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Optimality Conditions for Properly Efficient Solutions of Nonsmooth Multiobjective GSIP

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Abstract. This paper aims to establish first-order necessary optimality conditions for non-smooth multi-objective generalized semi-infinite programming problems. These problems involve inequality constraints whose index set depends on the decision vector, and all emerging functions are assumed to be locally Lipschitz. We introduce a new constraint qualification for these problems. Building upon this qualification, we derive an upper estimate for the Clarke sub-differential of the value function of the problem. Furthermore, we demonstrate the necessary optimality conditions for properly efficient solutions to the problem.

Keywords. Constraint qualification, Generalized semi-infinite optimization, Clarke subdifferential, Marginal function.

MSC. 90C46; 90C33; 49K10; 49J52.

1 Introduction

In this paper, we address the problem of multi-objective generalized semi-infinite programming (MGSIP). The MGSIP can be formulated as follows:

$$\begin{aligned} \text{(MGSIP)}: \quad & \inf \quad \varphi(x) := (\varphi_1(x), \ldots, \varphi_p(x)) \\ & \text{s.t.} \quad x \in \mathcal{F} := \big\{x \in \mathbb{R}^n \mid \psi(x, y) \geq 0, \quad \text{for all} \quad y \in \Sigma(x)\big\}, \end{aligned}$$

where the index set is defined as:

$$\Sigma(x) := \big\{y \in \mathbb{R}^m \mid \nu_i(x, y) \leq 0, \quad \text{for all} \quad i \in I\big\},$$

where the appearing functions $\varphi_j : \mathbb{R}^n \to \mathbb{R}$ as $j \in J := \{1, \ldots, p\}$ and $\psi, \nu_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ as $i \in I := \{1, \ldots, q\}$ are assumed to be locally Lipschitz. Additionally, the set-valued mapping $x \mapsto \Sigma(x)$ is uniformly bounded, meaning that for each $x_0 \in \mathcal{F}$ there exists a neighborhood U of x_0 such that the set $\bigcup_{x \in U} \Sigma(x)$ is bounded. This assumption implies that the mapping $x \mapsto \Sigma(x)$ is compact-valued and upper semi-continuous at each $x_0 \in \mathcal{F}$ (see Proposition 2.5.21 in [5]). We recall the definition of upper semi-continuity of set-valued mapping in Section 2. Throughout the paper, these assumptions are consistently maintained. In the case where the functions φ_j, ψ, and ν_i as $(i, j) \in I \times J$ are regular or convex on their domains, the MGSIP is referred to as regular or convex MGSIP, respectively. Smooth generalized semi-infinite programming (GSIP) is a special case of MGSIP when $p = 1$ and all the functions involved are smooth. GSIP has found applications in various fields such as design problems, robot maneuverability problems, reverse Chebyshev approximation problems, and robust optimization (see [11, 14]). Previous works have studied first-order optimality conditions of the smooth GSIPs [28, 29, 30, 32].

When $p = 1$ and the appearing functions are convex (resp. locally Lipschiz), the MGSIP coincides with convex GSIP (resp. nonsmooth GSIP), which have been analyzed in [16] (resp. [18]). For the case where the functions φ and ψ are differences of convex functions, the MGSIP reduces to DC GSIP for which constraint qualifications and optimality conditions have been presented in [15] (resp. [3, 12, 18, 27]).

Motivated by the above, it is both useful and interesting to study optimality conditions for MGSIPs with $p > 1$. While necessary first-order conditions have been addressed for the continuously differentiable case in [31], our current knowledge indicates that no efforts have been made to address the non-smooth case. Therefore, the primary objective of this paper is to fill this gap. Specifically, we introduce a constraint qualification and present optimality conditions for properly efficient solutions of non-smooth regular MGSIPs.

We organize the paper as follows. In the next section, we provide the necessary notations and preliminaries that will be used throughout the paper. Section 3 is devoted to investigating first-order necessary optimality conditions for non-smooth regular MGISPs. In Section 4, we provide several applications of the derived results to convex MGSIP. Finally, Section 5 concludes this paper.

2 Preliminaries

This section presents introductory material on convex analysis and non-smooth analysis, which are widely used in the subsequent discussions. For further details and additional resources, we recommend the following books written by Rockafellar and Wets [24], Hiriart-Urruty and Lemarechal [13], and Clarke [6].

Given a nonempty set $Y \subseteq \mathbb{R}^n$, the notations $\overline{Y}$, $conv(Y)$, and $cone(Y)$ represent the closure of Y, the convex hull of Y, and the convex cone generated by Y (including the origin), respectively. The zero vector in $\mathbb{R}^n$ is denoted by 0_n, and the standard inner product of x and $y \in \mathbb{R}^n$ is denoted by $\langle x, y\rangle$. Let us define:

$$Y^{\odot} := \big\{x \in \mathbb{R}^n \mid \langle x, y\rangle \leq 0, \quad \forall y \in Y\big\},$$
$$Y^{\ominus} := \big\{x \in \mathbb{R}^n \mid \langle x, y\rangle < 0, \quad \forall y \in Y\big\}.$$

The validity of the following equalities can be easily verified:

$$Y^{\odot} = \big(conv(Y)\big)^{\odot}, \quad \text{and} \quad Y^{\ominus} = \big(conv(Y)\big)^{\ominus}.$$

Furthermore, it is evident that if $Y^{\ominus} \neq \emptyset$, then $\overline{Y^{\ominus}} = Y^{\odot}$.

Theorem 1. [13, 24] Let $Y \subseteq \mathbb{R}^n$ be a compact set. Then,

i. $conv(Y)$ is compact.

ii. $cone(Y)$ is closed, provided that $0_n \notin conv(Y)$.

Theorem 2. [13, 24] Let $\{Y_\ell \mid \ell \in L\}$ be an arbitrary collection of nonempty convex sets in $\mathbb{R}^n$ and let $\mathcal{Y} = conv\Big(\bigcup_{\ell\in L} Y_\ell\Big)$. Then, every non-zero vector in $\mathcal{Y}$ can be expressed as a convex combination of $n+1$ or fewer linearly independent vectors, each belonging to a different Y_ℓ.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function and $\hat{x} \in \mathbb{R}^n$. The generalized Clarke directional derivative of f at $\hat{x}$ in the direction $d \in \mathbb{R}^n$ is defined by

$$f^\circ(\hat{x}; d) := \limsup_{y\to\hat{x},\ t\downarrow 0} \frac{f(y+td) - f(y)}{t},$$

and the Clarke subdifferential of f at $\hat{x}$ is defined as:

$$\partial^c f(\hat{x}) := \{\xi \in \mathbb{R}^n \mid\ f^\circ(\hat{x}; d) \geq \langle \xi, d\rangle, \quad \forall d \in \mathbb{R}^n\}.$$

We say that f is regular at $\hat{x}$ if

$$f^\circ(\hat{x}; d) = f'(\hat{x}, d), \quad \forall d \in \mathbb{R}^n,$$

where $f'(\hat{x}, d)$ denotes the classical directional derivative of f at $\hat{x}$ in the direction d, i.e.,

$$f'(\hat{x}, d) := \lim_{t\to 0^+} \frac{f(\hat{x}+td) - f(\hat{x})}{t}.$$

It is known from [6] that the Clarke sub-differential of a locally Lipschitz function at each point is always a non-empty convex compact set. Moreover, if the function $f(\cdot)$ is continuously differentiable at $\hat{x}$, then $\partial^c f(\hat{x}) = \{\nabla f(\hat{x})\}$, where $\nabla f(\hat{x})$ denotes the gradient of f at $\hat{x}$). If the function $f(\cdot)$ is convex, then $\partial^c f(\hat{x}) = \partial f(\hat{x})$, where $\partial f(\hat{x})$ denotes the sub-differential of f at $\hat{x}$ in the convex analysis sense:

$$\partial f(\hat{x}) := \{\xi \in \mathbb{R}^n \mid\ f(y) - f(\hat{x}) \geq \langle \xi, y - \hat{x}\rangle, \quad \forall y \in \mathbb{R}^n\}.$$

A locally Lipshitz function $f : \mathbb{R}^n \to \mathbb{R}$ is called Clarke-convex (denoted C-convex) at $\hat{x} \in \mathbb{R}^n$ if for each $\xi \in \partial^c f(\hat{x})$ the following inequality holds:

$$f(y) - f(\hat{x}) \geq \langle \xi, y - \hat{x}\rangle, \quad \forall y \in \mathbb{R}^n.$$

The properties and applications of C-convex functions in non-smooth optimization can be found in [17]. Hereafter, we will use the following important relations for two locally Lipschitz functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$:

$$\begin{aligned} &f_i^\circ(\hat{x};d) = \max\{\langle \zeta, d\rangle \mid \zeta \in \partial^c f_i(\hat{x})\}, \quad i \in \{1,2\},\\ &\partial^c\big(\max\{f_1,f_2\}\big)(\hat{x}) \subseteq conv\big(\partial^c f_1(\hat{x}) \cup \partial^c f_2(\hat{x})\big), \\ &\partial^c\big(\alpha_1 f_1 + \alpha_2 f_2\big)(\hat{x}) \subseteq \alpha_1 \partial^c f_1(\hat{x}) + \alpha_2 \partial^c f_2(\hat{x}), \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}. \end{aligned} \tag{1}$$

Furthermore, if $f_1,\ f_2$ are regular at $\hat{x}$ and $\alpha_1,\ \alpha_2$ are non-negative, then $\alpha_1 f_1 + \alpha_2 f_2$ is also regular at $\hat{x}$, and equality holds in the above inequalities. For a locally Lipschitz function $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and a point $(\hat{x},\hat{y}) \in \mathbb{R}^n \times \mathbb{R}^m$, let $\partial^c_x \psi(\hat{x},\hat{y})$ and $\partial^c_y \psi(\hat{x},\hat{y})$ denote the partial Clarke sub-differential of $\psi(\cdot,\cdot)$ at $(\hat{x},\hat{y})$, which are defined as $\partial^c \psi(\cdot,\hat{y})(\hat{x})$ and $\partial^c \psi(\hat{x},\cdot)(\hat{y})$, respectively.

Theorem 3. [6] If $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is regular at $(\hat{x},\hat{y})$, then

$$\partial^c \psi(\hat{x},\hat{y}) \subseteq \partial^c_x \psi(\hat{x},\hat{y}) \times \partial^c_y \psi(\hat{x},\hat{y}).$$

Let $Y \subset \mathbb{R}^n$ and $\hat{x} \in \overline{Y}$ be given. The Clarke tangent cone of Y at $\hat{x}$ is defined as:

$$\Gamma_Y(\hat{x}) := \{u \in \mathbb{R}^n \mid d^\circ_Y(\hat{x};u) = 0\},$$

where $d_Y(x) = \inf_{y \in Y} \|x - y\|$ is distance function related to Y. It is worth noting that $\Gamma_Y(\hat{x})$ is always a closed convex cone. The Clarke normal cone of Y at $\hat{x}$ is defined by

$$N_Y(\hat{x}) := \Big(\Gamma_Y(\hat{x})\Big)^{\odot}.$$

Theorem 4. [6] Let $Y \subset \mathbb{R}^n$ and φ be a locally Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}$. Suppose that φ attains its minimum on Y at $\hat{x}$, then

$$0_n \in \partial^c \varphi(\hat{x}) + N_Y(\hat{x}).$$

We require the following definition, which is extensively used in this paper.

Definition 1. For a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, and a point $\hat{x} \in \mathbb{R}^n$ with $F(\hat{x}) \neq \emptyset$:

- F is said to be inner semi-continuous at $(\hat{x},\hat{y})$ in

  $$gph(F) := \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^m \mid v \in F(u)\},$$

  if for every sequence $x_k \to \hat{x}$ as $k \to \infty$ with $F(x_k) \neq \emptyset$ for $k \in \mathbb{N}$, there exists a sequence $y_k \in F(x_k)$ as $k \in \mathbb{N}$ converging to $\hat{y}$.
- F said to satisfy the Lipschitz-like property (or Aubin property) around the point $(\hat{x},\hat{y}) \in gph(F)$ if there exist neighborhoods U of $\hat{x}$ and V of $\hat{y}$, and a constant $\kappa \geq 0$ such that

  $$F(x) \cap V \subset F(u) + \kappa \|x - u\| \mathbb{B}, \quad \text{for all } x, u \in U,$$

  where $\mathbb{B}$ denotes the closed unit ball of $\mathbb{R}^m$.
- F is said to be upper semi-continuous (u.s.c) at $\hat{x}$ if for every sequence $x_k \to \hat{x}$ as $k \to \infty$ and for every open set $V \subseteq \mathbb{R}^m$ with $F(\hat{x}) \subseteq V$, there exists a positive number κ such that $F(x_k) \subseteq V$ for all $k \geq \kappa$.

3 Necessary Optimality Conditions

In this section, we recall our standing assumptions made throughout the paper:

- The appearing functions φ_j, ψ, and ν_i for $(i,j) \in I \times J$ are locally Lipschitz and regular on $\mathbb{R}^n$.

- The index set I is finite.
- The set-valued mapping $x \mapsto \Sigma(x)$ is uniformly bounded.

We now provide the following definition, which is taken from [9].

Definition 2. A feasible point $\hat{x} \in \mathcal{F}$ is called a properly efficient solution to MGSIP if there exist positive scalars $\gamma_j > 0$ for $j \in J$ such that

$$\sum_{j=1}^{p} \gamma_j \varphi_j(\hat{x}) \leq \sum_{j=1}^{p} \gamma_j \varphi_j x), \quad \forall x \in \mathcal{F}.$$

For $\hat{x} \in \mathcal{F}$, we define the index set of active constraints and the lower-level problem at $\hat{x}$ as follows:

$$\begin{array}{c} \Sigma_0(\hat{x}) := \{y \in \Sigma(\hat{x}) \mid \psi(\hat{x}, y) = 0\}, \\ \min \quad \psi(\hat{x}, y), \quad \text{subject to} \quad y \in \Sigma(\hat{x}). \end{array} \tag{2}$$

Furthermore, the set (which may be empty) of active inequalities of Problem (2) at each $\hat{y} \in \Sigma(\hat{x})$ is denoted by $I_0(\hat{x}, \hat{y})$:

$$I_0(\hat{x}, \hat{y}) := \{i \in I \mid \nu_i(\hat{x}, \hat{y}) = 0\}.$$

Associated with the lower-level problem (2) is its optimal value function

$$\mu(x) := \begin{cases} \inf\{\psi(x, y) \mid y \in \Sigma(x)\}, & \text{if } \Sigma(x) \neq \emptyset, \\ \\ +\infty, & \text{if } \Sigma(x) = \emptyset. \end{cases} \tag{3}$$

The Lagrangian of the lower-level problem is defined as:

$$\mathfrak{L}(x, y, \alpha) := \psi(x, y) + \sum_{i=1}^{q} \alpha_i \nu_i(x, y),$$

where $\alpha := (\alpha_1, \dots, \alpha_q) \in \mathbb{R}^q$ and $\alpha_i \geq 0$ for all $i \in I$. It is well known from [6] that for $\hat{y} \in \Sigma_0(\hat{x})$, the corresponding set of Karush-Kuhn-Tucker (KKT in brief) multipliers, denoted by $K(\hat{x}, \hat{y})$:

$$K(\hat{x}, \hat{y}) := \Big\{\alpha \in \mathbb{R}^q \mid 0_m \in \partial_y^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha), \;\; \alpha_i \nu_i(\hat{x}, \hat{y}) = 0, \;\; \forall\, i \in I\Big\},$$

is nonempty if the following constraint qualification (CQ1) holds:

$$\left.\begin{array}{l} (\rho, 0_m) \in \displaystyle\sum_{i \in I_0(\hat{x}, \hat{y})} \alpha_i \partial^c \nu_i(\hat{x}, \hat{y}) \\ \\ \alpha_i \geq 0, \quad \forall i \in I_0(\hat{x}, \hat{y}) \end{array}\right\} \implies \alpha_i = 0, \;\; \forall i \in I_0(\hat{x}, \hat{y}).$$

Remark 1. If all the lower-level constraint functions ν_i for $i \in I$ are continuously differentiable, we can decompose the full derivative into the partial derivatives as follows:

$$\nabla \nu_i(\hat{x}, \hat{y}) = \nabla_x \nu_i(\hat{x}, \hat{y}) \times \nabla_y \nu_i(\hat{x}, \hat{y}), \quad \forall i \in I_0(\hat{x}, \hat{y}),$$

In this case, the constraint CQ1 is equivalent to the following implication:

$$\sum_{i \in I_0(\hat{x}, \hat{y})} \alpha_i \nabla_y \nu_i(\hat{x}, \hat{y}) = 0_n, \; \alpha_i \geq 0 \quad \implies \quad \alpha_i = 0, \;\; \forall\, i \in I_0(\hat{x}, \hat{y}).$$

By applying the Gordan alternative theorem (see [4]), we conclude that the above implication is equivalent to Cottle constraint qualification for the lower-level problem (2) at $\hat{y}$, which can be expressed as:

$$\Big\{z \in \mathbb{R}^m \mid \langle z, \nabla_y \nu_i(\hat{x}, \hat{y})\rangle < 0, \;\; i \in I_0(\hat{x}, \hat{y})\Big\} \neq \emptyset.$$

Remark 2. Due to the continuity of functions ν_i for $i \in I$, the upper semi-continuity of the mapping Σ, and [1, Proposition 1.7], we can conclude that the set-valued mapping $\Sigma(\cdot)$ is compact-valued and the optimal valued function $\mu(\cdot)$ is lower semi-continuous at $\hat{x}$. As a result, the infimum in (3) is attained.

Theorem 5. In addition to the standing assumptions, let us suppose that the set-valued mapping $\Sigma_0(\cdot)$ is inner semi-continuous at $(\hat{x}, \hat{y})$ for some $\hat{x} \in \digamma$ and $\hat{y} \in \Sigma_0(\hat{x})$. If CQ1 holds at $(\hat{x}, \hat{y})$, then we get:

$$\partial^c \mu(\hat{x}) \subseteq \bigcup_{\alpha \in K(\hat{x}, \hat{y})} \partial_x^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha).$$

Proof. Firstly, according to [20, Corollary 4.43] and under the assumption of satisfying CQ1, we can deduce that the set-valued mapping $\Sigma(\cdot)$ is Lipschitz-like around $(\hat{x}, \hat{y})$. Therefore, [21, Theorem 5.2)] implies that the marginal function $\mu(\cdot)$ is locally Lipschitz (not necessarily regular) around $\hat{x}$, and hence, the left side of the above inclusion is meaningful.

Secondly, by utilizing the inner semi-continuity of $\Sigma_0(\cdot)$ and the satisfaction of CQ1 at $(\hat{x}, \hat{y})$, we can conclude from [22, Theorem 8] the following upper estimate of the Clarke sub-differential of the value function $\mu(\cdot)$ at $\hat{x}$:

$$\begin{aligned}\partial^c \mu(\hat{x}) \quad \subseteq \quad \bigcup_{(\alpha_1, \ldots, \alpha_q)} \Big\{ u \in \mathbb{R}^n \mid (u, 0_m) \in \partial^c \psi(\hat{x}, \hat{y}) + \sum_{i=1}^{q} \alpha_i \partial^c \nu_i(\hat{x}, \hat{y}), \\ \alpha_i \nu_i(\hat{x}, \hat{y}) = 0, \quad \alpha_i \geq 0 \quad \forall\, i \in I \Big\}.\end{aligned}$$

Since the functions ψ and ν_is for $i \in I$ are assumed to be regular, the above inclusion implies that

$$\partial^c \mu(\hat{x}) \subseteq \bigcup_{\alpha := (\alpha_1, \ldots, \alpha_q)} \Big\{ u \in \mathbb{R}^n \mid (u, 0_m) \in \partial^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha),\ \alpha_i \nu_i(\hat{x}, \hat{y}) = 0,\ \forall\, i \in I \Big\}. \tag{4}$$

Finally, we can establish the following inclusion based on the regularity of $\mathfrak{L}$ and Theorem 3:

$$\partial^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha) \subseteq \partial_x^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha) \times \partial_y^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha).$$

Thus, from $(u, 0_m) \in \partial^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha)$ and $\alpha_i \nu_i(\hat{x}, \hat{y}) = 0$ $(i \in I)$, we can conclude that

$$\left\{ \begin{array}{l} u \in \partial_x^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha) \\ \left. \begin{array}{l} 0_m \in \partial_y^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha) \\ \alpha_i \nu_i(\hat{x}, \hat{y}) = 0,\ \forall i \in I \end{array} \right\} \Longrightarrow \alpha \in K(\hat{x}, \hat{y}) \end{array} \right\} \Longrightarrow u \in \bigcup_{\alpha \in K(\hat{x}, \hat{y})} \partial_x^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha).$$

Therefore, the above relation and (4) imply that

$$\partial^c \mu(\hat{x}) \subseteq \bigcup_{\alpha} \Big\{ \partial_x^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha) \mid \alpha \in K(\hat{x}, \hat{y}) \Big\} = \bigcup_{\alpha \in K(\hat{x}, \hat{y})} \partial_x^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha),$$

as required. □ □

The following example illustrates that the inner semi-continuity assumption of the set-valued mapping $\Sigma_0(\cdot)$ in Theorem 5 can be held at certain points, such as $(\hat{x}, \hat{y})$, while it can be invalid at other points, such as $(\tilde{x}, \tilde{y})$.

Example 1. Let us consider the following functions:

$$\psi(x, y) = \min \{2 - y_1, 2 + y_1, 2 - y_2, 2 + y_2\},$$

$$\nu_1(x,y) = (y_1 - x_1)^2 + (y_2 - x_2)^2 - 1.$$

This results in the set $\mathcal{F} = [-1,1] \times [-1,1]$. Choosing $\hat{x} = (1,1)$ and $\tilde{x} = (1, \frac{1}{2})$, we have:

$$\Sigma_0(\tilde{x}) = \{y \in \mathbb{R}^2 \mid \psi(\tilde{x}, y) = 0,\ (y_1 - 1)^2 + (y_2 - \frac{1}{2})^2 \leq 1\} = \{(2, \frac{1}{2})\},$$
$$\Sigma_0(\hat{x}) = \{y \in \mathbb{R}^2 \mid \psi(\hat{x}, y) = 0,\ (y_1 - 1)^2 + (y_2 - 1)^2 \leq 1\} = \{(2,1),(1,2)\}.$$

Considering $\tilde{x}$: Since $\Sigma_0(\tilde{x})$ is singleton, we can conclude that Σ_0 is inner semi-continuous at $(\tilde{x}, \tilde{y})$ with $\tilde{y} = (2, \frac{1}{2})$.

Considering $\hat{x}$: Put $\hat{y}^{(1)} = (1,2)$ and $\hat{y}^{(2)} = (2,1)$. Since there exists a sequence

$$\{(1, 1 - \frac{1}{k})\}_{k=1}^{\infty},$$

converging to $\hat{x}$ as $k \to \infty$, and

$$\Sigma_0\big((1, 1 - \frac{1}{k})\big) = \Big\{(2, 1 - \frac{1}{k})\Big\} \to (2,1) \neq \hat{y}^{(1)},$$

we can conclude that the mapping $\Sigma_0(\cdot)$ is not inner semi-continuous at $(\hat{x}, \hat{y}^{(1)})$. Similarly, the sequence $\{(1 - \frac{1}{k}, 1)\}_{k=1}^{\infty}$ demonstrates that $\Sigma_0(\cdot)$ is not inner semi-continuous at $(\hat{x}, \hat{y}^{(2)})$, as

$$\Sigma_0\big((1 - \frac{1}{k}, 1)\big) = \Big\{(1 - \frac{1}{k}, 2)\Big\} \to (1,2) \neq \hat{y}^{(2)}.$$

The first-order optimality condition for regular MGSIP is stated as follows:

Theorem 6. In addition to under the standing assumptions, let $\hat{x}$ be a properly efficient solution of regular (MGSIP) and CQ1 holds at $(\hat{x}, \hat{y})$ for some $\hat{y} \in \Sigma_0(\hat{x})$. If the set-valued mapping $\Sigma_0(\cdot)$ is inner semi-continuous at $(\hat{x}, \hat{y})$, then there exist non-negative scalars $\lambda_j \geq 0$ as $j \in J$, and $\alpha^{(\ell)} \in K(\hat{x}, \hat{y})$ and $\beta_\ell \geq 0$ for $\ell = 1, \ldots, n+1$, such that the following system holds:

$$\begin{cases} 0_n \in \displaystyle\sum_{j=1}^{p} \lambda_j \partial^c \varphi_j(\hat{x}) - \sum_{\ell=1}^{n+1} \beta_\ell \partial_x^c\, \mathfrak{L}(\hat{x}, \hat{y}, \alpha^{(\ell)}), \\ \displaystyle\sum_{j=1}^{p} \lambda_j + \sum_{\ell=1}^{n+1} \beta_\ell = 1. \end{cases} \tag{5}$$

Proof. Since $\hat{x}$ is a properly efficient solution of MGSIP, we can find some $\gamma_j > 0$ for $j \in J$ such that $\hat{x}$ is a minimizer of $\sum_{j=1}^{p} \gamma_j \varphi_j(x)$ on $\mathcal{F}$.

Case 1: If $\mu(\hat{x}) = 0$, then we consider the following function:

$$\Psi(x) := \max\Big\{\sum_{j=1}^{p} \gamma_j \varphi_j(x) - \sum_{j=1}^{p} \gamma_j \varphi_j(\hat{x}),\ -\mu(x)\Big\}.$$

If $x^* \notin \mathcal{F}$, there exist some $y^* \in \Sigma(x^*)$ with $\nu_i(x^*, y^*) < 0$, and hence $-\mu(x^*) > 0$. Therefore, $\Psi(x^*) > 0$.

If $x^* \in \mathcal{F}$, there exists $\displaystyle\sum_{j=1}^{p} \gamma_j \varphi_j(x^*) - \sum_{j=1}^{p} \gamma_j \varphi_j(\hat{x}) \geq 0$ due to the proper efficiency of $\hat{x}$, and so $\Psi(x^*) \geq 0$. Hence, we have

$$\Psi(x) \geq 0, \quad \forall x \in \mathbb{R}^n, \quad \text{and} \quad \Psi(\hat{x}) = 0.$$

This means that $\hat{x}$ is a global minimizer of $\Psi(\cdot)$, and therefore, $0_n \in \partial^c \Psi(\hat{x})$ by Theorem 4. By using (7) and Theorem 5, we obtain:

$$\partial^c \Psi(\hat{x}) \subseteq conv\Big(- \bigcup_{\alpha \in K(\hat{x},\hat{y})} \partial_x^c \mathfrak{L}(\hat{x},\hat{y},\alpha) \cup \partial^c\Big(\sum_{j=1}^{p} \gamma_j \varphi_j\Big)(\hat{x})\Big),$$

which implies

$$0_n \in conv\Big(- \bigcup_{\alpha \in K(\hat{x},\hat{y})} \partial_x^c \mathfrak{L}(\hat{x},\hat{y},\alpha) \cup \partial^c\Big(\sum_{j=1}^{p} \gamma_j \varphi_j\Big)(\hat{x})\Big).$$

Based on the above inclusion, Theorem 2, and the convexity of the Clarke subdifferential, there exist some non-negative scalars $\tau_0, \dots, \tau_{n+1}$ and some vectors $\alpha^{(1)}, \dots, \alpha^{(n+1)}$ in $K(\hat{x},\hat{y})$ such that

$$\begin{cases} 0_n \in \tau_0 \partial^c\Big(\sum_{j=1}^{p} \gamma_j \varphi_j\Big)(\hat{x}) - \sum_{\ell=1}^{n+1} \tau_\ell \mathfrak{L}(\hat{x},\hat{y},\alpha^{(\ell)}), \\ \tau_0 + \sum_{\ell=1}^{n+1} \tau_\ell = 1, \end{cases}$$

i.e.,

$$\begin{cases} 0_n \in \sum_{j=1}^{p} \tau_0 \gamma_j \partial^c \varphi_j(\hat{x}) - \sum_{\ell=1}^{n+1} \tau_\ell \mathfrak{L}(\hat{x},\hat{y},\alpha^{(\ell)}), \\ \tau_0 + \sum_{\ell=1}^{n+1} \tau_\ell = 1. \end{cases} \tag{6}$$

Our assertion is that

$$\mathcal{A} := \sum_{j=1}^{p} \tau_0 \gamma_j + \sum_{\ell=1}^{n+1} \tau_\ell \neq 0.$$

Indeed, if $\tau_0 = 0$, then $\mathcal{A} = \sum_{\ell=1}^{n+1} \tau^{(\ell)} = 1$ according to (6). If $\tau_0 \neq 0$, then $\sum_{j=1}^{p} \tau_0 \gamma_j > 0$ due to the positivity of γ_j as $j \in J$, resulting in $\mathcal{A} \neq 0$. By defining

$$\lambda_j := \frac{\tau_0 \gamma_j}{\mathcal{A}}, \quad j \in J, \quad \text{and} \quad \beta_\ell := \frac{\tau_\ell}{\mathcal{A}}, \quad \ell = 1, \dots, n+1,$$

we can deduce from (6) that

$$\begin{cases} 0_n \in \sum_{j=1}^{p} \lambda_j \partial^c \varphi_j(\hat{x}) - \sum_{\ell=1}^{n+1} \beta_\ell \partial_x^c\, \mathfrak{L}(\hat{x},\hat{y},\alpha^{(\ell)}), \\ \sum_{j=1}^{p} \lambda_j + \sum_{\ell=1}^{n+1} \beta_\ell = 1. \end{cases}$$

Case 2: If $\mu(\hat{x}) > 0$, then $\hat{x}$ is an interior point of $\mathcal{F}$ because of the lower semi-continuity of $\mu(\cdot)$. Therefore, based on Theorem 4, we can conclude that

$$0_n \in \partial^c\Big(\sum_{j=1}^{p} \gamma_j \varphi_j\Big)(\hat{x}) = \sum_{j=1}^{p} \gamma_j \partial^c \varphi_j(\hat{x}),$$

which can be expressed in the form (5) by setting

$$\lambda_j := \frac{\gamma_j}{\sum_{j=1}^{p} \gamma_j}, \quad j \in J, \quad \text{and} \quad \beta_\ell = 0, \quad \ell = 1, \ldots, n+1.$$

The proof is complete. □

The following example illustrates that the assumption of inner semi-continuity of the set-valued function $\Sigma_0(\cdot)$ cannot be neglected in Theorem 6.

Example 2. Consider the problem MGSIP with the following data:

$$\begin{aligned} \varphi(x) &= -x_1 - x_2 \\ \psi(x,y) &= \min \{2 - y_1, 2 + y_1, 2 - y_2, 2 + y_2\}, \\ \nu_1(x,y) &= (y_1 - x_1)^2 + (y_2 - x_2)^2 - 1. \end{aligned}$$

Since $\mathcal{F} = [-1,1] \times [-1,1]$, the unique optimal solution of the problem is $\hat{x} = (1,1)$. As demonstrated in Example 1,

$$\Sigma_0(\hat{x}) = \{\hat{y}^{(1)}, \hat{y}^{(2)}\}, \quad \hat{y}^{(1)} = (1,2), \quad \hat{y}^{(2)} = (2,1), \quad I_0(\hat{x}, \hat{y}^{(1)}) = I_0(\hat{x}, \hat{y}^{(2)}) = \{1\}.$$

The set-valued mapping $\Sigma(\cdot)$ is evidently uniformly bounded, and CQ1 is satisfied at $(\hat{x}, \hat{y}^{(1)})$ and $(\hat{x}, \hat{y}^{(2)})$. A brief calculation shows that

$$K(\hat{x}, \hat{y}^{(1)}) = \left\{ \frac{1}{2} \right\}, \quad \text{and} \quad \partial_x^c \mathfrak{L}(\hat{x}, \hat{y}^{(1)}, \frac{1}{2}) = \{(0,-1)\}.$$

Thus, it is evident that there are no scalars λ^1 and β^1 as stated in Theorem 6 satisfying (5). Similarly, it shows that (5) is not valid at $\hat{y}^{(2)}$.

Note that, as demonstrated in Example 1, the set-valued mapping $\Sigma_0(\cdot)$ is not inner semi-continuous at $(\hat{x}, \hat{y}^{(1)})$ and $(\hat{x}, \hat{y}^{(2)})$.

It is worth mentioning that Theorem 6 has a very restrictive assumption, which is the inner semi-continuity of the set-valued mapping $\Sigma_0(\cdot)$ at the point $(\hat{x}, \hat{y})$. Several conditions can ensure this inner semi-continuity, some of which are as follows:

1. $\Sigma_0(\hat{x}) = \{\hat{y}\}$ is a singleton, while the map Σ_0 may be multi-valued at any point other than $\hat{x}$ (see [7]).
2. The lower-level constraint functions $\nu_i(x,y)$, where $i \in I$, are weakly analytic according to Klatte and Kummer (see [2]).
3. The lower-level objective function $\psi(x,y)$ is strictly convex with respect to y for every $x \in \mathcal{F}$ (see [8, 23]).

The proof of Theorem 6 relies on the upper estimate of $\partial^c \mu(\hat{x})$, as presented in Theorem 5, and this estimate is based on the inner semi-continuity of $\Sigma_0(\cdot)$ at $(\hat{x}, \hat{y})$. It is worth mentioning that there are special cases where can calculate $\partial^c \mu(\hat{x})$ without requiring the inner semi-continuity of $\Sigma_0(\cdot)$ at $(\hat{x}, \hat{y})$.

1. If all the functions $\psi(x,y)$ and $\nu_i(x,y)$, where $i \in I$, are continuously differentiable, and the Cottle constraint qualification is satisfied at $(\hat{x}, \hat{y})$, then according to [10], we have:

$$\partial^c \mu(\hat{x}) \subseteq conv\Big(\Big\{ \nabla_x \mathfrak{L}(\hat{x}, y, \alpha) \mid y \in \Sigma_0(\hat{x}), \ \alpha \in K(\hat{x}, y) \Big\}\Big).$$

2. If all the functions $\psi(x,y)$ and $\nu_i(x,y)$, where $i \in I$, are convex, and the Farkas-Minkowski property holds at $(\hat{x},\hat{y})$ in sense of [16], then $\mu(\cdot)$ is a convex function, and we have (see [16]):

$$\partial\mu(\hat{x}) \subseteq \bigcup \Big\{\partial_x \mathfrak{L}(\hat{x},y,\alpha) \mid \alpha \in K(\hat{x},y)\Big\}.$$

3. All the functions $\psi(x,y)$ and $\nu_i(x,y)$, where $i \in I$, are D.C. (difference of convex functions), and the closed qualification condition defined in [15] holds at $(\hat{x},\hat{y})$, then the Mordukhovich subdifferential of $\mu(\cdot)$ at $\hat{x}$, denoted by $\partial^M\mu(\hat{x})$, is estimated as follows (see [15]):

$$\partial^M\mu(\hat{x}) \subseteq \bigcup_{\alpha \in K(\hat{x},y)} \partial_x^M \mathfrak{L}(\hat{x},y,\alpha).$$

In the following, we will introduce another class of problems where $\partial^c\mu(\hat{x})$ can be estimated without the inner semi-continuity of $\Sigma_0(\cdot)$. We will introduce another constraint qualification.

Definition 3. We say that MGSIP satisfies CQ2 at $(\hat{x},\hat{y}) \in \Lambda$ if the cone $\Big(\bigcup_{i\in I_0(\hat{x},\hat{y})} \partial^c\nu_i(\hat{x},\hat{y})\Big)$ is closed and

$$\Big(\bigcup_{i\in I_0(\hat{x},\hat{y})} \partial^c\nu_i(\hat{x},\hat{y})\Big)^{\odot} \subseteq \Gamma_\Lambda(\hat{x},\hat{y}), \tag{7}$$

where,

$$\Lambda := \big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in \digamma \text{ and } y \in \Sigma(x)\big\} = \bigcup_{x\in\digamma} \Big(\{x\} \times \Sigma(x)\Big).$$

Observe that CQ2 was initially introduced in [13] in the context of convex optimization problems. It was later extended to the framework of convex semi-infinite programming problems (SIP) in [19] and extensively studied for non-convex SIPs in [17].

Remark 3. The inclusion (7) is referred to as the Abadie constraint qualification. CQ2, also known as the basic constraint qualification, is equivalent to the following inclusion (refer to [17] for proof and more details).

$$N_\Lambda(\hat{x},\hat{y}) \subseteq cone\Big(\bigcup_{i\in I_0(\hat{x},\hat{y})} \partial^c\nu_i(\hat{x},\hat{y})\Big).$$

Theorem 7. Under the standing assumptions, let us assume that the regular MGSIP satisfies CQ2 at $(\hat{x},\hat{y})$ for some $\hat{y} \in \Sigma_0(\hat{x})$. If $\mu(\cdot)$ is a C-convex function, then

$$\partial^c\mu(\hat{x}) \subseteq \bigcup_{\alpha\in K(\hat{x},\hat{y})} \partial_x^c \mathfrak{L}(\hat{x},\hat{y},\alpha).$$

Proof. Let $\xi \in \partial^c\mu(\hat{x})$ be arbitrarily chosen. Due to the C-convexity assumption of $\mu(\cdot)$, the definition of $\mu(\cdot)$, and $\hat{y} \in \Sigma_0(\hat{x})$, we have

$$\begin{cases} \mu(x) \le \psi(x,y), & \forall x \in \digamma,\ \forall y \in \Sigma(x), \\ \mu(\hat{x}) = 0 = \psi(\hat{x},\hat{y}), & \\ \mu(x) - \mu(\hat{x}) \ge \langle \xi, x-\hat{x}\rangle, & \forall x \in \mathbb{R}^n, \end{cases}$$

which implies

$$\psi(x,y) - \psi(\hat{x},\hat{y}) \ge \langle \xi, x\rangle - \langle \xi, \hat{x}\rangle, \quad \forall x \in \digamma,\ \forall y \in \Sigma(x) \implies$$

$$\psi(x,y) - \langle \xi, x\rangle \geq \psi(\hat{x},\hat{y}) - \langle \xi, \hat{x}\rangle, \quad \forall x \in F, \ \forall y \in \Sigma(x).$$

This means that $(\hat{x},\hat{y})$ is a minimizer of $\psi(x,y) - \langle \xi, x\rangle$ on Λ, and thus

$$0_{n+m} \in \partial^c\big(\psi(x,y) - \langle \xi, x\rangle\big)(\hat{x},\hat{y}) + N_\Lambda(\hat{x},\hat{y}).$$

From this, the CQ2 assumption, and Remark 3, we obtain

$$(\xi, 0_m) \in \partial\psi(\hat{x},\hat{y}) + cone\Big(\bigcup_{i\in I_0(\hat{x},\hat{y})} \partial^c\nu_i(\hat{x},\hat{y})\Big).$$

Thus, we can find non-negative numbers $\gamma_i \geq 0$ (for $i \in I_0(\hat{x},\hat{y})$), such that

$$(\xi, 0_m) \in \partial^c\psi(\hat{x},\hat{y}) + \sum_{i\in I_0(\hat{x},\hat{y})} \gamma_i \partial^c\nu_i(\hat{x},\hat{y}).$$

By defining $\alpha_i := \gamma_i$ for $i \in I_0(\hat{x},\hat{y})$ and $\alpha_i := 0$ for $i \in I \setminus I_0(\hat{x},\hat{y})$, and introducing $\alpha := (\alpha_1,\dots,\alpha_q) \in \mathbb{R}^q$, we can conclude from regularity of functions ψ and ν_i for $i \in I$, and the inclusion mentioned above that

$$\begin{cases} (\xi, 0_m) \in \partial^c\mathfrak{L}(\hat{x},\hat{y},\alpha) \subseteq \partial^c_x\mathfrak{L}(\hat{x},\hat{y},\alpha) \times \partial^c_y\mathfrak{L}(\hat{x},\hat{y},\alpha) \\ \alpha_i\nu_i(\hat{x},\hat{y}) = 0, \quad \forall i \in I, \end{cases}$$

This implies that

$$xi \in \bigcup_{\alpha\in K(\hat{x},\hat{y})} \partial^c_x\mathfrak{L}(\hat{x},\hat{y},\alpha).$$

Since ξ was chosen as arbitrary from $\partial^c\mu(\hat{x})$, we have

$$\partial^c\mu(\hat{x}) \subseteq \bigcup_{\alpha\in K(\hat{x},\hat{y})} \partial^c_x\mathfrak{L}(\hat{x},\hat{y},\alpha),$$

and the proof is complete. □

Note that providing sufficient conditions for C-convexity of $\mu(\cdot)$ is an important and separate research topic that may be of interest to researchers.

Theorem 8. Under the standing assumptions, let $\hat{x}$ be a properly efficient solution of regular MGSIP, and assume that CQ2 holds at $(\hat{x},\hat{y})$ for some $\hat{y} \in \Sigma_0(\hat{x})$. If $\mu(\cdot)$ is a C-convex function, then there exist $\lambda_j \geq 0$ for $j \in J$, as well as $\alpha^{(\ell)} \in K(\hat{x},\hat{y})$ and $\beta_\ell \geq 0$ for $\ell = 1,\dots,n+1$, such that

$$\begin{cases} 0_n \in \sum_{j=1}^{p} \lambda_j \partial^c\varphi_j(\hat{x}) - \sum_{\ell=1}^{n+1} \beta_\ell \partial^c_x\, \mathfrak{L}(\hat{x},\hat{y},\alpha^{(\ell)}), \\ \sum_{j=1}^{p} \lambda_j + \sum_{\ell=1}^{n+1} \beta_\ell = 1. \end{cases}$$

Proof. Employing Theorem 7 and the following proof of Theorem 6, we obtain the desired result. □

To illustrate the significance of Theorem 8, we provide an example.

Example 3. Consider the problem (MGSIP) with the following data:

$$\begin{aligned}
&\varphi_1(x) = |x|, && \varphi_2(x) = x^2, \\
&\psi(x, y) = x - 2y, \\
&\nu_1(x, y) = |x| + y + |y|, && \nu_2(x, y) = x, \qquad \nu_3(x, y) = x^2 + 3y,
\end{aligned}$$

Since the above functions are linear or convex, they are regular and C-convex. Furthermore, $\hat{x} = 0$ is an optimal solution to the problem, and we have:

$$\begin{aligned}
&\Lambda = \{0\} \times (-\infty, 0], \\
&\Sigma_0(\hat{x}) = \{y \in \mathbb{R}_- \mid 0 - 2y = 0\} = \{0\} \implies \hat{y} = 0, \\
&I_0(0_2) = \{1, 2\}, \\
&\partial^c \nu_1(0_2) = [-1, 1] \times [0, 2], \qquad \partial^c \nu_2(0_2) = \big\{(1, 0)\big\}.
\end{aligned}$$

Consequently,

$$N_\Lambda(0_2) = \mathbb{R} \times [0, +\infty) = cone\Big(\partial^c \nu_0(0_2) \cup \partial^c \nu_2(0_2)\Big),$$

and thus, the problem satisfies the CQ2 at $(\hat{x}, \hat{y})$. Since

$$\mathfrak{L}(x, y, \alpha) = x - 2y + \sum_{i=1}^{3} \alpha_i \nu_i(x, y),$$

we have

$$\begin{cases}
\partial_x^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha) = \{1\} + \alpha_1[-1, 1] + \alpha_2\{1\} + \alpha_3\{0\} \\[2ex]
\partial_y^c \mathfrak{L}(x, y, \alpha) = \{-2\} + \alpha_1[0, 2] + \alpha_2\{0\} + \alpha_3\{3\}.
\end{cases}$$

Thus, $\alpha^{(1)} := (1, 0, 0) \in K(\hat{x}, \hat{y})$, and with $\lambda_1 := \frac{1}{2}$, $\lambda_2 := 0$, $b_1 := \frac{1}{2}$, we have

$$\begin{cases}
0 \in \lambda_1[-1, 1] + \lambda_2\{0\} - \beta_1\big(\{1\} + [-1, 1]\big) = \displaystyle\sum_{j=1}^{2} \lambda_j \partial^c \varphi_j(\hat{x}) - \beta_1 \partial_x^c \mathfrak{L}(\hat{x}, \hat{y}, \alpha^{(1)}), \\[3ex]
\lambda_1 + \lambda_2 + \beta_1 = 1.
\end{cases}$$

4 Application to Convex MGSIP

The Slater constraint qualification plays a significant role in the analysis of convex optimization problems (refer to [13]). In the context of convex GSIPs, researchers have considered two types of Slater constraint qualifications (see e.g., [26, 28].

Definition 4. For a convex MGSIP, we define the following:

(a) The problem satisfies the first Slater constraint qualification (FSCQ) at $(\hat{x}, \hat{y}) \in \Lambda$ if there exists $y^* \in \Sigma(\hat{x})$ such that
$$\nu_i(\hat{x}, y^*) < 0, \quad \text{for all} \quad i \in I_0(\hat{x}, \hat{y}).$$

(b) The problem satisfies the second Slater constraint qualification (SSCQ) at $(\hat{x}, \hat{y}) \in \Lambda$ if there exists $(x^*, y^*) \in \Lambda$ such that
$$\nu_i(x^*, y^*) < 0, \quad \text{for all} \quad i \in I_0(\hat{x}, \hat{y}).$$

The following two theorems establish the relationships between these Slater constraint qualifications and CQ1 and CQ2.

Theorem 9. For convex (MGSIP), CQ1 holds at $(\hat{x}, \hat{y}) \in \Lambda$ if FSCQ is satisfied at there.

Proof. Since FSCQ holds at $(\hat{x}, \hat{y})$, there exists a vector $y^* \in \Sigma(\hat{x})$ such that $\nu_i(\hat{x}, y^*) < 0$ for all $i \in I_0(\hat{x}, \hat{y})$. Thus,

$$\langle \xi_i, y^* - \hat{y} \rangle \le \nu_i(\hat{x}, y^*) - \underbrace{\nu_i(\hat{x}, \hat{y})}_{=0} < 0, \quad \forall i \in I_0(\hat{x}, \hat{y}),\ \xi_i \in \partial_y \nu_i(\hat{x}, \hat{y}). \tag{8}$$

Now, suppose that

$$(\rho, 0_m) \in \sum_{i \in I_0(\hat{x}, \hat{y})} \beta_i \partial \nu_i(\hat{x}, \hat{y}), \quad \text{and} \quad \beta_i \ge 0,\ \ \forall i \in I_0(\hat{x}, \hat{y}).$$

This inclusion, along with Theorem 3 concludes that

$$(\rho, 0_m) \in \partial\Big(\sum_{i \in I_0(\hat{x}, \hat{y})} \beta_i \partial \nu_i\Big)(\hat{x}, \hat{y}) \subseteq \partial_x\Big(\sum_{i \in I_0(\hat{x}, \hat{y})} \beta_i \partial \nu_i\Big)(\hat{x}, \hat{y}) \times \partial_y\Big(\sum_{i \in I_0(\hat{x}, \hat{y})} \beta_i \partial \nu_i\Big)(\hat{x}, \hat{y}),$$

and so,

$$0_m \in \partial_y\Big(\sum_{i \in I_0(\hat{x}, \hat{y})} \beta_i \partial \nu_i\Big)(\hat{x}, \hat{y}) = \sum_{i \in I_0(\hat{x}, \hat{y})} \beta_i \partial_y \nu_i(\hat{x}, \hat{y}).$$

Hence, for each $i \in I_0(\hat{x}, \hat{y})$, we can find $\xi_i \in \partial_y \nu_i(\hat{x}, \hat{y})$ such that

$$\sum_{i \in I_0(\hat{x}, \hat{y})} \beta_i \xi_i = 0_m \implies \sum_{i \in I_0(\hat{x}, \hat{y})} \beta_i \langle \xi_i, \hat{y} - y^* \rangle = \langle 0_m, \hat{y} - y^* \rangle = 0.$$

Owing to above equality, $\beta_i \ge 0$ for all $i \in I_0(\hat{x}, \hat{y})$, and (8), we conclude that $\beta_i = 0$ for all $i \in I_0(\hat{x}, \hat{y})$. Therefore, CQ1 holds at $(\hat{x}, \hat{y})$. □

Theorem 10. For convex (MGSIP), CQ2 holds at $(\hat{x}, \hat{y}) \in \Lambda$ if SSCQ holds at there.

Proof. Since SSCQ holds at $(\hat{x}, \hat{y})$, there exists a vector $(x^*, y^*) \in \Lambda$ such that $\nu_i(x^*, y^*) < 0$ for all $i \in I_0(\hat{x}, \hat{y})$. From this and the definition of convex subdifferential, we obtain

$$\langle \zeta_i, (x^*, y^*) - (\hat{x}, \hat{y}) \rangle \le \nu_i(x^*, y^*) - \underbrace{\nu_i(\hat{x}, \hat{y})}_{=0} < 0, \quad \forall i \in I_0(\hat{x}, \hat{y}),\ \zeta_i \in \partial \nu_i(\hat{x}, \hat{y}),$$

and thus

$$(x^* - \hat{x}, y^* - \hat{y}) \in \Big(\bigcup_{i \in I_0(\hat{x}, \hat{y})} \partial^c \nu_i(\hat{x}, \hat{y})\Big)^{\ominus} \implies \Big(\bigcup_{i \in I_0(\hat{x}, \hat{y})} \partial^c \nu_i(\hat{x}, \hat{y})\Big)^{\ominus} \ne \emptyset.$$

From this and

$$\Bigg(conv\Big(\bigcup_{i \in I_0(\hat{x}, \hat{y})} \partial^c \nu_i(\hat{x}, \hat{y})\Big)\Bigg)^{\ominus} = \Big(\bigcup_{i \in I_0(\hat{x}, \hat{y})} \partial^c \nu_i(\hat{x}, \hat{y})\Big)^{\ominus},$$

we deduce that

$$\Bigg(conv\Big(\bigcup_{i \in I_0(\hat{x}, \hat{y})} \partial^c \nu_i(\hat{x}, \hat{y})\Big)\Bigg)^{\ominus} \ne \emptyset \implies 0_{n+m} \ne conv\Big(\bigcup_{i \in I_0(\hat{x}, \hat{y})} \partial^c \nu_i(\hat{x}, \hat{y})\Big).$$

The above relation, Theorem 1, and the compactness of $\bigcup_{i\in I_0(\hat{x},\hat{y})} \partial^c \nu_i(\hat{x},\hat{y})$ imply that $cone\Big(\bigcup_{i\in I_0(\hat{x},\hat{y})} \partial^c \nu_i(\hat{x},\hat{y})\Big)$ is closed.

On the other hand, let $w \in \Big(\bigcup_{i\in I_0(\hat{x},\hat{y})} \partial^c \nu_i(\hat{x},\hat{y})\Big)^{\ominus}$ be arbitrarily given. Thus, $\langle w, \zeta_i\rangle < 0$ for all $i \in I_0(\hat{x},\hat{y})$ and $\zeta_i \in \partial \nu_i(\hat{x},\hat{y})$. Therefore,

$$\nu^0\big((\hat{x},\hat{y}); w\big) = \max_{\zeta_i \in \partial \nu_i(\hat{x},\hat{y})} \langle w, \zeta_i\rangle < 0, \quad \forall i \in I_0(\hat{x},\hat{y}),$$

and thus, for each $i \in I_0(\hat{x},\hat{y})$ there exists $\delta_i > 0$ such that

$$\nu_i\big((\hat{x},\hat{y}) + tw\big) - \underbrace{\nu_i(\hat{x},\hat{y})}_{=0} < 0, \quad \forall t \in (0,\delta_i).$$

Hence, $\nu_i\big((\hat{x},\hat{y}) + tw\big) < 0$ for all $i \in I_0(\hat{x},\hat{y})$ and $t \in (0,\delta)$ where $\delta := \min\{\delta_i \mid i \in I_0(\hat{x},\hat{y})\}$. This means $(\hat{x},\hat{y}) + tw \in \Lambda$ for all $t \in (0,\delta)$, so $w \in \Gamma_\Lambda(\hat{x},\hat{y})$. Since $w \in \Big(\bigcup_{i\in I_0(\hat{x},\hat{y})} \partial^c \nu_i(\hat{x},\hat{y})\Big)^{\ominus}$ was arbitrary chosen, we have proved that

$$\Big(\bigcup_{i\in I_0(\hat{x},\hat{y})} \partial^c \nu_i(\hat{x},\hat{y})\Big)^{\ominus} \subseteq \Gamma_\Lambda(\hat{x},\hat{y}).$$

Consequently,

$$\Big(\bigcup_{i\in I_0(\hat{x},\hat{y})} \partial^c \nu_i(\hat{x},\hat{y})\Big)^{\odot} = \overline{\Big(\bigcup_{i\in I_0(\hat{x},\hat{y})} \partial^c \nu_i(\hat{x},\hat{y})\Big)^{\ominus}} \subseteq \overline{\Gamma_\Lambda(\hat{x},\hat{y})} = \Gamma_\Lambda(\hat{x},\hat{y}),$$

where the final equality holds due to the closedness of $\Gamma_\Lambda(\hat{x},\hat{y})$. The above inclusion and the closedness of $cone\Big(\bigcup_{i\in I_0(\hat{x},\hat{y})} \partial^c \nu_i(\hat{x},\hat{y})\Big)$ conclude that CQ2 holds at $(\hat{x},\hat{y})$, as required. □

The theorem presented below is an immediate consequence of Theorem 6, and Theorems 8-10, and the fact that for convex MGSIPs, the value function $\mu(\cdot)$ is convex refer to [16] for further details).

Theorem 11. Assuming the standing assumptions, let $\hat{x}$ be a properly efficient solution of a convex MGSIP, and suppose that one of following statements is true:

- FSCQ holds at $(\hat{x},\hat{y})$ and the set-valued mapping $\Sigma_0(\cdot)$ is inner semi-continuous at $(\hat{x},\hat{y})$.
- SSCQ holds at $(\hat{x},\hat{y})$.

Then, there exist some $\lambda_j \geq 0$ for $j \in J$, as well as $\alpha^{(\ell)} \in K(\hat{x},\hat{y})$ and $\beta_\ell \geq 0$ for $\ell = 1,\ldots,n+1$, such that

$$\begin{cases} 0_n \in \sum_{j=1}^{p} \lambda_j \partial \varphi_j(\hat{x}) - \sum_{\ell=1}^{n+1} \beta_\ell \partial_x \mathfrak{L}(\hat{x},\hat{y},\alpha^{(\ell)}), \\ \sum_{j=1}^{p} \lambda_j + \sum_{\ell=1}^{n+1} \beta_\ell = 1. \end{cases}$$

Finally, we note that the theorem above represents a generalization of first-order optimality conditions for smooth convex GSIPs, proved in [25, 28].

5 Conclusion

This paper is focused on the analysis of non-smooth multi-objective generalized semi-infinite programming problems (MGSIP), where all functions involved are assumed to be locally Lipschitz. The properties of the value function of MGSIP have been investigated, and a Mangasarian-Fromovitz type constraint qualification for MGSIP has been introduced in terms of Clarke subdifferential. An upper bound for the subdifferential of the value function of MGSIP has been derived, and optimality conditions have been established for a properly efficient solution of MGSIP.

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