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Solving Fractional Optimal Control-Affine Problems via Fractional-Order Hybrid Jacobi Functions

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Abstract. This paper proposes and analyzes an applicable approach for numerically computing the solution of fractional optimal control-affine problems. The fractional derivative in the problem is considered in the sense of Caputo. The approach is based on a fractional-order hybrid of block-pulse functions and Jacobi polynomials. First, the corresponding Riemann-Liouville fractional integral operator of the introduced basis functions is calculated. Then, an approximation of the fractional derivative of the unknown state function is obtained by considering an approximation in terms of these basis functions. Next, using the dynamical system and applying the fractional integral operator, an approximation of the unknown control function is obtained based on the given approximations of the state function and its derivatives. Subsequently, all the given approximations are substituted into the performance index. Finally, the optimality conditions transform the problem into a system of algebraic equations. An error upper bound of the approximation of a function based on the fractional hybrid functions is provided. The method is applied to several numerical examples, and the experimental results confirm the efficiency and capability of the method. Furthermore, they demonstrate a good agreement between the approximate and exact solutions.

Keywords. Fractional optimal control-affine problems, Fractional-order hybrid functions, Caputo derivative, Riemann-Liouville integral.

MSC. 65K10; 26A33.

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1 Introduction

The main aim of an optimal control problem is to determine control signals that make a process satisfy some specific constraints and simultaneously optimize a chosen functional (performance index or cost function) [17]. Fractional optimal control problems (FOCPs) can be seen as a generalization of classical integer order optimal control problems in which the fractional differential equations are used as the dynamic constraints [1]. Many scholars have applied different analytical and numerical procedures for solving various kinds of FOCPs. Agrawal [2] have solved the Hamiltonian system of equations for FOCPs approximately. Lotfi et al. [12] presented a numerical direct method for solving a general class of FOCPs. In [9], a numerical scheme based on Legendre orthonormal polynomials has been presented. Postavaru and Toma [21] employed a fractional-order hybrid of block-pulse functions and Bernoulli polynomials to solve a class of FOCP. Nemati et al. [19] applied modified hat functions for the numerical solution of FOCP. Bhrawy et al. [7] proposed a Chebyshev-Legendre operational technique for approximating the numerical solution of FOCPs. Hybrid Chelyshkov functions have been utilized to solve a class of FOCP in [16]. Also, Alizadeh and Effati [4] solved a class of FOCP employing a variational iteration method. The interested reader can also see [3, 10, 27, 28, 29].

Almost from the beginning of the current century, hybrid functions have been introduced and used as an efficient mathematical tool for solving various problems (e.g. [13, 24, 25]). By passing time, many researchers have been attracted to using these types of basis functions for solving fractional models. To mention a few, one can refer to [14, 15, 30, 31]. Recently, a new generalization of hybrid functions to fractional-order was introduced by changing t to t^{λ} $(\lambda > 0)$. Postavaru and Toma have used fractional-order Bernoulli functions for solving two-dimensional fractional-order partial differential equations [21]. Moreover, they have employed these basis functions to solve FOCPs in [22].

The objective of this paper is to present a numerical method for solving fractional-order optimal control-affine problems (FOCAPs). To achieve this, we utilize a combination of block-pulse functions and Jacobi polynomials with a fractional-order component, which we refer to as fractional hybrid Jacobi functions (FHJFs) for brevity. The FOCAP in question is as follows:

$$\min \ \ J = \int_0^{t_f} f(t, x(t), u(t))dt, \tag{1}$$

subject to:

$$ {}_0^C D_t^{\alpha} x(t) = g(t, x(t), {}_0^C D_t^{\alpha_1} x(t), {}_0^C D_t^{\alpha_2} x(t), \cdots, {}_0^C D_t^{\alpha_r} x(t)) + b(t)u(t), \tag{2}$$

$$x^{(k)}(0) = x_k, \quad k = 0, 1, \ldots, \lceil \alpha \rceil - 1, \tag{3}$$

where f and g are smooth linear or nonlinear functions of their arguments, $b(t) \neq 0$ for all $t \in [0, t_f]$, $0 < \alpha_1 < \cdots < \alpha_r < \alpha$, ${}_0^C D_t^{\alpha}$ denotes the Caputo fractional derivative operator and $\lceil \cdot \rceil$ is the ceiling function.

The outline of this paper is as follows: In Section 2, some basic definitions and preliminary concepts on the fractional calculus are presented, and FHJFs are introduced, too. We derive the Riemann-Liouville fractional integral operator of the basis functions in Section 3. Section 4 is devoted to introducing a numerical method for solving problem (1)–(3). In Section 5, an error analysis is given. Then, we report our numerical findings and demonstrate the efficiency of the proposed method by considering some numerical examples in Section 6. Finally, concluding remarks are presented in Section 7.

2 Preliminaries

In this section, we recall definitions and properties of two important fractional operators called as fractional Caputo derivative and Riemann-Liouville integral. Furthermore, FHJFs are introduced and the method of expanding a function based on these basis functions is discussed.

2.1 Fractional Calculus

Definition 1. Let $\alpha \in \mathbb{R}$, $n-1<\alpha \leq n$, $n \in \mathbb{N}$, and f be a real-valued continuous function defined on $[0,\infty)$, then the Caputo fractional derivative of order $\alpha > 0$ of f is defined by [20]:

$$ {}_0^C D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1<\alpha<n, \\ f^{(n)}(t), & \alpha = n, \end{cases} $$

where $\Gamma(\cdot)$ is Gamma function given by

$$ \text{Gamma}(s) = \int_0^\infty t^{s-1} e^{-t} dt. $$

For $\mathbb{N}_0 = \{0,1,2,\ldots\}$, the Caputo fractional derivative operator ${}_0^C D_t^\alpha$ satisfies the following useful and practical property [11]

$$ {}_0^C D_t^\alpha t^\beta = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \alpha, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \text{otherwise.} \end{cases} $$

Definition 2. The left Riemann-Liouville fractional integral operator of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ of a function f is defined by [20]:

$$ {}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t>0. $$

We have the following useful property for the Riemann-Liouville integral operator:

$$ {}_0I_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha}, \quad \alpha>0, \quad \beta>-1, \quad t>0. \tag{4} $$

Also, for fractional Riemann-Liouville integral and Caputo derivative operators, the following properties hold [11]:

$$ {}_0I_t^\alpha({}_0^C D_t^\alpha f(t)) = f(t) - \sum_{k=0}^{\lceil\alpha\rceil-1} f^{(k)}(0)\frac{t^k}{k!}, \tag{5} $$

$$ {}_0I_t^{\alpha-\beta}({}_0^C D_t^\alpha f(t)) = {}_0^C D_t^\beta f(t) - \sum_{k=\lceil\beta\rceil}^{\lceil\alpha\rceil-1} f^{(k)}(0)\frac{t^{k-\beta}}{\Gamma(k-\beta+1)}, \quad 0<\beta<\alpha, \quad t>0. \tag{6} $$

2.2 Fractional-Order Hybrid Jacobi Functions

The well-known Jacobi polynomials are generated through the following recurrence relation [26]

$$J_{m+1}^{(\nu,\gamma)}(t) = (a_m^{(\nu,\gamma)} t + b_m^{(\nu,\gamma)}) J_m^{(\nu,\gamma)}(t) - c_m^{(\nu,\gamma)} J_{m-1}^{(\nu,\gamma)}(t), \quad m \geq 1, \quad \nu, \gamma > -1,$$
$$J_0^{(\nu,\gamma)}(t) = 1, \quad J_1^{(\nu,\gamma)}(t) = \frac{1}{2}(\nu+\gamma+2)t + \frac{1}{2}(\nu-\gamma),$$

in which

$$a_m^{(\nu,\gamma)}(t) = \frac{(2m+\nu+\gamma+1)(2m+\nu+\gamma+2)}{2(m+1)(m+\nu+\gamma+1)},$$
$$b_m^{(\nu,\gamma)}(t) = \frac{(\nu^2-\gamma^2)(2m+\nu+\gamma+1)}{2(m+1)(m+\nu+\gamma+1)(2m+\nu+\gamma)},$$
$$c_m^{(\nu,\gamma)}(t) = \frac{(m+\nu)(m+\gamma)(2m+\nu+\gamma+2)}{(m+1)(m+\nu+\gamma+1)(2m+\nu+\gamma)}.$$

Also, the analytic form of the Jacobi polynomial of degree m, $J_m^{(\nu,\gamma)}(t)$, is given by

$$J_m^{(\nu,\gamma)}(t) = \frac{\Gamma(m+\nu+1)}{m!\Gamma(m+\nu+\gamma+1)} \sum_{k=0}^{m} \sum_{r=0}^{k} \binom{m}{k} \binom{k}{r} (-1)^{k-r} \times \frac{\Gamma(m+k+\nu+\gamma+1)}{2^k \Gamma(k+\nu+1)} t^r. \tag{7}$$

Jacobi polynomials are orthogonal functions on the interval $[-1, 1]$ concerning the weight function

$$w^{(\nu,\gamma)}(t) = (1-t)^\nu (1+t)^\gamma,$$

and the orthogonality property holds as follows:

$$\int_{-1}^{1} w^{(\nu,\gamma)}(t) J_i^{(\nu,\gamma)}(t) J_j^{(\nu,\gamma)}(t) dt = h_i^{(\nu,\gamma)} \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta and

$$h_i^{(\nu,\gamma)} = \frac{2^{\nu+\gamma+1}\Gamma(\nu+i+1)\Gamma(\gamma+i+1)}{i!(\nu+\gamma+2i+1)\Gamma(\nu+\gamma+i+1)}.$$

Definition 3. Fractional hybrid Jacobi functions $b_{n,m}^{\lambda}(t)$, for $\lambda > 0$, $n = 1, 2, \ldots, N$ and $m = 0, 1, \ldots, M$, are defined on the interval $[0, t_f)$ as follows:

$$b_{n,m}^{\lambda}(t) = \begin{cases} J_m^{(\nu,\gamma)}\left(2N\left(\frac{t}{t_f}\right)^\lambda - 2n + 1\right), & t \in \left[(\frac{n-1}{N})^{\frac{1}{\lambda}} t_f, (\frac{n}{N})^{\frac{1}{\lambda}} t_f\right), \\ 0, & \text{otherwise}, \end{cases}$$

where n and m are the order of block-pulse functions and Jacobi polynomials, respectively.

Fractional-order hybrid Jacobi functions construct a set of orthogonal functions concerning the following weight function:

$$w_N^{(\nu,\gamma)}(t) = \begin{cases} w_{1,N}^{(\nu,\gamma)}(t), & 0 \leq t < (\frac{1}{N})^{\frac{1}{\lambda}} t_f, \\ w_{2,N}^{(\nu,\gamma)}(t), & (\frac{1}{N})^{\frac{1}{\lambda}} t_f \leq t < (\frac{2}{N})^{\frac{1}{\lambda}} t_f, \\ \vdots \\ w_{N,N}^{(\nu,\gamma)}(t), & (\frac{N-1}{N})^{\frac{1}{\lambda}} t_f \leq t < t_f, \end{cases}$$

where

$$w_{n,N}^{(\nu,\gamma)}(t) = \frac{2N\lambda}{t_f^\lambda} t^{\lambda-1} w^{(\nu,\gamma)} \left(2N \left(\frac{t}{t_f} \right)^\lambda - 2n + 1 \right), \quad n = 1, 2, \ldots, N.$$

2.3 Function Approximation

Any square integrable function f defined on $[0, t_f)$ might be expanded by the FHJFs as:

$$f(t) = \sum_{n=1}^{N} \sum_{m=0}^{\infty} c_{n,m} b_{n,m}^\lambda(t).$$

The corresponding coefficients $c_{n,m}$, $n = 1, 2, \cdots N$, $m = 0, 1, 2, \cdots$, are evaluated as

$$c_{n,m} = \frac{\langle f(t), b_{n,m}^\lambda(t) \rangle_{w_N^{(\nu,\gamma)}}}{\langle b_{n,m}^\lambda(t), b_{n,m}^\lambda(t) \rangle_{w_N^{(\nu,\gamma)}}},$$

with

$$\langle f(t), g(t) \rangle_{w_N^{(\nu,\gamma)}} = \int_0^{t_f} w_N^{(\nu,\gamma)}(t) f(t) g(t) dt.$$

Considering a finite expansion of a function f in terms of the FHJFs, one obtains

$$f(t) \simeq f_{N,M}(t) = \sum_{n=1}^{N} \sum_{m=0}^{M} c_{n,m} b_{n,m}^\lambda(t) = C^T B^\lambda(t), \tag{8}$$

where

$$C = [c_{1,0}, c_{1,1}, \ldots, c_{1,M}, c_{2,0}, c_{2,1}, \ldots, c_{2,M}, \ldots, c_{N,0}, c_{N,1}, \ldots, c_{N,M}]^T$$

and

$$B^\lambda(t) = \left[b_{1,0}^\lambda(t), \ldots, b_{1,M}^\lambda(t), b_{2,0}^\lambda(t), \ldots, b_{2,M}^\lambda(t), \ldots, b_{N,0}^\lambda(t), \ldots, b_{N,M}^\lambda(t) \right]^T. \tag{9}$$

3 Riemann-Liouville Integral Operator of FHJFs

This section is devoted to the construction method of the Riemann-Liouville fractional integral operator of the FHJFs. For this purpose, we apply the Riemann-Liouville fractional integral operator ${}_0I_t^\alpha$ to $B^\lambda(t)$ defined by (9). Let us consider

$${}_0I_t^\alpha B^\lambda(t) = \overline{B}^\lambda(t, \alpha), \tag{10}$$

in which

$$\overline{B}^{\lambda}(t,\alpha) = \big[{}_0I_t^{\alpha} b_{1,0}^{\lambda}(t), {}_0I_t^{\alpha} b_{1,1}^{\lambda}(t), \ldots, {}_0I_t^{\alpha} b_{1,M}^{\lambda}(t), \\ \ldots, {}_0I_t^{\alpha} b_{N,0}^{\lambda}(t), {}_0I_t^{\alpha} b_{N,1}^{\lambda}(t), \ldots, {}_0I_t^{\alpha} b_{N,M}^{\lambda}(t)\big]^T.$$

To compute the elements of $\overline{B}^{\lambda}(t,\alpha)$, we rewrite each basis function in the following more appropriate form

$$b_{n,m}^{\lambda}(t) = \mu_{\left(\frac{n-1}{N}\right)^{\frac{1}{\lambda}} t_f}(t) J_m^{(\nu,\gamma)}\left(2N\left(\frac{t}{t_f}\right)^{\lambda} - 2n + 1\right) \\ - \mu_{\left(\frac{n}{N}\right)^{\frac{1}{\lambda}} t_f}(t) J_m^{(\nu,\gamma)}\left(2N\left(\frac{t}{t_f}\right)^{\lambda} - 2n + 1\right),$$

where μ_c is the unit step function defined as

$$\mu_c(t) = \begin{cases} 1, & t \geq c, \\ 0, & t < c. \end{cases}$$

Taking into account the analytic form of Jacobi polynomials given by (7), we get

$$b_{n,m}^{\lambda}(t) = \frac{\Gamma(m+\nu+1)}{m!\Gamma(m+\nu+\gamma+1)} \sum_{k=0}^{m}\sum_{r=0}^{k} \binom{m}{k}\binom{k}{r}(-1)^{k-r} \\ \times \frac{\Gamma(m+k+\nu+\gamma+1)}{2^k\Gamma(k+\nu+1)}\left(2N\left(\frac{t}{t_f}\right)^{\lambda} - 2n + 1\right)^r \\ \times \left(\mu_{\left(\frac{n-1}{N}\right)^{\frac{1}{\lambda}} t_f}(t) - \mu_{\left(\frac{n}{N}\right)^{\frac{1}{\lambda}} t_f}(t)\right),$$

which is equivalent to:

$$b_{n,m}^{\lambda}(t) = \frac{\Gamma(m+\nu+1)}{m!\Gamma(m+\nu+\gamma+1)} \sum_{k=0}^{m}\sum_{r=0}^{k}\sum_{l=0}^{r} \binom{m}{k}\binom{k}{r}\binom{r}{l}(-1)^{k-r} \\ \times \left(\frac{2N}{t_f^{\lambda}}\right)^{l}(1-2n)^{r-l}\frac{\Gamma(m+k+\nu+\gamma+1)}{2^k\Gamma(k+\nu+1)} t^{\lambda l} \\ \times \left(\mu_{\left(\frac{n-1}{N}\right)^{\frac{1}{\lambda}} t_f}(t) - \mu_{\left(\frac{n}{N}\right)^{\frac{1}{\lambda}} t_f}(t)\right). \tag{11}$$

Applying the Riemann-Liouville integral operator of order α to both sides of the above resultant relation implies that

$${}_0I_t^{\alpha} b_{n,m}^{\lambda}(t) = \frac{\Gamma(m+\nu+1)}{m!\Gamma(m+\nu+\gamma+1)} \sum_{k=0}^{m}\sum_{r=0}^{k}\sum_{l=0}^{r} \binom{m}{k}\binom{k}{r}\binom{r}{l}(-1)^{k-r} \\ \times \left(\frac{2N}{t_f^{\lambda}}\right)^{l}(1-2n)^{r-l}\frac{\Gamma(m+k+\nu+\gamma+1)}{2^k\Gamma(k+\nu+1)} \\ \times \left({}_0I_t^{\alpha}(t^{\lambda l}\mu_{\left(\frac{n-1}{N}\right)^{\frac{1}{\lambda}} t_f}(t)) - {}_0I_t^{\alpha}(t^{\lambda l}\mu_{\left(\frac{n}{N}\right)^{\frac{1}{\lambda}} t_f}(t))\right). \tag{12}$$

Suppose that c, λ and α are arbitrary positive numbers. Using property (4), we have

$$
\begin{aligned}
{}_0I_t^{\alpha}(t^{\lambda}\mu_c(t)) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\lambda}\mu_c(s)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\lambda}ds - \frac{1}{\Gamma(\alpha)}\int_0^c (t-s)^{\alpha-1}s^{\lambda}ds \\
&= {}_0I_t^{\alpha}(t^{\lambda}) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^c s^{\lambda}(1-\frac{s}{t})^{\alpha-1}ds \\
&= t^{\lambda+\alpha}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\frac{c^{\lambda+1}}{\lambda+1}{}_2F_1(\lambda+1,1-\alpha;\lambda+2;\frac{c}{t}),
\end{aligned} \tag{13}
$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function defined by

$$
{}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},
$$

with

$$
(x)_n \equiv \frac{\Gamma(x+n)}{\Gamma(x)}.
$$

Utilizing (13) in (12) yields

$$
{}_0I_t^{\alpha}b_{n,m}^{\lambda}(t) = \begin{cases} 0, & t \in \left(-\infty, \left(\frac{n-1}{N}\right)^{\frac{1}{\lambda}}t_f\right), \\ \Omega_1(t), & t \in \left[\left(\frac{n-1}{N}\right)^{\frac{1}{\lambda}}t_f, \left(\frac{n}{N}\right)^{\frac{1}{\lambda}}t_f\right), \\ \Omega_1(t)-\Omega_2(t), & t \in \left[\left(\frac{n}{N}\right)^{\frac{1}{\lambda}}t_f, \infty\right), \end{cases}
$$

with

$$
\begin{aligned}
\Omega_1(t) = &\frac{\Gamma(m+\nu+1)}{m!\Gamma(m+\nu+\gamma+1)}\sum_{k=0}^{m}\sum_{r=0}^{k}\sum_{l=0}^{r}\binom{m}{k}\binom{k}{r}\binom{r}{l}(-1)^{k-r} \\
&\times\left(\frac{2N}{t_f^{\lambda}}\right)^{l}(1-2n)^{r-l}\frac{\Gamma(m+k+\nu+\gamma+1)}{2^k\Gamma(k+\nu+1)} \\
&\times\left(t^{\lambda l+\alpha}\frac{\Gamma(\lambda l+1)}{\Gamma(\lambda l+\alpha+1)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)(\lambda l+1)}\left(\left(\frac{n-1}{N}\right)^{\frac{1}{\lambda}}t_f\right)^{\lambda l+1}\right. \\
&\left.\times\ {}_2F_1\left(\lambda l+1, 1-\alpha; \lambda l+2; \frac{t_f}{t}\left(\frac{n-1}{N}\right)^{\frac{1}{\lambda}}\right)\right), \\
\Omega_2(t) = &\frac{\Gamma(m+\nu+1)}{m!\Gamma(m+\nu+\gamma+1)}\sum_{k=0}^{m}\sum_{r=0}^{k}\sum_{l=0}^{r}\binom{m}{k}\binom{k}{r}\binom{r}{l}(-1)^{k-r} \\
&\times\left(\frac{2N}{t_f^{\lambda}}\right)^{l}(1-2n)^{r-l}\frac{\Gamma(m+k+\nu+\gamma+1)}{2^k\Gamma(k+\nu+1)} \\
&\times\left(t^{\lambda l+\alpha}\frac{\Gamma(\lambda l+1)}{\Gamma(\lambda l+\alpha+1)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)(\lambda l+1)}\left(\left(\frac{n}{N}\right)^{\frac{1}{\lambda}}t_f\right)^{\lambda l+1}\right. \\
&\left.\times\ {}_2F_1\left(\lambda l+1, 1-\alpha; \lambda l+2; \frac{t_f}{t}\left(\frac{n}{N}\right)^{\frac{1}{\lambda}}\right)\right).
\end{aligned}
$$

4 Numerical Method

In this section, we employ the FHJFs and their properties to present a computational technique for solving FOCAP (1)-(3). At first, we approximate the fractional state rate ${}_{0}^{C}D_{t}^{\alpha}x(t)$ in terms of the FHJFs as

$$ {}_{0}^{C}D_{t}^{\alpha}x(t) \simeq C^{T}B^{\lambda}(t), \tag{14} $$

where C is a vector with unknown entries $c_{n,m}$, $n = 1,2,\dots,N$, $m = 0,1,\dots,M$, and $B^{\lambda}(t)$ is defined by (9). Then, using Equations (5), (10) and (14), we have

$$ x(t) \simeq C^{T}\overline{B}^{\lambda}(t,\alpha) + \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t^{k}}{k!}x_{k}. \tag{15} $$

From Equations (14) and (6), we obtain

$$ {}_{0}^{C}D_{t}^{\alpha_i}x(t) \simeq C^{T}\overline{B}^{\lambda}(t,\alpha-\alpha_i) + \sum_{k=\lceil\alpha_i\rceil}^{\lceil\alpha\rceil-1} \frac{t^{k-\alpha_i}}{\Gamma(k-\alpha_i+1)}x_{k}, \quad \text{for } i = 1,2,\cdots,r. $$

Then, utilizing the dynamical system in Eq. (2) and substituting the above approximations in it yield

$$ \begin{aligned} u(t) \simeq \frac{1}{b(t)}\Bigg[& C^{T}B^{\lambda}(t) - g\Bigg(t, C^{T}\overline{B}^{\lambda}(t,\alpha) + \sum_{k=0}^{\lceil\alpha\rceil-1}\frac{t^{k}}{k!}x_{k}, C^{T}\overline{B}^{\lambda}(t,\alpha-\alpha_1) \\ & + \sum_{k=\lceil\alpha_1\rceil}^{\lceil\alpha\rceil-1}\frac{t^{k-\alpha_1}}{\Gamma(k-\alpha_1+1)}x_{k}, \cdots, C^{T}\overline{B}^{\lambda}(t,\alpha-\alpha_r) \\ & + \sum_{k=\lceil\alpha_r\rceil}^{\lceil\alpha\rceil-1}\frac{t^{k-\alpha_r}}{\Gamma(k-\alpha_r+1)}x_{k}\Bigg)\Bigg]. \end{aligned} \tag{16} $$

By substituting (15) and (16) into (1), we gain

$$ \begin{aligned} J[C] = \int_{0}^{t_f} f\Bigg(& t, C^{T}\overline{B}^{\lambda}(t,\alpha) + \sum_{k=0}^{\lceil\alpha\rceil-1}\frac{t^{k}}{k!}x_{k}, \frac{1}{b(t)}[C^{T}B^{\lambda}(t) - g(t, C^{T}\overline{B}^{\lambda}(t,\alpha) \\ & + \sum_{k=0}^{\lceil\alpha\rceil-1}\frac{t^{k}}{k!}x_{k}, C^{T}\overline{B}^{\lambda}(t,\alpha-\alpha_1) + \sum_{k=\lceil\alpha_1\rceil}^{\lceil\alpha\rceil-1}\frac{t^{k-\alpha_1}}{\Gamma(k-\alpha_1+1)}x_{k}, \cdots \\ & , C^{T}\overline{B}^{\lambda}(t,\alpha-\alpha_r) + \sum_{k=\lceil\alpha_r\rceil}^{\lceil\alpha\rceil-1}\frac{t^{k-\alpha_r}}{\Gamma(k-\alpha_r+1)}x_{k})]\Bigg) dt. \end{aligned} \tag{17} $$

Let us set

$$ \begin{aligned} F(C,t) = f\Bigg(& t, C^{T}\overline{B}^{\lambda}(t,\alpha) + \sum_{k=0}^{\lceil\alpha\rceil-1}\frac{t^{k}}{k!}x_{k}, \frac{1}{b(t)}[C^{T}B^{\lambda}(t) - g(t, C^{T}\overline{B}^{\lambda}(t,\alpha) \\ & + \sum_{k=0}^{\lceil\alpha\rceil-1}\frac{t^{k}}{k!}x_{k}, C^{T}\overline{B}^{\lambda}(t,\alpha-\alpha_1) + \sum_{k=\lceil\alpha_1\rceil}^{\lceil\alpha\rceil-1}\frac{t^{k-\alpha_1}}{\Gamma(k-\alpha_1+1)}x_{k}, \cdots \end{aligned} $$

$$, C^T \overline{B}^{\lambda}(t, \alpha - \alpha_r) + \sum_{k=\lceil \alpha_r \rceil}^{\lceil \alpha \rceil - 1} \frac{t^{k-\alpha_r}}{\Gamma(k - \alpha_r + 1)} x_k)] \Bigg).$$

Then, we have

$$J[C] = \int_0^{t_f} F(C, t) dt.$$

In the cases where it is difficult to compute the above integral by the analytic methods, we can use Gauss-Legendre quadrature formula as

$$J[C] \simeq \frac{t_f}{2} \sum_{j=1}^{K} w_j F\left(\frac{t_f}{2}(t_j + 1), C\right),$$

where $\{t_j\}_{j=1}^K$ are zeros of the Legendre polynomial of degree K and $\{w_j\}_{j=1}^K$ are the corresponding weights [8].

The necessary conditions for the extremum of the performance index are

$$\frac{\partial J}{\partial c_{n,m}}[C] = 0, \quad n = 1, 2, \ldots, N, \quad m = 0, 1, \cdots, M. \tag{18}$$

Equation (18) builds a system of algebraic equations that can be solved using Newton's iterative method for finding the elements of C. Finally, we can obtain approximations of x, u and the optimal value of J using (15), (16) and (17), respectively.

5 Error Bound

This section aims to give an error bound for the expansion of a given function in terms of the FHJFs. To do this, for the sake of simplicity, we set:

$$w_N(t) := w_N^{(\nu,\gamma)}(t) \quad \text{and} \quad w_{n,N}(t) := w_{n,N}^{(\nu,\gamma)}(t).$$

Let us consider $\Pi_M(I_{n,\lambda})$, $n = 1, 2, \ldots, N$, as the space of all polynomials of degree at most M on the subinterval $I_{n,\lambda}$, with $I_{n,\lambda}$ is defined as

$$I_{n,\lambda} = \left[t_{n-1}^{N,\lambda}, t_n^{N,\lambda}\right),$$

where

$$t_i^{N,\lambda} = \left(\frac{i}{N}\right)^{\frac{1}{\lambda}} t_f, \quad i = 0, 1, \ldots, N.$$

Furthermore, we consider $L^2_{w_N}[0, t_f)$ and $L^2_{w_{n,N}}(I_{n,\lambda})$ as the spaces of all measurable functions whose square is Lebesgue integrable on $[0, t_f)$ and $I_{n,\lambda}$ concerning the weight functions w_N and $w_{n,N}$, respectively. The corresponding norms are defined as follows:

$$\|f\|_{w_N} = \left(\int_0^{t_f} w_N(t)|f(t)|^2 dt\right)^{\frac{1}{2}},$$

$$\|f\|_{w_{n,N}} = \left(\int_{t_{n-1}^{N,\lambda}}^{t_n^{N,\lambda}} w_{n,N}(t)|f(t)|^2 dt\right)^{\frac{1}{2}}.$$

Suppose that f is a sufficiently smooth function on $I_{n,\lambda}$ and $P_{n,M}$ is its interpolating polynomial at $(M+1)$ arbitrary distinct points $t_{n,m}$, $m = 0, 1, \ldots, M$, belonging to $I_{n,\lambda}$. Then, we have

$$f(t) - P_{n,M}(t) = \frac{f^{(M+1)}(\eta)}{(M+1)!} \prod_{m=0}^{M} (t - t_{n,m}), \quad \eta \in I_{n,\lambda}.$$

Therefore,

$$\begin{aligned} |f(t) - P_{n,M}(t)| &\leq \frac{1}{(M+1)!} \max_{t \in I_{n,\lambda}} |f^{(M+1)}(t)| \max_{t \in I_{n,\lambda}} \prod_{m=0}^{M} |t - t_{n,m}| \\ &\leq \frac{1}{(M+1)!} \max_{t \in I_{n,\lambda}} |f^{(M+1)}(t)| \left(\left(\frac{n}{N}\right)^{\frac{1}{\lambda}} t_f - \left(\frac{n-1}{N}\right)^{\frac{1}{\lambda}} t_f \right)^{M+1} \\ &= \frac{1}{(M+1)!} t_f^{M+1} N^{-\left(\frac{M+1}{\lambda}\right)} \max_{t \in I_{n,\lambda}} \left| f^{(M+1)}(t) \right| \\ &\quad \times \sum_{k=0}^{M+1} (-1)^k \binom{M+1}{k} n^{\frac{M+1-k}{\lambda}} (n-1)^{\frac{k}{\lambda}}. \end{aligned} \tag{19}$$

The above inequality results to the following lemma.

Lemma 1. Let N be a positive integer number and $x_n : I_{n,\lambda} \longrightarrow \mathbb{R}$, $n = 1, \ldots, N$ be a sufficiently smooth function defined on $I_{n,\lambda}$. Suppose that $\tilde{x}_n(t) = \sum_{m=0}^{M} c_{n,m} b_{n,m}^{\lambda}(t)$ denotes the truncated FHJFs series of x_n, then

$$\begin{aligned} \|x_n - \tilde{x}_n\|_{w_{n,N}} \leq \frac{C_{n,M}}{(M+1)!} \sqrt{\Lambda^{(\nu,\gamma)}} t_f^{M+1} N^{-\left(\frac{M+1}{\lambda}\right)} \\ \times \sum_{k=0}^{M+1} (-1)^k \binom{M+1}{k} n^{\frac{M+1-k}{\lambda}} (n-1)^{\frac{k}{\lambda}}, \end{aligned}$$

where $C_{n,M} = \max_{t \in I_{n,\lambda}} |x_n^{(M+1)}(t)|$ and

$$\Lambda^{(\nu,\gamma)} = \frac{1}{1+\gamma} ({}_2F_1(1, -\nu; 2+\gamma; -1)) + \frac{1}{1+\nu} ({}_2F_1(1, -\gamma; 2+\nu; -1)).$$

Proof. It is clear that $\tilde{x}_n$ is the best approximation of x_n in $\Pi_M(I_{n,\lambda})$, that is

$$\|x_n - \tilde{x}_n\|_{w_{n,N}} \leq \|x_n - p\|_{w_{n,N}},$$

where p is any arbitrary polynomial in $\Pi_M(I_{n,\lambda})$. Specially, we have

$$\|x_n - \tilde{x}_n\|_{w_{n,N}} \leq \|x_n - P_{n,M}\|_{w_{n,N}}, \tag{20}$$

where $P_{n,M}$ is the interpolating polynomial of x_n. Using (19) and (20), we get

$$\begin{aligned} \|x_n - \tilde{x}_n\|_{w_{n,N}}^2 &\leq \int_{t_{n-1}^{N,\lambda}}^{t_n^{N,\lambda}} w_{n,N}(t) |x_n - P_{n,M}|^2 dt \\ &\leq \left(\frac{C_{n,M}}{(M+1)!} t_f^{M+1} N^{-\left(\frac{M+1}{\lambda}\right)} \sum_{k=0}^{M+1} (-1)^k \binom{M+1}{k} n^{\frac{M+1-k}{\lambda}} (n-1)^{\frac{k}{\lambda}} \right)^2 \\ &\quad \times \int_{t_{n-1}^{N,\lambda}}^{t_n^{N,\lambda}} w_{n,N}(t) dt \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{C_{n,M}}{(M+1)!} t_f^{M+1} N^{-\left(\frac{M+1}{\lambda}\right)} \sum_{k=0}^{M+1} (-1)^k \binom{M+1}{k} n^{\frac{M+1-k}{\lambda}} (n-1)^{\frac{k}{\lambda}} \right)^2 \\
&\times \int_{-1}^{1} w^{(\nu,\gamma)}(t) dt \\
&= \left(\frac{C_{n,M}}{(M+1)!} t_f^{M+1} N^{-\left(\frac{M+1}{\lambda}\right)} \sum_{k=0}^{M+1} (-1)^k \binom{M+1}{k} n^{\frac{M+1-k}{\lambda}} (n-1)^{\frac{k}{\lambda}} \right)^2 \\
&\times \left(\frac{1}{1+\gamma} ({}_2F_1(1,-\nu;2+\gamma;-1)) + \frac{1}{1+\nu} ({}_2F_1(1,-\gamma;2+\nu;-1)) \right). \quad (21)
\end{aligned}
$$

Taking square root of both sides of (21) completes the proof. □

Theorem 1. Let x be a continuous function on $[0, t_f]$ and its restrictions on each subinterval $I_{n,\lambda}$ are sufficiently smooth. Let $x_n : I_{n,\lambda} \longrightarrow \mathbb{R}$, $n = 1, \dots, N$ be defined by $x_n(t) = x(t)$ for all $t \in I_{n,\lambda}$. If $x_{N,M}$, given by (8), is the expansion of x based on the FHJFs, then

$$
\|x - x_{N,M}\|_{w_N} \leq \frac{C_M}{(M+1)!} t_f^{M+1} N^{-\left(\frac{M+1}{\lambda}\right)} \sqrt{\Lambda^{(\nu,\gamma)} \Upsilon_{N,M}},
$$

in which $C_M = \max_{1 \leq n \leq N} C_{n,M}$ and

$$
\Upsilon_{N,M} = \sum_{n=1}^{N} \left(\sum_{k=0}^{M+1} (-1)^k \binom{M+1}{k} n^{\frac{M+1-k}{\lambda}} (n-1)^{\frac{k}{\lambda}} \right)^2 .
$$

Proof. Using the notations introduced in Lemma 1, we have

$$
\begin{aligned}
&\|x - x_{N,M}\|_{w_N}^2 \\
&\quad = \int_0^{t_f} w_N(t) |x(t) - x_{N,M}(t)|^2 dt \\
&\quad = \sum_{n=1}^{N} \int_{t_{n-1}^{N,\lambda}}^{t_n^{N,\lambda}} w_{n,N}(t) |x_n(t) - \tilde{x}_n(t)|^2 dt \\
&\quad = \sum_{n=1}^{N} \|x_n - \tilde{x}_n\|_{w_{n,N}}^2 \\
&\quad \leq \sum_{n=1}^{N} \left(\frac{C_{n,M}}{(M+1)!} t_f^{M+1} N^{-\left(\frac{M+1}{\lambda}\right)} \sum_{k=0}^{M+1} (-1)^k \binom{M+1}{k} n^{\frac{M+1-k}{\lambda}} (n-1)^{\frac{k}{\lambda}} \right)^2 \Lambda^{(\nu,\gamma)} \\
&\quad \leq \left(\frac{C_M}{(M+1)!} t_f^{M+1} N^{-\left(\frac{M+1}{\lambda}\right)} \right)^2 \Lambda^{(\nu,\gamma)} \sum_{n=1}^{N} \left(\sum_{k=0}^{M+1} (-1)^k \binom{M+1}{k} n^{\frac{M+1-k}{\lambda}} (n-1)^{\frac{k}{\lambda}} \right)^2 ,
\end{aligned}
$$

which completes the proof by taking the square root of both sides. □

6 Illustrative Examples

In this section, we apply the proposed method to several examples of FOCAPs to show its accuracy and efficiency. All the numerical computations have been implemented by Mathematica software. Moreover, for solving the resulting system (18), the function FindRoot was used. To show the accuracy of the present method, we define the absolute error functions as follows:

$$e(x_{N,M},\lambda) = |x - x^{\lambda}_{N,M}|,$$
$$e(u_{N,M},\lambda) = |u - u^{\lambda}_{N,M}|,$$

where x and u are the optimum state and control variables and $x^{\lambda}_{N,M}$ and $u^{\lambda}_{N,M}$ are their approximations, respectively.

Example 1. As the first example, let us consider the following FOCAP [12, 18, 19, 23]:

$$\min \quad J = \int_0^1 \left[e^t (x(t) - t^4 + t - 1)^2 + (t^2+1)\left(u(t) + 1 - t + t^4 - \frac{8000 t^{\frac{21}{10}}}{77\Gamma(\frac{1}{10})} \right)^2 \right] dt,$$

subject to the dynamic constraints:

$$ {}^C_0D^{1.9}_t x(t) = x(t) + u(t),$$
$$x(0) = 1, \quad x'(0) = -1.$$

The optimal value for the performance index is $J = 0$ which occurs when $x(t) = 1 - t + t^4$ and

$$u(t) = -t^4 + t - 1 + \frac{8000}{77\Gamma(\frac{1}{10})} t^{\frac{21}{10}}.$$

For solving this problem using the proposed method, we consider $M = N = 1$ and $\nu = \gamma = 0.5$. Taking into account the power function with the largest non-integer power existing in the problem, we choose $\lambda = 2.1$. Therefore, we have

$$ {}^C_0D^{1.9}_t x(t) = C^T B^{2.1}(t),$$

where

$$C = [c_{1,0}, c_{1,1}]^T, \qquad B^{2.1}(t) = [b^{2.1}_{1,0}(t), b^{2.1}_{1,1}(t)]^T,$$

with

$$b^{2.1}_{1,0}(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{otherwise}, \end{cases} \qquad b^{2.1}_{1,1}(t) = \begin{cases} 2t^{2.1} - 1, & 0 \le t < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then, utilizing the initial conditions gives

$$x(t) = C^T \overline{B}^{2.1}(t, 1.9) + 1 - t, \tag{22}$$

where

$$\overline{B}^{2.1}(t, 1.9) = [{}_0I^{1.9}_t b^{2.1}_{1,0}(t), {}_0I^{1.9}_t b^{2.1}_{1,1}(t)]^T,$$

in which writing the results for $0 \le t < 1$ yields

$$\overline{B}^{2.1}(t, 1.9) = [0.547239 t^{1.9}, 0.183135 t^4 - 0.547239 t^{1.9}]^T.$$

Moreover, we use the dynamical system of the problem and get

$$u(t) = C^T B^{2.1}(t) - C^T \overline{B}^{2.1}(t, 1.9) - 1 + t. \tag{23}$$

Table 1: Comparison of J (Example 1).

Method	J
Legendre polynomials [12]	
$M = 4$	5.42×10^{-7}
$M = 8$	8.22×10^{-10}
Second-kind Chebyshev polynomials [18]	
$M = 4$	5.33×10^{-7}
$M = 8$	2.77×10^{-10}
Modified hat functions [19]	
$M = 4$	9.64×10^{-7}
$M = 8$	1.00×10^{-8}
Fractional-order Boubaker wavelets [23]	
$k = 1, M = 4$	6.05×10^{-7}
$k = 1, M = 6$	1.64×10^{-11}
$k = 2, M = 4$	2.08×10^{-8}
$k = 2, M = 6$	8.12×10^{-13}
Presented method ($\nu = \gamma = 0.5$)	
$N = 1, M = 4, \lambda = 1$	5.41×10^{-7}
$N = 1, M = 6, \lambda = 1$	5.98×10^{-9}
$N = 2, M = 4, \lambda = 1$	1.86×10^{-13}
$N = 2, M = 6, \lambda = 1$	1.08×10^{-14}
$N = 1, M = 4, \lambda = 0.5$	3.94×10^{-7}
$N = 1, M = 6, \lambda = 0.5$	1.25×10^{-11}
$N = 2, M = 4, \lambda = 0.5$	6.01×10^{-12}
$N = 2, M = 6, \lambda = 0.5$	2.68×10^{-22}

By substituting approximations given by (22) and (23) into the performance index and applying the Gauss-Legendre quadrature formula with $K = 8$ and using the optimality condition, we obtain the following system of algebraic equations

$$\begin{cases} 1.9663c_{1,0} - 0.646188c_{1,1} = 7.20843, \\ 1.64025c_{1,1} - 0.646188c_{1,0} = 5.42804. \end{cases}$$

By solving this system, we have

$$c_{1,0} = c_{1,1} = 5.46045.$$

With these values of $c_{1,0}$ and $c_{1,1}$, we obtain the exact state and control variables. It should be noted that all the numbers in our computations have been represented with six significant digits.

Although we obtain the exact solution with $\lambda = 2.1$ and only two basis functions, for checking the rule of choosing λ in the accuracy of the proposed method and to have a comparison between the accuracy of the present method and some other existing ones introduced for solving FOCPs, we implemented the method with different values of M, N, λ and $\nu = \gamma = 0.5$. The numerical results of J are reported in Table 1. Moreover, plots of the absolute error functions with $N = 1, 2$, $M = 4, 6$, $\nu = \gamma = -0.5$ and $\lambda = 0.5$ are displayed in Figure 1.

Example 2. Now we concentrate on the following minimization problem [6, 12, 23]

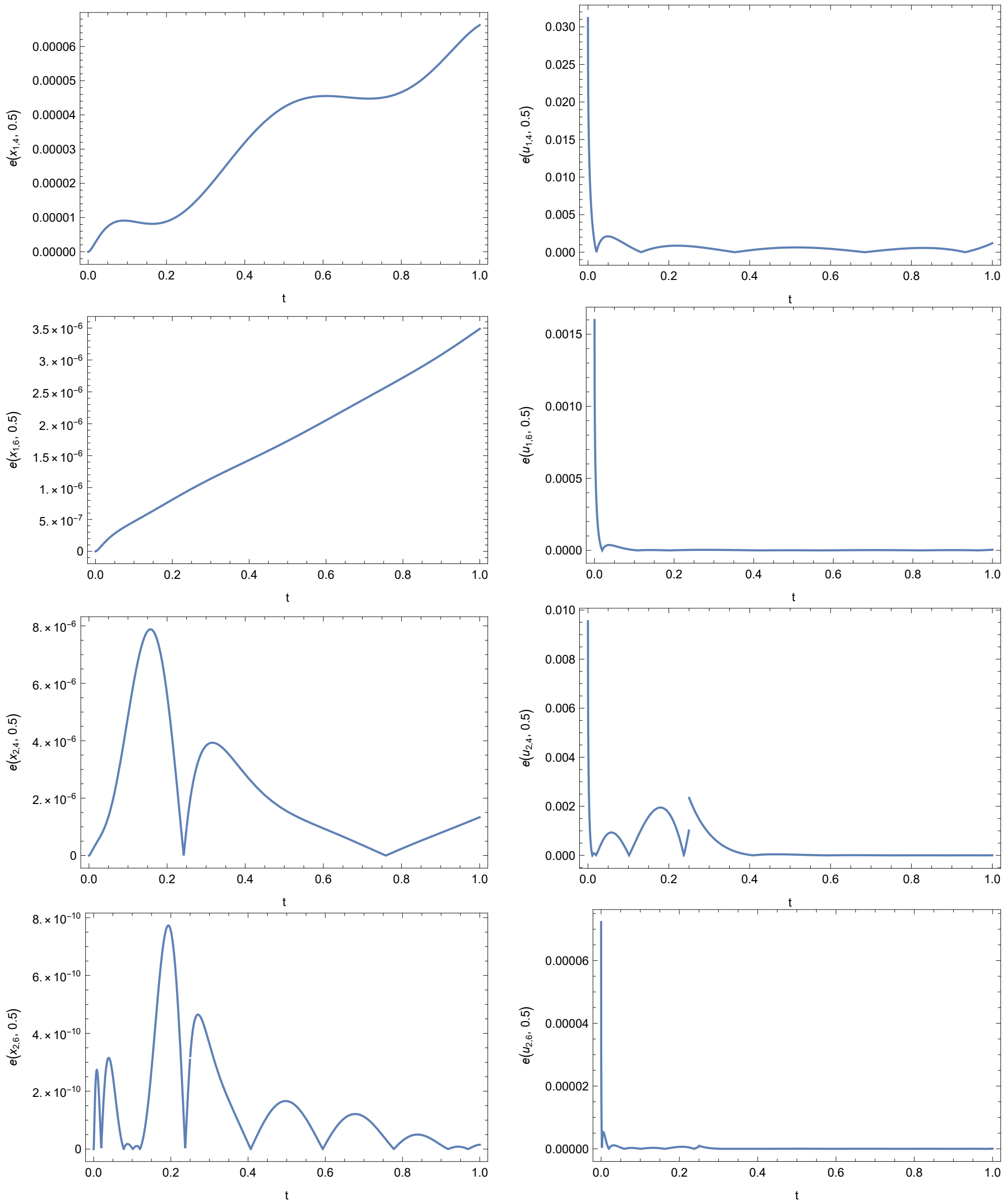

Figure 1: Plots of the absolute error functions with $N = 1, 2$, $M = 4, 6$, $\nu = \gamma = 0.5$ and $\lambda = 0.5$ in Example 1.

$$
\begin{aligned}
min \quad J = \int_0^1 \Big[& x^2(t) - 2t^{\frac{3}{2}}x(t) + u^2(t) - \frac{3\sqrt{\pi}}{4}e^{-t}u(t) + e^{-t+t^{\frac{3}{2}}}u(t) \\
& + t^3 + \frac{9\pi}{64}e^{-2t} - \frac{3\sqrt{\pi}}{8}e^{-2t+t^{\frac{3}{2}}} + \tfrac{1}{4}e^{-2t+2t^{\frac{3}{2}}} + e^{2t} \Big] \, dt,
\end{aligned}
$$

subject to:

$$
\begin{aligned}
& {}_0^C D_t^{1.5} x(t) = e^{x(t)} + 2e^t u(t), \\
& x(0) = x'(0) = 0,
\end{aligned}
$$

In this problem, the optimal state and control variables are $x(t) = t^{\frac{3}{2}}$ and

Table 2: Comparison of J (Example 1).

Method	J
Legendre polynomials [12]	
$M = 4$	1.67×10^{-6}
$M = 7$	1.22×10^{-7}
Second-kind Chebyshev polynomials [18]	
$M = 4$	1.46×10^{-6}
$M = 7$	3.55×10^{-8}
Bernoulli wavelets [5]	
$k = 2, M = 5$	1.19×10^{-7}
$k = 2, M = 7$	1.76×10^{-8}
Presented method ($\nu = \gamma = 0$)	
$N = 1, M = 4, \lambda = 1$	1.17×10^{-6}
$N = 1, M = 6, \lambda = 1$	3.93×10^{-8}
$N = 2, M = 4, \lambda = 1$	2.21×10^{-10}
$N = 2, M = 6, \lambda = 1$	4.63×10^{-11}
$N = 1, M = 4, \lambda = 0.5$	3.03×10^{-9}
$N = 1, M = 6, \lambda = 0.5$	3.51×10^{-11}
$N = 2, M = 4, \lambda = 0.5$	1.08×10^{-12}
$N = 2, M = 6, \lambda = 0.5$	7.08×10^{-21}

$$u(t) = \frac{1}{2} e^{-t} \left(-e^{t^{\frac{3}{2}}} + \frac{3\sqrt{\pi}}{4} \right),$$

respectively. Furthermore, the minimum value for the performance index with sixteen significant digits is $J = 3.194528049465325$. A comparison between the results obtained by the present method, method of [6] based on Bernoulli polynomials and method of [23] based on fractional-order Boubaker wavelets is shown in Table 3. From the results of this table, it is found that regardless of choosing the values of λ and admissible ν and γ, the present method gives the exact solution with only one basis function which shows its superiority compared to the two other methods. To show this result in more detail, we set, for example, $M = 0$, $N = 1$, $\lambda = 1$ and $\nu = \gamma = 0$. According to our suggested method, we have

$${}_0^C D_t^{1.5} x(t) = c_{1,0}.$$

On the other hand, we get

$${}_0 I_t^{\alpha} c_{1,0} = \frac{4}{3\sqrt{\pi}} c_{1,0} t^{1.5}.$$

By implementing the method and solving the final system, we obtain

$$c_{1,0} = \frac{3\sqrt{\pi}}{4},$$

which gives the exact solution.

Example 3. Consider the FOCP in the form [5, 12, 18]:

$$\min \quad J = \int_0^1 \left[(x(t) - t^2)^2 + \left(u(t) + t^4 - \frac{20 t^{\frac{9}{10}}}{9\Gamma(\frac{9}{10})} \right)^2 \right] dt,$$

Table 3: Comparison of J (Example 2).

Method	J
Legendre polynomials [12]	
$M = 3$	3.19453
Bernoulli polynomials [6]	
$M = 8$	3.19453
Fractional-order Boubaker wavelets [23]	
$k = 1$, $M = 3$, $\alpha = 0.5$	3.1945280495
Presented method	
$N = 1$, $M = 0$, $\lambda = 0.5$	3.194528049465325
$N = 1$, $M = 0$, $\lambda = 1$	3.194528049465325
$N = 1$, $M = 0$, $\lambda = 1.5$	3.194528049465325
$N = 1$, $M = 0$, $\lambda = 2$	3.194528049465325

subject to:
$${}_0^C D_t^{1.1} x(t) = t^2 x(t) + u(t),$$
$$x(0) = x'(0) = 0.$$

The exact state and control functions are

$$x(t) = t^2, \quad u(t) = -t^4 + \frac{20 t^{\frac{9}{10}}}{9\Gamma(\frac{9}{10})},$$

and the minimum performance index is $J = 0$.

First, we choose λ equal to the largest non-integer power of t in the performance index ($\lambda = 0.9$) and implement the proposed method for solving the above problem. By this choice, we get the exact solution with every admissible Jacobi parameters ν and γ and $M = N = 1$.

Similar to Example 1, to check the consequence of choosing different values of λ in our implementation, we consider different values for this parameter and display the results in Table 2. Besides the results of our method with $N = 1, 2$, $M = 4, 6$ and $\nu = \gamma = 0$, the results of J obtained by Legendre polynomials method [12], second-kind Chebyshev polynomials method [18] and Bernoulli wavelets method [5] are reported in Table 2. It is seen that our method yields more accurate results with the same number of basis functions compared to the other methods applied for solving this problem. Moreover, the absolute error functions of state and control variables obtained by $N - 1, 2$, $M = 4, 6$, $\nu = \gamma = 0.5$, and $\lambda = 0.5$ are plotted in Figure 2. The results confirm the convergence of the numerical solutions to the exact ones.

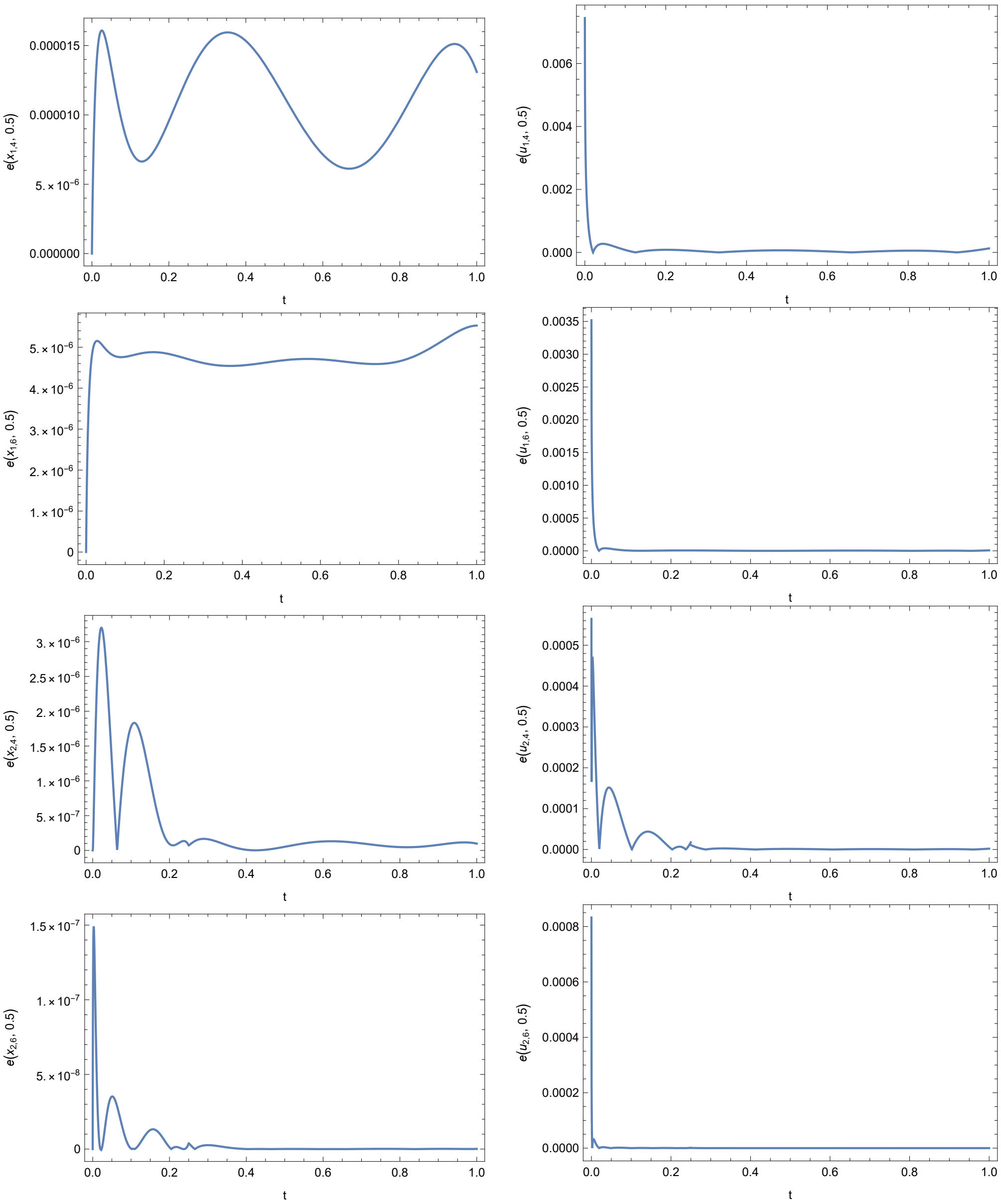

Figure 2: Plots of the absolute error functions with $N = 1, 2$, $M = 4, 6$, $\nu = \gamma = 0.5$ and $\lambda = 0.5$ in Example 3.

7 Concluding Remarks

In this paper, we proposed a novel numerical technique for solving a specific class of fractional-order optimal control-affine problems. The method utilized fractional-order hybrid Jacobi functions, which are composed of a combination of block-pulse functions and fractional-order Jacobi polynomials. To compute the Riemann-Liouville fractional integral of each basis function, we directly employ the analytic form of Jacobi polynomials. By expanding the fractional state rate in terms of the basis functions and leveraging the properties of the Riemann-Liouville integral, we can approximate the state variable and its derivatives. Utilizing the dynamic system constraint of the problem, we can then determine an approximation of the control variable. These approximations are subsequently substituted into the cost function formula, and the optimality conditions transform the main problem into a system of algebraic equations. An error analysis is provided, examining the accuracy of our approach. We solved three fractional optimal control-affine problems previously addressed by other methods in the literature, using our scheme. The numerical results validate the efficiency and high accuracy of our technique when compared to other available methods. Moreover, the examples demonstrate that by appropriately selecting the parameter of the fractional-order of the basis functions, our method can yield exact solutions for problems in which the optimal state and control variables are represented as power functions.

Declarations

Availability of supporting data
All data generated or analyzed during this study are included in this published paper.

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