset.$$

Definition 11. The Tolerable Solution Set (TSS) of the hesitant fuzzy equation in (5) is a hesitant fuzzy set $X = \langle x, h_X \rangle$ that satisfies the condition:

$$\left\{h_i h'_i | h'_i \in h_X \in [0, 1], h_i \in h_A\right\} \subset \left\{\mu_{\tilde{B}}\right\}.$$

Definition 12. The Controllable Solution Set (CSS) of the hesitant fuzzy equation in (5) is a hesitant fuzzy set $X = \langle x, h_X \rangle$ that satisfies the condition:

$$\left\{h_i h'_i | h'_i \in h_X \in [0, 1], h_i \in h_A\right\} \supset \left\{\mu_{\tilde{B}}\right\}.$$

Definition 13. The Algebraic Solution Set (ASS) of the hesitant fuzzy set $X = \langle x, h_X \rangle$ that satisfies the condition:

$$\left\{h_i h'_i | h'_i \in h_X \in [0, 1], h_i \in h_A\right\} = \left\{\mu_{\tilde{B}}\right\}.$$

Definition 14. The multiplication of a hesitant fuzzy set $B = \langle b, \{h_1, h_2, \ldots, h_n\}\rangle$ and a fuzzy set $\tilde{A} = (a, \mu_{\tilde{A}})$ is defined as:

$$\langle ab, \{\mu_{\tilde{A}} h_1, \mu_{\tilde{A}} h_2, \ldots, \mu_{\tilde{A}} h_n\}\rangle. \tag{6}$$

Definition 15. Two hesitant fuzzy sets $\langle a, \{h_1, h_2, \ldots, h_n\}\rangle$ and $\langle b, \{h'_1, h'_2, \ldots, h'_n\}\rangle$ are considered equal if:

i. $a = b$,

ii. $\{h'_1, h'_2, \ldots, h'_n\} = \{h_1, h_2, \ldots, h_n\}$.

Remark 3. It is worth mentioning that the equality of two sets:

$$\{h_1, h_2, \ldots, h_n\} \text{ and } \{h'_1, h'_2, \ldots, h'_n\},$$

depends only on having the same members, regardless of the repetition or order of placement of members.

4.1 Solving partial hesitant fuzzy equation

Let $B = \langle b, \{h_1, h_2, \ldots, h_n\}\rangle$, and $a = (a, 1)$. We define

$$X = \langle x, \{h'_1, h'_2, \ldots, h'_n\}\rangle.$$

By substituting them into Equation (3), we obtain:

$$(a, 1)\langle x, \{h'_1, h'_2, \ldots, h'_n\}\rangle = \langle b, \{h_1, h_2, \ldots, h_n\}\rangle.$$

Using multiplication Equation (6), we can simplify it as follows:

$$\langle ax, \{h'_1, h'_2, \ldots, h'_n\}\rangle = \langle b, \{h_1, h_2, \ldots, h_n\}\rangle.$$

Based on Definition 15, we can obtain:

$$ax = b, \tag{7}$$

$$\{h'_1, h'_2, \ldots, h'_n\} = \{h_1, h_2, \ldots, h_n\}. \tag{8}$$

Therefore, we can conclude from Equations (7) and (8) that Equation (3) has a hesitant fuzzy solution given by:

$$X = \left\langle \frac{b}{a}, \{h_1, h_2, \ldots, h_n\} \right\rangle. \tag{9}$$

Theorem 1. The equation (3), always has an algebraic solution set.

Proof. By substituting $a = (a, 1)$ and Equation (9) into (3), we have:

$$aX = (a, 1)\left\langle \frac{b}{a}, \{h_1, h_2, \ldots, h_n\} \right\rangle = \langle b, \{h_1, h_2, \ldots, h_n\}\rangle = B.$$

Therefore, X is an algebraic solution for Equation (3). □

Example 1. Let's consider the case where $a = 9$ in Equation (3) and $B = \langle 27, \{0.9, 0.5, 1\}\rangle$. From (9), we have $X = \langle 3, \{0.9, 0.5, 1\}\rangle$. According to the previous theorem, this is an algebraic solution.

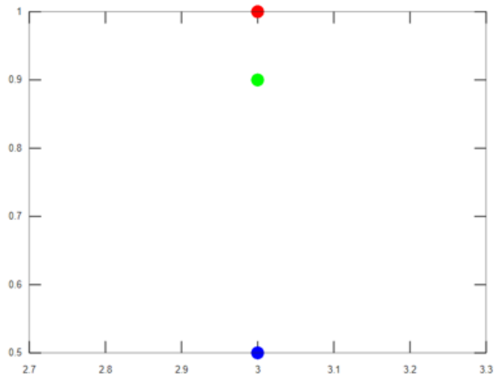

Figure 1: The algebraic solution of Example 1.

4.2 Solving half hesitant fuzzy equation

Solving a half-hesitant fuzzy equation in general is a challenging task. To simplify the problem, we consider Equation (4) and impose the condition that there exists $h' \in h_B$ such that $h' \leq \mu_{\tilde{A}}$, and for Equation (5), there exists $h' \in h_A$ such that $h' \geq \mu_{\tilde{B}}$. Under this limitation, the algebraic solution set, the controllable solution set, and the tolerable solution set coincide.

To solve Equation (4), let $B = \langle b, \{h_1, h_2, \ldots, h_n\}\rangle$ and $\tilde{A} = \left(a, \mu_{\tilde{A}}\right)$. We assume $X = \left\langle \frac{b}{a}, h_X \right\rangle$, with $h_t = \{h'_1, h'_2, \ldots, h'_n\}$, to determine $h'_1, h'_2, \ldots, h'_n$. The following equations are formed:

$$\mu_{\tilde{A}} h'_i = h_i, \forall\, 1 \leq i \leq n. \tag{10}$$

By solving these equations and calculating "h_t", the parameter h_X is defined as follow:

$$h_X \subseteq \left\{h'_i \in h_t : h'_i \leq 1,\ 1 \leq i \leq n\right\}. \tag{11}$$

Theorem 2. If there is at least one member in h_B, whose degree of membership is less than $\mu_{\tilde{A}}$, then, Equation (4) does not have a CSS solution.

Proof. If there is at least one member in h_B with a degree of membership is less than $\mu_{\tilde{A}}$, then, the number of members of $\mu_{\tilde{A}} h_X$ is always less than or equal to the number of members of the set h_B. If the number of members in $\mu_{\tilde{A}} h_X$ is less than the number of members in h_B, Equation (4) has a TSS. If the number of members in $\mu_{\tilde{A}} h_X$ is equal to the number of members in h_B, Equation (4) has an ASS. Therefore , Equation (4) does not have a CSS. □

Example 2. Consider the following half-hesitant fuzzy equation: $\tilde{A}X = B$, where $\tilde{A} = (2, 0.6)$ and $B = \langle 5, \{0.1, 0.3, 0.7\}\rangle$. Let $X = \left\langle \frac{5}{2}, h_X \right\rangle$. We consider $h_t = \{h_1, h_2, h_3\}$. For determining h_1, h_2, h_3, the following equations are introduced:

$$0.6h_1 = 0.1 \quad 0.6h_2 = 0.3 \quad 0.6h_3 = 0.7.$$

By solving the above equations, we obtain $h_t = \left\{\frac{1}{6}, \frac{3}{6}, \frac{7}{6}\right\}$. Therefore, we have

$$h_X \subseteq \left\{\frac{1}{6}, \frac{3}{6}\right\}.$$

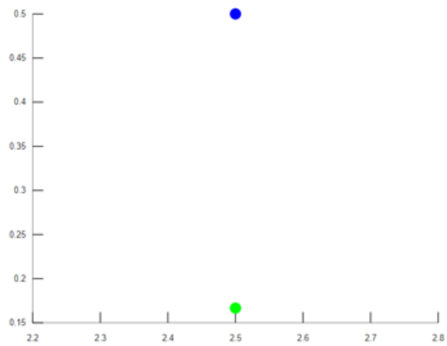

Figure 2: Visual display of Example 2.

It can be observed that $0.7 > 0.6$. According to Theorem 2, the equation does not have a CSS solution.

Following by presenting an example, we demonstrate the converse of Theorem 2 does not hold.

Example 3. Consider Equation (4), where $\tilde{A} = (5, 0.2)$ and $B = \langle 45, \{0.5, 0.6, 0.9\}\rangle$. Then, there is $X = \langle 9, h_X\rangle$. We consider $h_t = \{h_1, h_2, h_3\}$.. For determining h_1, h_2, h_3, the following equations are introduced:

$$0.2h_1 = 0.5 \quad 0.2h_2 = 0.6 \quad 0.2h_3 = 0.9.$$

By solving the above equations, we obtain $h_t = \left\{\frac{5}{2}, \frac{6}{2}, \frac{5}{2}\right\}$. Therefore, since $\frac{5}{2}$, $\frac{6}{2}$, and $\frac{5}{2}$ are all greater than 1, this equation does not have any solution category, including the CSS solution. Additionally, there is no member with a degree of membership is less than 0.2.

Theorem 3. Let, $B = \langle b, \{h_1, h_2, \ldots, h_n\}\rangle$ and $\tilde{A} = (a, \mu_{\tilde{A}})$. For each $h_i \in h_B$ we have $h_i \leq \mu_{\tilde{A}}$, and $h_X = \left\{\frac{h_i}{\mu_{\tilde{A}}}\right\}$ for all $1 \leq i \leq n$, if and only if the equation has algebraic solution.

Proof. After calculating $h_X = \{h'_1, h'_2, \ldots, h'_n\}$ from Equation (11), and calculating $\mu_{\tilde{A}}h'_i$, it can be observed that if for each $h_i \in h_B$, we have $h_i \leq \mu_{\tilde{A}}$. Then, $h_X = h_t$ and $\{\mu_{\tilde{A}}h'_i\} = \{h_1, h_2, \ldots, h_n\}$, which means that Equation (4) has an algebraic solution.

Now, to prove the converse of the theorem, suppose that the Equation (4), has an algebraic solution. From Definition 4, we have:

$$\{\mu_{\tilde{A}}h'_i | h'_i \in h_X \in [0, 1]\} = \{h_i | h_i \in h_B\}.$$

Since two sets are equal if they have the same members, the above equality implies:

$$h'_i = \frac{h_i}{\mu_{\tilde{A}}}, \forall 1 \leq i \leq n.$$

Therefore, $h_X = \left\{\frac{h_i}{\mu_{\tilde{A}}}\right\}$. Furthermore, $h_X \in [0, 1]$. So, $\frac{h_i}{\mu_{\tilde{A}}} \leq 1$. Hence , $h_i \leq \mu_{\tilde{A}}$. Thus , the proof is complete. □

Example 4. Suppose in Equation (4) that $\tilde{A} = (4, 0.7)$ and $B = \langle 12, \{0.2, 0.4, 0.6\}\rangle$. According to the suggested method, we obtain $X = \langle 3, h_X\rangle$, where

$$h_X \subseteq \left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}.$$

According to Theorem 3, since $0.7 < 0.2, 0.4, 0.6$, when considering $h_X = \left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}$, we have an algebraic solution $X = \left\langle 3, \left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}\right\rangle$ for Equation (4).

Theorem 4. Let $B = \langle b, \{h_1, h_2, \dots, h_n\}\rangle$ and $\tilde{A} = \left(a, \mu_{\tilde{A}}\right)$. There exists only one $h_i \in h_B$ such that $h_i \leq \mu_{\tilde{A}}$ for all $1 \leq i \leq n$, or a one-member set $\left\{\frac{h_i}{\mu_{\tilde{A}}} : \frac{h_i}{\mu_{\tilde{A}}} \leq 1, 1 \leq i \leq n\right\}$ is considered if and only if Equation (4) has a fuzzy solution.

Proof. Assume that there exists only one $h_i \in h_B$ such that $h_i \leq \mu_{\tilde{A}}$ for all $1 \leq i \leq n$, or a one-member set $\left\{\frac{h_i}{\mu_{\tilde{A}}} : \frac{h_i}{\mu_{\tilde{A}}} \leq 1, 1 \leq i \leq n\right\}$ is considered. Therefore, based on these assumptions, h_X has only one member. Consequently, Equation (4) has a fuzzy solution.

The inverse of the theorem can be proved similarly. □

Example 5. Suppose in Equation (4) that $\tilde{A} = (9, 0.5)$ and $B = \langle 18, \{0.3, 0.8, 0.6\}\rangle$. Then, $X = \left\langle 2, \left\{\frac{3}{5}\right\}\right\rangle$, is the solution of Equation (4). We observe that the conditions of Theorem 4 are satisfied; therefore, Equation (4) has a fuzzy solution.

Theorem 5. Suppose $B = \langle b, \{h_1, h_2, \dots, h_n\}\rangle$ and $\tilde{A} = \left(a, \mu_{\tilde{A}}\right)$. If $h_i > \mu_{\tilde{A}}$ for all $i = 1, 2, \dots, n$, then, Equation (4) does not have a hesitant fuzzy solution.

Proof. If $h_i > \mu_{\tilde{A}}$ for all $i = 1, 2, \dots, n$, then, $\frac{h_i}{\mu_{\tilde{A}}} > 1$. Therefore, from (11) we obtain $h_X = \emptyset$. Hence, Equation (4) dose not have a hesitant fuzzy solution.

For solving Equation (5), assuming that $A = \langle a, \{h_1, h_2, \dots, h_n\}\rangle$ and $B = \left(b, \mu_{\tilde{B}}\right)$ hold. We have $X = \left\langle \frac{b}{a}, h_X\right\rangle$. Assume $h_t = \{h'_1, h'_2, \dots, h'_n\}$, the following equations determine $h'_1, h'_2, \dots, h'_n$:

$$h_i h'_i = \mu_{\tilde{B}}, \quad \forall 1 \leq i \leq n. \tag{12}$$

By solving the above equations and determining h_t, the value of h_X is determined as follows:

$$h_X \subseteq \left\{h'_i \in h_t : h'_i \leq 1, 1 \leq i \leq n\right\}. \tag{13}$$

□

Theorem 6. If $h' \geq \mu_{\tilde{B}}$ for $h' \in h_A$, then, Equation (5) has ASS and CSS, but does not have TSS.

Proof. The proof is similar to that of Theorem 2 and is omitted. □

Remark 4. The inverse of Theorem 6, dose not hold.

Theorem 7. Let $A = \langle a, \{h_1, h_2, \dots, h_n\}\rangle$ and $\tilde{B} = \left(b, \mu_{\tilde{B}}\right)$. For each $h_i \in h_A$ where $1 \leq i \leq n$, we have $\mu_{\tilde{B}} \leq h_i$ and $h_X = \left\{\frac{\mu_{\tilde{B}}}{h_i}\right\}$, if and only if, the equation has an algebraic solution.

Proof. This theorem can be proved similarly to Theorem 3. □

Example 6. Consider the half hesitant fuzzy equation $AX = \tilde{B}$, where

$$A = \langle 7, \{0.3, 0.6, 0.5, 0.4, 0.8, 0.9\}\rangle,$$

and

$$\tilde{B} = (14, 0.1).$$

There is $X = \langle 2, h_X\rangle$. By considering $h_t = \{h_1, h_2, h_3\}$, the following equations are used to determine h_1, h_2, h_3:

$$\begin{array}{lll} 0.3h_1 = 0.1 & 0.6h_1 = 0.1 & 0.5h_1 = 0.1 \\ 0.4h_1 = 0.1 & 0.8h_1 = 0.1 & 0.9h_1 = 0.1. \end{array}$$

By solving the above equations, we have $h_t = \{\frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{8}, \frac{1}{9}\}$. Therefore, $h_X \subseteq \{\frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{8}, \frac{1}{9}\}$, where, according to Theorem 7, $X = \langle 2, \{\frac{1}{3}, \frac{1}{6}, \frac{1}{9}\}\rangle$ is an algebraic solution to this equation.

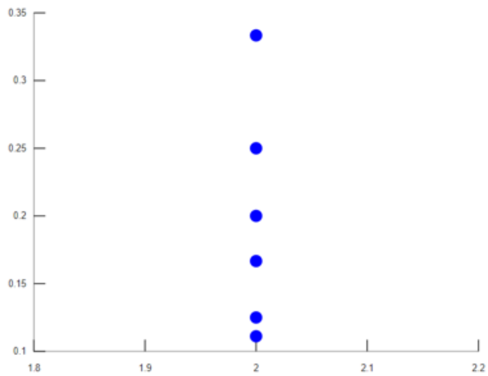

Figure 3: The image of Example 6.

Theorem 8. There exists only one $h' \in h_A$, such that $\mu_{\tilde{B}} \leq h'$, if and only if Equation (5) has a fuzzy solution.

Proof. The proof is similar to that of Theorem 4. □

Theorem 9. Suppose $A = \langle a, \{h_1, h_2, \ldots, h_n\}\rangle$ and $\tilde{B} = \left(b, \mu_{\tilde{B}}\right)$. If $\mu_{\tilde{B}} > h_i$ for all $i = 1, \ldots, n$, then the Equation (5) does not have a hesitant fuzzy solution.

Proof. The proof is omitted due to its similarity to the proof of Theorem 5. □

5 Conclusion

This paper introduces two novel categories of hesitant fuzzy equations, namely partial hesitant fuzzy equations and half hesitant fuzzy equations. The paper proceeds to address and resolve these problems. Specifically, it explores key aspects such as verifying the product of a real number within a hesitant fuzzy set (partial hesitant fuzzy equation), demonstrating how the product of a fuzzy set within a hesitant fuzzy set can yield a hesitant fuzzy set (Equation (4)), and illustrating how the product of two hesitant fuzzy sets can result in a fuzzy set (Equation (5)). Theorems are presented to support these findings. In future research, we intend to extend our investigation to partial hesitant fuzzy equations and half hesitant fuzzy equations with inputs and outputs consisting of n element and employ the α-cut method to solve them.

Declarations

Availability of supporting data

All data generated or analyzed during this study are included in this published paper.

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Competing interests

The authors have no competing interests to declare that are relevant to the content of this paper.

Authors' contributions

The main manuscript text is written collectively by the authors.

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