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Solving Linear Fractional Programming Problems in Uncertain Environments: A Novel Approach with Grey Parameters

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Abstract. Fractional programming is a significant nonlinear planning tool within operation research. It finds applications in diverse domains such as resource allocation, transportation, production programming, performance evaluation, and finance. In practical scenarios, uncertainties often make it challenging to determine precise coefficients for mathematical models. Consequently, utilizing indefinite coefficients instead of definite ones is recommended in such cases. Grey systems theory, along with probability theory, randomness, fuzzy logic, and rough sets, is an approach that addresses uncertainty. In this study, we address the problem of linear fractional programming with grey coefficients in the objective function. To tackle this problem, a novel approach based on the variable change technique proposed by Charnes and Cooper, along with the convex combination of intervals, is employed. The article presents an algorithm that determines the solution to the grey fractional programming problem using grey numbers, thus capturing the uncertainty inherent in the objective function. To demonstrate the effectiveness of the proposed method, an example is solved using the suggested approach. The result is compared with solutions obtained using the whitening method, employing Hu and Wong's technique and the Center and Greyness Degree Ranking method. The comparison confirms the superiority of the proposed method over the whitening method, thus it suggests that adopting the grey system approach is preferable in such situations.

Keywords. Uncertainty, Optimization, Fractional programming, Grey system, Grey interval numbers.

MSC. 90C32.

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1 Introduction

The fractional programming problem (FPP) is a significant nonlinear planning tool widely used in operation research. While linear programming problems (LPP) are insufficient to explain many real-world models, the linear fractional programming problem (LFPP) emerged as a branch of nonlinear programming in the 1960s. LFPP is particularly suitable for optimizing the efficiency of various activities, such as maximizing company profit per unit cost of manpower, minimizing production cost per unit of product, or maximizing dietary calories per unit cost. For more details, refer to [6, 9, 34]. Due to its practical application, LFPP has attracted considerable attention from researchers [2, 3, 5, 8, 23, 36], who have proposed different methods to solve LFPP. Charnes and Cooper [5] introduced a variable change-based method. Bitran and Novaes [3] presented an approach that updates the objective function and solves a sequence of LPPs to tackle LFPP. Das and Mandal [12] employed various solving methods based on the simplex method for LFPP.

In real-life scenarios, precise decision-making is often hindered by incomplete or inaccurate information. In such cases, dealing with uncertainty using theories like fuzzy set theory (FST) and grey system theory (GST) can be beneficial for expressing imprecise coefficients. It is crucial to identify the uncertainty in the problem and apply these theories accordingly, considering their specific characteristics in handling real data uncertainties. FST was first introduced by Professor Zadeh [38]. Recently, researchers have shown interest in solving fuzzy linear fractional programming problems (FLFPPs) [4, 7, 13, 14, 15, 16, 25, 29, 31]. Chinnadurai and Muthokumar [8] proposed a fuzzy mathematical programming approach to solve the FLFPP. Das et al. [15] developed an algorithm based on the multi-objective FPP for FLFPP. Srinivasan [35] investigated the FLFPP with fuzzy number coefficients and proposed an algorithm utilizing ranking methods. However, when there is a scarcity of experts or limited experience, making it difficult to obtain membership functions or gather sufficient data, FST may not apply to solve FPPs with imprecise coefficients. In such cases, Deng introduced GST in 1982 [16]. The GST focuses on studying small samples and systems with limited information, where some information is known while others are unknown. With its development, GST has become a distinct scientific branch encompassing systems analysis, modeling, prediction, decision-making, control, and optimization techniques. The GST finds practical applications in solving real-world problems in various fields, including social sciences, and engineering (e.g., metallurgy, petroleum, chemical industry, electronics, lighting industries, energy sources, transportation, pharmaceuticals, health and health).

The application of GST has resulted in significant economic and social benefits in society, highlighting its wide-ranging utility, particularly in situations where available information is incomplete or collected data is imprecise [22]. Currently, existing methods for tackling the grey linear fractional programming problem (GLFPP) primarily rely on converting the grey parameters into exact ones via a whitening process, thereby transforming the problem into an FPP with precise parameters. However, these methods only yield exact solutions for the GLFPP. Therefore, in this study, we propose a direct approach to address the uncertainty inherent in real-world problems by leveraging the advantages of GST. In this method, the solution to the GLFPP is expressed as grey numbers (GN), allowing for the representation of uncertainty within the objective function, and subsequently reflected in the final result.

This paper is organized as follows: The second section presents an overview of GST and relevant concepts essential to comprehend the paper. The third section introduces the FPP in general and outlines Charnes and Cooper's method for solving them. In the fourth section, the GLFPP is introduced. The fifth section presents the proposed solution method for the GLFPP and the corresponding algorithm. Furthermore, the sixth section, demonstrates the effectiveness of the proposed method by solving an example using it, and the obtained results are compared with those derived from solving the problem using the whitening method. Finally, the seventh section provides the conclusion of the paper.

2 Grey Systems Theory

The GST is widely recognized as a significant scientific advancement in utilizing uncertain information. This theory offers a novel approach for investigating problems characterized by limited data and restricted information, resulting in high levels of inaccuracy. A grey system refers to a system that encompasses uncertain information. Numerous researchers have been drawn to the exploration of GST [10, 20, 28, 27, 26, 31, 32, 30, 33].

Definition 1. GN denotes an interval number with known upper and lower bounds, but its location within these bounds remains uncertain [1]. Grey numbers find diverse applications in various mathematical disciplines [19, 18, 21, 37].

Definition 2. The whitened GN $\otimes x \in [\underline{x}, \bar{x}]$ is denoted by the symbol $\otimes\tilde{x}$ and can be computed as follows:

$$\otimes\tilde{x} = \alpha\overline{x} + (1-\alpha)\,\underline{x}, \quad \alpha \in [0,1]\,, \tag{1}$$

where α represents the weight employed for whitening the GN. When $\alpha = \frac{1}{2}$, equal weight is assigned to the lower and upper values of the interval, resulting in the whitened average with equal weight [24].

Definition 3. For any IGN $\otimes x \in [\underline{x}, \bar{x}]$, the center, $\otimes\hat{x}$ and width, $\otimes x'$ are defined as follows[18].

$$\otimes\hat{x} = \frac{\underline{x}+\overline{x}}{2}, \quad \otimes x' = \frac{\overline{x}-\underline{x}}{2}. \tag{2}$$

Definition 4. The relationships between interval grey numbers (IGN), pertaining to two GN $\otimes x_1 \in [\underline{x}_1, \bar{x}_1]$ and $\otimes x_2 \in [\underline{x}_2, \bar{x}_2]$, can be expressed through the following concepts [24]:

$$\begin{aligned}
&\otimes x_1 + \otimes x_2 = [\underline{x}_1 + \underline{x}_2, \bar{x}_1 + \bar{x}_2]\,,\\
&\otimes x_1 - \otimes x_2 = \otimes x_1 + (-\otimes x_2) = [\underline{x}_1 - \bar{x}_2, \bar{x}_1 - \underline{x}_2]\,,\\
&\otimes x_1 \times \otimes x_2 = [\min\{\underline{x}_1\underline{x}_2, \bar{x}_1\bar{x}_2, \bar{x}_1\underline{x}_2, \underline{x}_1\bar{x}_2\}, \max\{\underline{x}_1\underline{x}_2, \bar{x}_1\bar{x}_2, \bar{x}_1\underline{x}_2, \underline{x}_1\bar{x}_2\}]\,,\\
&\frac{\otimes x_1}{\otimes x_2} = \otimes x_1 \times \otimes x_2{}^{-1} = \left[\min\left\{\frac{\underline{x}_1}{\underline{x}_2}, \frac{\underline{x}_1}{\bar{x}_2}, \frac{\bar{x}_1}{\underline{x}_2}, \frac{\bar{x}_1}{\bar{x}_2}\right\}, \max\left\{\frac{\underline{x}_1}{\underline{x}_2}, \frac{\underline{x}_1}{\bar{x}_2}, \frac{\bar{x}_1}{\underline{x}_2}, \frac{\bar{x}_1}{\bar{x}_2}\right\}\right],\\
&\quad 0 \notin [\underline{x}_2, \bar{x}_2]\,.
\end{aligned} \tag{3}$$

2.1 Grey numbers ranking

The ranking of GN holds significant importance in decision-making and the practical applications of GST. Darvishi et al. [11] conducted a detailed comparison of GN. Several methods have been proposed for ranking IGN, in the meantime.

2.1.1 Hu and Wang approach

Hu and Wang identified the limitations of existing methods for comparing IGN and introduced a new order relationship as follows [17].
Let $\otimes x = [\underline{x}, \bar{x}]$ and $\otimes y = [\underline{y}, \bar{y}]$ represent two IGN, with centers $\otimes\hat{x}$ and$\otimes\hat{y}$ and widths $\otimes x'$ and $\otimes y'$, respectively. In this case, the order relationship based on the method proposed by Hu and Wang can be expressed as follows:

$$1. \otimes x \leq_G \otimes y \quad \Leftrightarrow \begin{cases} 1) \otimes \hat{x} \neq \otimes \hat{y} \Rightarrow \otimes \hat{x} < \otimes \hat{y}, \\ 2) \otimes \hat{x} = \otimes \hat{y} \Rightarrow \otimes x' \geq \otimes y', \end{cases}$$

$$2. \otimes x <_G \otimes y \Leftrightarrow (\otimes x \leq_G \otimes y, \ \ \otimes x \neq_G \otimes y) .$$

The center and width of IGN are regarded as the expected value and uncertainty of the parameters, respectively. Consequently, when the centers of IGN are equal, the widths of IGN are employed for comparison.

Example 1. Compare IGN $\otimes x = [-3, -1]$ and $\otimes y = [2, 4]$ using Hu and Wang's ranking method. By calculating the centers of the given IGN, we obtain:

$$\otimes \hat{y} = 3, \quad \otimes \hat{x} = -2.$$

According to the first case, we have

$$\otimes \hat{x} = -2 < \otimes \hat{y} = 3 \Rightarrow \otimes x = [-3, -1] \leq_G \otimes y = [2, 4] .$$

Example 2. Compare IGN $\otimes x = [1, 5]$ and $\otimes y = [2, 4]$ using Hu and Wang's ranking method. By calculating the centers and widths of the given IGN, we have:

$$\otimes \hat{y} = 3, \quad \otimes \hat{x} = 3,$$
$$\otimes y' = 1, \quad \otimes x' = 2.$$

According to the second case, we can obtain

$$\otimes \hat{x} = 3 = \otimes \hat{y} = 3 \ , \quad \otimes x' = 2 \ \geq \ \otimes y' = 1 \ \Rightarrow \otimes x = [1, 5] \leq_G \otimes y = [2, 4] .$$

2.1.2 Center and degree of greyness of grey numbers approach

Definition 5. [20] The length of IGN $\otimes x \in [\underline{x}, \bar{x}]$ is defined as:

$$\mu (\otimes x) = |\bar{x} - \underline{x}| . \tag{4}$$

Definition 6. [20] With IGN background(Ω), the degree of greyness $g^0(\otimes x)$ is introduced as follows.

$$g^0(\otimes x) = \frac{\mu(\otimes x)}{\mu(\Omega)}.$$

Definition 7. [20] For IGN $\otimes x_1, \otimes x_2$, we define their ranking based on the Center and degree of greyness is defined as follows:

$$\otimes \hat{x}_1 < \otimes \hat{x}_2 \Rightarrow \otimes x_1 <_G \otimes x_2,$$

$$\otimes \hat{x}_1 = \otimes \hat{x}_2 \Rightarrow \begin{cases} \text{if } g^0 (\otimes x_1) = g^0 (\otimes x_2) \Rightarrow \otimes x_1 =_G \otimes x_2, \\ \text{if } g^0 (\otimes x_1) < g^0 (\otimes x_2) \Rightarrow \otimes x_1 >_G \otimes x_2, \\ \text{if } g^0 (\otimes x_1) > g^0 (\otimes x_2) \Rightarrow \otimes x_1 <_G \otimes x_2. \end{cases} \tag{5}$$

Example 3. Let us assume that the IGNs $\otimes x_1 = [-4, -2] \,, \ \otimes x_2 = [1, 7] \,, \ \otimes x_3 = [1, 5]$ belong to the field $\Omega \in [-5, 20]$.

Calculate the measures of field $\Omega, \ \otimes x_1, \ \otimes x_2, \ \otimes x_3$ and

$$\mu (\Omega) = 25, \ \ \mu (\otimes x_1) = 2, \ \ \mu (\otimes x_2) = 6, \ \ \mu (\otimes x_3) = 12,$$

respectively. The centers and the degrees of greyness of these three GN are as follows:

$$\begin{aligned}
&\otimes \hat{x}_1 = -3,\ \otimes\hat{x}_2 = 4,\ \otimes\hat{x}_3 = 4,\\
&g^0(\otimes x_1) = \frac{\mu(\otimes x_1)}{\mu(\Omega)} = \frac{2}{25} = 0.08,\ g^0(\otimes x_2) = \frac{\mu(\otimes x_2)}{\mu(\Omega)} = \frac{6}{25} = 0.24,\\
&< g^0(\otimes x_3) = \frac{\mu(\otimes x_3)}{\mu(\Omega)} = \frac{12}{25} = 0.48.
\end{aligned}$$

The first mode:

$$\otimes\hat{x}_1 = -3 < \otimes\hat{x}_2 = 4 \ \Rightarrow \otimes x_1 <_G \otimes x_2.$$

The second mode:

$$\otimes\hat{x}_2 = 4 = \otimes\hat{x}_3 = 4,$$

therefore,

$$g^0(\otimes x_2) = 0.24 < g^0(\otimes x_3)0.48 \Rightarrow \otimes x_2 >_G \otimes x_3.$$

3 Fractional Programming Problem

The FPP is a topic extensively studied in operations research. It serves as a modeling tool for real-world problems in various fields, such as business, economics, engineering, economics, etc. In an FPP, the objective function is a ratio of two functions, typically nonlinear that needs to be optimized. This ratio often represents the efficiency of a system.

$$\begin{aligned}
&\text{Min(Max)} \quad W(x) = \frac{f(x)}{g(x)}\\
&s.t.\\
&g(x) > 0,\\
&x \in S = \{x \in S_0 \subset R^n : h_j(x) \le 0,\ j = 1, \ldots, m\}.
\end{aligned}$$

In [36], Stancu-Minassian conducted a thorough investigation into deficit planning, its applications, solution methods, and related issues. The LFPP is a special case of FPP. The general form of an LFPP problem can be expressed as follows:

$$\begin{aligned}
&\text{Min } W(x) = \frac{a_1x_1 + \cdots + a_kx_k + a_{k+1}}{c_1x_1 + \cdots + c_kx_k + c_{k+1}}\\
&s.t.\\
&\qquad A_1x_1 + \cdots + A_kx_k \le b,\\
&\qquad x_1 \ge 0, \ldots, x_k \ge 0,
\end{aligned} \tag{6}$$

subject to $c_1x_1 + \cdots + c_kx_k + c_{k+1} > 0$ for each for each$(x_1, \ldots, x_k) \in X$, where X represents the feasible region of problem (6).

An LFPP can be transformed into a LFPP using the method proposed by Charnes and Cooper's [5].

3.1 Charnes and Cooper's method for solving the LFPP

Let us assume $z = \frac{1}{c_1x_1+\cdots+c_kx_k+c_{k+1}}$, in this case, Problem (6) transforms into the following LPP.

$$\begin{array}{ll} \text{Min} & W'(x) = a_1x_1z + \cdots + a_kx_kz + a_{k+1}z \\ \text{s.t.} & \\ & c_1x_1z + \cdots + c_kx_kz + c_{k+1}z = 1, \\ & A_1x_1 + \cdots + A_kx_k \leq b, \\ & x_1 \geq 0 \ , \ldots , x_k \geq 0 \ , \ z \geq 0. \end{array} \tag{7}$$

By introducing $y_i = x_iz$ for $i = 1, \ldots, k$, Problem (7) takes the following form:

$$\begin{array}{ll} \text{Min} & W'(x) = a_1y_1 + \cdots + a_ky_k + a_{k+1}z \\ \text{s.t.} & \\ & c_1y_1 + \cdots + c_ky_k + c_{k+1}z = 1, \\ & A_1y_1 + \cdots + A_ky_k \leq bz, \\ & y_1 \geq 0 \ , \ldots , y_k \geq 0 \ , \ z \geq 0. \end{array} \tag{8}$$

In practical real-life scenarios, decisions are not always made based on for explicit data. In such cases, uncertainty theories such as GST can be employed to represent imprecise coefficients.

4 Grey Linear Fractional Programming Problem

In general, the LFPP with grey coefficients in the objective function can be expressed in the following form:

$$\begin{array}{ll} \text{Min} & \otimes F =_G \frac{\sum_{j=1}^{k} \otimes a_j x_j + \otimes a_{j+1}}{\sum_{j=1}^{k} \otimes c_j x_j + \otimes c_{j+1}} \\ \text{s.t.} & \\ & \sum_{j=1}^{k} A_j x_j \leq b, \\ & x_j \geq 0 \ j = 1, \ldots, k. \end{array} \tag{9}$$

In other words,

$$\begin{array}{ll} \text{Min} & \otimes F =_G \frac{\otimes[\underline{a}_1, \bar{a}_1]x_1 + \cdots + \otimes[\underline{a}_k, \bar{a}_k]x_k + \otimes[\underline{a}_{k+1}, \bar{a}_{k+1}]}{\otimes[\underline{c}_1, \bar{c}_1]x_1 + \cdots + \otimes[\underline{c}_k, \bar{c}_k]x_k + \otimes[\underline{c}_{k+1}, \bar{c}_{k+1}]} \\ \text{s.t.} & \\ & A_1x_1 + \cdots + A_kx_k \leq b, \\ & x_1 \geq 0, \ldots, x_k \geq 0, \end{array} \tag{10}$$

under the assumption $\otimes[\underline{c}_1, \bar{c}_1]\, x_1 + \cdots + \otimes[\underline{c}_k, \bar{c}_k]\, x_k + \otimes\left[\underline{c}_{k+1}, \bar{c}_{k+1}\right] > 0$ for each $(x_1, \ldots, x_k) \in X$, where X represents the feasible region of Problem (10).

Example 4. Here is an illustration of an LFPP with grey coefficients in the objective function:

$$\begin{array}{ll} \text{Min} & \otimes F =_G \frac{\otimes[-3,-1]x_1 + \otimes[2,4]x_2 + \otimes[-2,-0.5]}{\otimes[0.5,1.5]x_1 + \otimes[0.5,1.5]x_2 + \otimes[3,5]} \\ \text{s.t.} & \\ & -x_1 + x_2 \leq 4, \\ & +2x_1 + 3x_2 \leq 14, \\ & +x_1 - x_2 \leq 5, \\ & x_1 \geq 0 \ , x_2 \geq 0. \end{array} \tag{11}$$

5 Proposed Method for Solving GLFPP

To address Problem (10) by incorporating the Charnes and Cooper variables transformation, the following approach is proposed:

$$z = \frac{1}{\otimes[\underline{c}_1, \bar{c}_1]\, x_1 + \cdots + \otimes[\underline{c}_k, \bar{c}_k]\, x_k + \otimes\left[\underline{c}_{k+1}, \bar{c}_{k+1}\right]} \quad , \quad y_i = x_i z \quad i = 1, \ldots, k.$$

In this scenario, Problem (5) takes the following form:

$$\begin{array}{ll} \text{Min} & \otimes F' =_G \otimes[\underline{a}_1, \bar{a}_1]\, y_1 + \cdots + \otimes[\underline{a}_k, \bar{a}_k]\, y_k + \otimes\left[\underline{a}_{k+1}, \bar{a}_{k+1}\right] z \\ \text{s.t.} & \\ & \otimes[\underline{c}_1, \bar{c}_1]\, y_1 + \cdots + \otimes[\underline{c}_k, \bar{c}_k]\, y_k + \otimes\left[\underline{c}_{k+1}, \bar{c}_{k+1}\right] z = 1, \\ & A_1 y_1 + \cdots + A_k y_k \le bz, \\ & y_1 \ge 0\ , \ldots, y_k \ge 0\ ,\ z \ge 0. \end{array} \tag{12}$$

By considering a convex combination of intervals for the limit

$$\otimes[\underline{c}_1, \bar{c}_1]\, y_1 + \cdots + \otimes[\underline{c}_k, \bar{c}_k]\, y_k + \otimes\left[\underline{c}_{k+1}, \bar{c}_{k+1}\right] z = 1,$$

we obtain:

$$\begin{array}{l} 0 \le \lambda_i \le 1 \quad , i = 1, \ldots, k+1 \quad , \quad y_1 \ge 0, \ldots, y_k \ge 0, \\ \left[\lambda_1 \underline{c}_1 + (1-\lambda_1)\, \bar{c}_1\right] y_1 + \cdots + \left[\lambda_k \underline{c}_k + (1-\lambda_k)\, \bar{c}_k\right] y_k + \\ \left[\lambda_{k+1} \underline{c}_{k+1} + (1-\lambda_{k+1})\, \bar{c}_{k+1}\right] z = 1. \end{array} \tag{13}$$

In simpler terms,

$$\begin{array}{l} \lambda_1 y_1(\underline{c}_1 - \bar{c}_1) + \cdots + \lambda_k y_k(\underline{c}_k - \bar{c}_k) + \lambda_{k+1} z(\underline{c}_{k+1} - \bar{c}_{k+1}) + \bar{c}_1 y_1 + \cdots + \\ \bar{c}_k y_k + \bar{c}_{k+1} z = 1. \end{array} \tag{14}$$

Consequently, we can derive:

$$\begin{array}{l} 1 \le 1 + \left[\lambda_1 y_1 (\bar{c}_1 - \underline{c}_1) + \cdots + \lambda_k y_k (\bar{c}_k - \underline{c}_k) + \lambda_{k+1} z \left(\bar{c}_{k+1} - \underline{c}_{k+1}\right)\right] \\ \quad \le 1 + y_1 (\bar{c}_1 - \underline{c}_1) + \cdots + y_k (\bar{c}_k - \underline{c}_k) + z \left(\bar{c}_{k+1} - \underline{c}_{k+1}\right). \end{array} \tag{15}$$

Now, combining Equations (14) and (15), we arrive at:

$$\begin{array}{l} 1 \le \lambda_1 y_1 (\underline{c}_1 - \bar{c}_1) + \cdots + \lambda_k y_k (\underline{c}_k - \bar{c}_k) + \lambda_{k+1} z \left(\underline{c}_{k+1} - \bar{c}_{k+1}\right) + \bar{c}_1 y_1 + \cdots \\ \quad + \bar{c}_k y_k + \bar{c}_{k+1} z + \lambda_1 y_1 (\bar{c}_1 - \underline{c}_1) + \cdots + \lambda_k y_k (\bar{c}_k - \underline{c}_k) + \lambda_{k+1} z \left(\bar{c}_{k+1} - \underline{c}_{k+1}\right) \\ \le 1 + y_1 (\bar{c}_1 - \underline{c}_1) + \cdots + y_k (\bar{c}_k - \underline{c}_k) + z \left(\bar{c}_{k+1} - \underline{c}_{k+1}\right). \end{array} \tag{16}$$

Hence,

$$\begin{array}{l} 1 \le \bar{c}_1 y_1 + \cdots + \bar{c}_k y_k + \bar{c}_{k+1} z \le 1 + y_1 (\bar{c}_1 - \underline{c}_1) + \cdots \\ \quad + y_k (\bar{c}_k - \underline{c}_k) + z \left(\bar{c}_{k+1} - \underline{c}_{k+1}\right), \end{array} \tag{17}$$

Based on Equation (17), we can further deduce:

$$\begin{array}{l} \bar{c}_1 y_1 + \cdots + \bar{c}_k y_k + \bar{c}_{k+1} z \ge 1, \\ \underline{c}_1 y_1 + \cdots + \underline{c}_k y_k + \underline{c}_{k+1} z \le 1. \end{array} \tag{18}$$

By utilizing the constraints expressed in Equation (18), Problem (10) can be reformulated as follows:

$$\begin{array}{ll} \text{Min} & \otimes F' =_G \otimes [\underline{a}_1, \bar{a}_1] y_1 + \cdots + \otimes [\underline{a}_k, \bar{a}_k] y_k + \otimes [\underline{a}_{k+1}, \bar{a}_{k+1}] z \\ \text{s.t.} & \\ & \bar{c}_1 y_1 + \cdots + \bar{c}_k y_k + \bar{c}_{k+1} z \geq 1, \\ & \underline{c}_1 y_1 + \cdots + \underline{c}_k y_k + \underline{c}_{k+1} z \leq 1, \\ & A_1 y_1 + \cdots + A_k y_k \leq bz, \\ & y_1 \geq 0, \ldots, y_k \geq 0, \; z \geq 0. \end{array} \tag{19}$$

5.1 The proposed method algorithm

The proposed algorithm for solving the GLFPP is outlined as follows.
Step 1. Consider the general form of the problem as:

$$\begin{array}{ll} \text{Min} & (or \text{ Max}) \otimes Z =_G \frac{\sum_{j=1}^{k} \otimes a_j x_j + \otimes a_{j+1}}{\sum_{j=1}^{k} \otimes c_j x_j + \otimes c_{j+1}} \\ \text{s.t.} & \\ & \sum_{j=1}^{k} A_j x_j \leq b, \\ & x_j \geq 0 \;\; j = 1, \ldots, k. \end{array} \tag{20}$$

Step 2. Apply the proposed method to convert the problem from the first step into a grey linear programming problem (GLPP) in the following form.

$$\begin{array}{ll} \text{Min} & (or \text{ Max}) \otimes F' =_G \otimes [\underline{a}_1, \bar{a}_1] y_1 + \cdots + \otimes [\underline{a}_k, \bar{a}_k] y_k + \otimes [\underline{a}_{k+1}, \bar{a}_{k+1}] z \\ \text{s.t.} & \\ & \bar{c}_1 y_1 + \cdots + \bar{c}_k y_k + \bar{c}_{k+1} z \geq 1, \\ & \underline{c}_1 y_1 + \cdots + \underline{c}_k y_k + \underline{c}_{k+1} z \leq 1, \\ & A_1 y_1 + \cdots + A_k y_k \leq bz, \\ & y_1 \geq 0, \ldots, y_k \geq 0, \; z \geq 0. \end{array} \tag{21}$$

Step 3: Solve the GLPP obtained in the second step using the simplex method for new variables and obtain the solution.
Step 4: Obtain the solution to the initial problem by utilizing constraints $y_i = x_i z$ for $i = 1, \ldots, k$.

6 Numerical Example

In this section, we demonstrate the efficiency of the proposed algorithm by solving a problem.

Example 5. Obtain the solution to the following GLFPP using the proposed algorithm.

$$\begin{array}{ll} \text{Min} & \otimes F =_G \frac{\otimes[-3,-1]x_1 + \otimes[2,4]x_2 + \otimes[-2,-0.5]}{\otimes[0.5,1.5]x_1 + \otimes[0.5,1.5]x_2 + \otimes[3,5]} \\ \text{s.t.} & \\ & -x_1 + x_2 \leq 4, \\ & +2x_1 + 3x_2 \leq 14, \\ & +x_1 - x_2 \leq 5, \\ & x_1 \geq 0, x_2 \geq 0. \end{array} \tag{22}$$

By utilizing the simplex method, we obtain:

$$
\begin{array}{ll}
\text{Min} & \otimes F' =_G \otimes[-3,-1]\, y_1 + \otimes[2,4]\, y_2 + \otimes[-2,-0.5]\, z \\
\text{s.t.} & \\
& 0.5y_1 + 0.5y_2 + 3z \leq 1, \\
& 1.5y_1 + 1.5y_2 + 5z \geq 1, \\
& -y_1 + y_2 - 4z \leq 0, \\
& +2y_1 + 3y_2 - 14z \leq 0, \\
& +y_1 - y_2 - 5z \leq 0, \\
& y_1 \geq 0\ , y_2 \geq 0\ , z \geq 0.
\end{array} \tag{23}
$$

By using the simplex method, we will have:

$$
\begin{array}{ll}
\text{Min} & \otimes F' =_G \otimes[-3,-1]\, y_1 + \otimes[2,4]\, y_2 + \otimes[-2,-0.5]\, z + MR_1 \\
\text{s.t.} & \\
& 0.5y_1 + 0.5y_2 + 3z + s_1 = 1, \\
& 1.5y_1 + 1.5y_2 + 5z - s_2 + R_1 = 1, \\
& -y_1 + y_2 - 4z + s_3 = 0, \\
& +2y_1 + 3y_2 - 14z + s_4 = 0, \\
& +y_1 - y_2 - 5z + s_5 = 0, \\
& y_1 \geq 0, y_2 \geq 0, z \geq 0.
\end{array} \tag{24}
$$

Table 1: Preliminary tableau of the GLPP.

Basic variables	y_1	y_2	z	s_1	s_2	R_1	s_3	s_4	s_5	RHS
F_0'	$-[-3,-1]$	$-[2,4]$	$-[-2,-0.5]$	0	0	$-M$	0	0	0	0
s_1	0.5	0.5	3	1	0	0	0	0	0	1
R_1	1.5	1.5	5	0	-1	1	0	0	0	1
s_3	-1	1	-4	0	0	0	1	0	0	0
s_4	2	3	-14	0	0	0	0	1	0	0
s_5	1	-1	-5	0	0	0	0	0	1	0

Table 2: Initial tableau of GLPP.

Basic variables	y_1	y_2	z	s_1	s_2	R_1	s_3	s_4	s_5	RHS
F_0'	$1.5M+[1,3]$	$1.5M-[2,4]$	$5M+[0.5,2]$	0	$-M$	0	0	0	0	M
s_1	0.5	0.5	3	1	0	0	0	0	0	1
R_1	1.5	1.5	5	0	-1	1	0	0	0	1
s_3	-1	1	-4	0	0	0	1	0	0	0
s_4	2	3	-14	0	0	0	0	1	0	0
s_5	1	-1	-5	0	0	0	0	0	1	0

Table 3 represents the optimal tableau of the problem, and thus, the optimal solution is as follows.

$$z = 0.18, \quad y_1 = 0.9, \quad y_2 = 0, \quad \otimes F' =_G -\otimes[0.59, 3.14]\,. \tag{25}$$

Using constraints $y_i = x_i z$ for $i = 1,\ldots,k$, the optimal solution to the primal problem is as follows.

$$x_1 = 5, \quad x_2 = 0, \quad \otimes F =_G -\otimes[0.44, 3.09] = \otimes[-3.09, -0.44]\,. \tag{26}$$

Table 3: Optimal tableau of GLPP.

Basic variables	y_1	y_2	z	s_1	s_2	R_1	s_3	s_4	s_5	RHS
F_0'	0	$-[1.12, 4.98]$	0	$-[0.59, 3.49]$	0	$-M$ $+[-1.28, 1.28]$	0	0	$-[0.25, 1.69]$	$-[0.59, 3.49]$
s_2	0	-0.72	0	2.27	1	-1	0	0	0.36	1.27
z	0	0.18	1	0.18	0	0	0	0	0.09	0.18
s_3	0	1.65	0	1.63	0	0	1	0	0.17	1.63
s_4	0	5.73	0	0.72	0	0	0	1	-2.37	0.72
y_1	1	-0.08	0	0.9	0	0	0	0	0.54	0.9

To showcase the effectiveness of the proposed method, we compare the results obtained from the proposed method for Example 5 with the solution obtained from the whitening method.

6.1 Solving Example using the whitening method

$$\begin{array}{ll} \text{Min} & \otimes F =_G \frac{\otimes[-3,-1]x_1 + \otimes[2,4]x_2 + \otimes[-2,-0.5]}{\otimes[0.5,1.5]x_1 + \otimes[0.5,1.5]x_2 + \otimes[3,5]} \\ \text{s.t.} & \\ & -x_1 + x_2 \leq 4, \\ & +2x_1 + 3x_2 \leq 14, \\ & +x_1 - x_2 \leq 5, \\ & x_1 \geq 0, x_2 \geq 0. \end{array} \tag{27}$$

By applying the whitening technique to the GLFPP using the center of the $\otimes C$, we obtain the following LFPP in canonical form:

$$\begin{array}{ll} \text{Min} & W = \frac{-2x_1 + 3x_2 - 1.25}{x_1 + x_2 + 4} \\ \text{s.t.} & \\ & -x_1 + x_2 \leq 4, \\ & +2x_1 + 3x_2 \leq 14, \\ & +x_1 - x_2 \leq 5, \\ & x_1 \geq 0, x_2 \geq 0. \end{array} \tag{28}$$

By employing the Charnes and Cooper method, we obtain:

$$\begin{array}{ll} \text{Min} & W' = -2y_1 + 3y_2 - 1.25z \\ \text{s.t.} & \\ & +y_1 + y_2 + 4z = 1, \\ & -y_1 + y_2 - 4z \leq 0, \\ & +2y_1 + 3y_2 - 14z \leq 0, \\ & +y_1 - y_2 - 5z \leq 0, \\ & y_1 \geq 0, y_2 \geq 0. \end{array} \tag{29}$$

Table 6 represents the optimal tableau of the problem, and thus, the optimal solution is as follows:

$$z = 0.111, \quad y_1 = 0.555, \quad y_2 = 0, \quad W' = -1.247. \tag{30}$$

Using these solutions, the solution to the main problem can be obtained as follows:

$$x_1 = 5, \quad x_2 = 0, \quad W = -1.25. \tag{31}$$

Table 4: Preliminary tableau of the LPP.

Basic variables	y_1	y_2	z	R_1	s_2	s_3	s_4	RHS
W'_0	2	-3	1.25	$-M$	0	0	0	0
R_1	1	1	4	1	0	0	0	1
s_2	-1	1	-4	0	1	0	0	0
s_3	2	3	-14	0	0	1	0	0
s_4	1	-1	-5	0	0	0	1	0

Table 5: Initial tableau of LPP.

Basic variables	y_1	y_2	z	R_1	s_2	s_3	s_4	RHS
W'_0	$2+M$	$-3+M$	$1.25+4M$	0	0	0	0	M
R_1	1	1	4	1	0	0	0	1
s_2	-1	1	-4	0	1	0	0	0
s_3	2	3	-14	0	0	1	0	0
s_4	1	-1	-5	0	0	0	1	0

Table 6: Optimal tableau of LPP.

Basic variables	y_1	y_2	z	R_1	s_2	s_3	s_4	RHS
W'_0	0	-3.49	0	$-1.23-M$	0	0	-0.74	-1.247
Z	0	-0.22	1	0.112	0	0	-0.11	0.111
s_2	0	2	0	1	1	0	0	1
s_3	0	5.89	0	0.475	0	1	-2.42	0.447
y_1	1	0.11	0	0.55	0	0	0.44	0.555

6.2 Discussions

In this section, we compare the results obtained from the proposed method and the whitening method.

Table 7: Comparison of the solution obtained from the proposed method and the whitening method.

proposed method	$y_1 = 0.9$	$y_2 = 0$	$z = 0.18$	$\otimes F' =_G -\otimes[0.59, 3.14]$
	$x_1 = 5$	$x_2 = 0$		$\otimes F =_G -\otimes[0.44, 3.09]$
whitening method	$y_1 = 0.555$	$y_2 = 0$	$z = 0.111$	$W' = -1.247$
	$x_1 = 5$	$x_2 = 0$		$W = -1.25$

When dealing with real-world problems that involve imprecise parameters, the LPP method cannot be used directly. Hence, the whitening method is employed to transform the problem into an exact one by handling the imprecise parameters. Consequently, the solution obtained through this method is only an approximation of the solution for the problem with imprecise parameters and does not capture the uncertainty associated with the imprecise parameters of the main problem in the final solution. As depicted in Table 7, $W = -1.25$ the exact solution for the GLFPP of Example 5, serves as an approximation to the grey solution obtained through the proposed method $\otimes F =_G - \otimes [0.44, 3.09]$.

Table 8: Comparison of the proposed method and the whitening method using Hu and Wang's and Center and greyness degree ranking methods.

proposed method solution (A)	Whitening method solution (B)	Hu and Wang method	center and greyness degree method
$\otimes[-3.09, -0.44]$	$\otimes[-1.25, -1.25]$	$A \leq B$	$A \leq B$

The outcomes presented in Table 8 demonstrate that the solution obtained through the proposed method, considering both Hu and Wang's, and Center and greyness degree ranking methods, outperforms the whitening method.

7 Conclusion

The linear fractional programming problem is a prominent area of focus in operations research, employed for modeling real-world problems. However, in many practical problem models, the determination of precise coefficients is often unattainable, necessitating the utilization of non-deterministic coefficients. The theory of grey systems provides an approach to address uncertainty in such scenarios. This research focuses on the problem of linear fractional programming with grey coefficients within the objective function. While existing approaches transform the grey linear fractional programming problem into one or multiple classical fractional programming problems to derive optimal solutions, this study introduces a novel method that directly solves the linear fractional programming problem while incorporating grey coefficients within the objective function. The proposed approach employs a combination of the Charnes and Cooper variable change technique and convex intervals. By converting the grey linear fractional programming problem into a grey linear programming problem, the solution is obtained as a range of interval grey numbers. Consequently, the uncertainty inherent in the objective function is accurately reflected in the final result. Evaluation using the ranking methods of Hu and Wang, as well as the Center and Greyness Degree method, demonstrates the superiority of the proposed method over the whitening method, further reinforcing its efficacy.

Declarations

Availability of supporting data

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Authors' contributions

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