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An Adaptive Time-Stepping Algorithm to Solve a Stochastic Lotka-Volterra Competition System with Time-Variable Delays

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Abstract. This paper introduces a variable step size strategy for a stochastic time-delays Lotka-Volterra competition system. This adaptive strategy utilizes the Milstein method for numerical solutions. It employs two local error estimates, corresponding to the diffusion and drift components of the model, to select and control the step sizes. The algorithm is described in detail, and numerical experiments are conducted to demonstrate the efficiency of the proposed method. The primary objective of this research is to propose a dynamic strategy for generating and controlling the step sizes in the finite difference algorithm employed. This adaptive approach accelerates the numerical procedure and improves efficiency compared to a constant-size scheme. As an analytical solution for the model is unavailable, a numerical estimation with a small fixed step size is considered a reference solution. The numerical results demonstrate the superior accuracy of the proposed strategy compared to a reference solution.

Keywords. Stochastic Lotka-Volterra model, Time-dependent delays, Milstein method, Variable step-size.

MSC. 92B05; 60h35; 60H10.

1 Introduction

The Lotka-Volterra (L-V) competition model plays an important role in mathematical biology [5]. The Lotka-Volterra equation is a pair of nonlinear differential equations used as a model for biological systems in which there are two species in the form of a predator and a prey. These equations were first established by Lotka [10] and Volterra [19]. The deterministic form of the L-V competition two-species model is expressed as

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_1(t)\left[\delta_1 - b_{11}x_1(t) - b_{12}x_2(t)\right], \\ \frac{dx_2(t)}{dt} &= x_2(t)\left[\delta_2 - b_{21}x_1(t) - b_{22}x_2(t)\right], \end{aligned} \tag{1}$$

in which $x_i(t)$ denotes the size of the population for the ith species at t; δ_i and b_{ij}, $i, j = 1, 2$, are some positive constants [9].

Due to the maturation time, there is a delay time for almost all species. Thus, it can be argued that delayed systems can better represent the reality of the species. Suppose $x^* = (x_1^*, x_2^*)$ is an equilibrium state of the system (1), the time-delayed form of this model can be represented as

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_1(t)\left[b_{11}(x_1^\star - x_1(t)) + b_{12}(x_1^\star - x_1(t-\tau_1(t))) + b_{13}(x_2^\star - x_2(t-\tau_2(t)))\right], \\ \frac{dx_2(t)}{dt} &= x_2(t)\left[b_{21}(x_2^\star - x_2(t)) + b_{22}(x_2^\star - x_2(t-\tau_3(t))) + b_{23}(x_1^\star - x_1(t-\tau_4(t)))\right], \end{aligned} \tag{2}$$

where b_{13} and b_{23} are some positive constants; $\tau_i(t)$, $i = 1, \ldots, 4$, are non-negative, and continuously differentiable bounded functions on the closed interval $[0, +\infty]$ [9].

Because of the random fluctuations in the various parameters that affect the behavior of real biological systems, some additional noises are considered in the modeling of the L-V competition problem. In the last two decades, many researchers have focused on several stochastic versions of systems (1) and (2); Bahar and Mao presented an introductory study on stochastic L-V system with delay in [1]; in [14], the authors constructed optimized harvesting techniques for stochastic L-V competition ecosystems; while in [16], the authors studied the coexistence and extinction of these problems; in [15] the authors proposed a numerical operational matrices approach to determine the solution of a stochastic L-V problem; the author in [17] studied analytically and numerically three different stochastic models of a deterministic L-V system; a backward Euler-Maruyama method was proposed in [22] for nonlinear hybrid time-dependent delay stochastic differential equations (SDEs); [20] aimed to analyze a stochastic L-V model with periodically distributed delay for three species; in [21] a stochastic L-V system with Levy noises was considered and a related parameter estimation problem was solved; the unique existence of a global solution and a numerical solution based on an optimized Euler-Maruyama technique for a stochastic age-structured cooperative L-V model with Poisson jumps were introduced in [23]; and in [2, 8, 11] some positivity preserving numerical approaches were surveyed to solve the n-dimensional stochastic L-V models.

In the present work, we study the Itô form of the stochastic L-V competition system with time delays, written as

$$\begin{aligned} dx_1(t) = {} & x_1(t)\left[\delta_1 - b_{11}x_1(t) - b_{12}x_1(t-\tau_1(t)) - b_{13}x_2(t-\tau_2(t))\right]dt \\ & + \mu_1 x_1(t)(x_1(t) - x_1^\star)dW_1(t), \end{aligned}$$

$$dx_2(t) = x_2(t)\left[\delta_2 - b_{21}x_2(t) - b_{22}x_2(t-\tau_3(t)) - b_{23}x_1(t-\tau_4(t))\right]dt + \mu_2 x_2(t)(x_2(t) - x_2^\star)dW_2(t), \tag{3}$$

where $\mu_i^2,\ i = 1, 2$, denote white noises affecting δ_i; and $W(t) = (W_1(t), W_2(t))^T$ denotes a Winner process on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\{\mathcal{F}_i\}_{i\in\mathbb{R}_+}$. More details about the coefficients and the interpretation of the model can be found in [9].

The initial value for system (3) is given as follows

$$N_0 = \{\zeta(\theta) = (\zeta_1(\theta), \zeta_2(\theta))^T, -\overline{\tau} \le \theta \le 0 \subset \mathcal{C}\}, \tag{4}$$

where $\mathcal{C} = C\left([-\overline{\tau}, 0]; \mathbb{R}_+^2\right)$ is the family of continuous functions ϕ from $[-\overline{\tau}, 0]$ to $\mathbb{R}_+^2 = \{(x, y)^T \in \mathbb{R}^2 : x > 0, y > 0\}$ with norm $\|\phi\| = sup_{-\overline{\tau}\le\theta\le 0}\ |\phi(\theta)|$ [9].

Lemma 1 in [9] proves the unique existence of a global positive solution of the system (3) for $t \ge -\overline{\tau}$ under the condition

$$\tau' = \max_{i=1,2,3,4} \sup_{t\ge 0}\{\tau_i'(t)\} < 1.$$

Also, in [9], the following theorem is presented to investigate the stability of the state (x_1^*, x_2^*) for system (3).

Theorem 1. Assume

$$A = -b_{11} + \frac{2-\tau'}{2(1-\tau')}b_{12} + \frac{b_{13}}{2} + \frac{b_{23}}{2(1-\tau')} + \frac{1}{2}\mu_1^2 x_1^*,$$
$$B = -b_{21} + \frac{2-\tau'}{2(1-\tau')}b_{22} + \frac{b_{23}}{2} + \frac{b_{13}}{2(1-\tau')} + \frac{1}{2}\mu_2^2 x_2^*.$$

The positive equilibrium state (x_1^*, x_2^*) of the system (3) subject to (4) is almost surely (a.s.) globally asymptotically stable if $A, B < 0$.

This work is about the application of the Milstein method [6] to obtain the numerical solution of the L-V system (3)-(4). The main challenge in using finite difference techniques to solve stochastic systems is selecting an appropriate step size. In other words, the accuracy of these methods essentially depends on the choice of the size of each step. In fixed step-size algorithms, a suitable size is first chosen for all steps and the recursion relation is performed based on this predetermined step-size. Thus, conventional constant step-size algorithms perform uniform steps throughout the problem domain. To increase the accuracy of the method, one usually has to reduce the step-size in these cases, which may lead to excessive computations and increase the computational time of the numerical process. By introducing an adaptive step-size strategy, the convergence behavior of the method can be improved. The most important benefit of a variable step-size algorithm is the reduction of the step-size when sufficient progress is not achieved. Therefore, using an adaptive procedure improves the convergence rate and the computational efficiency of the method significantly.

The most crucial concern with adaptive procedures is to determine an effective local error estimator to choose the optimal step size for each stage. In the following, we propose a variable step size algorithm for solving the system (3), where the numerical approach is adopted by the famous Milstein method, and the step sizes are determined by two local error estimators corresponding to the diffusion and drift components of the system at each recursion.

The subsequent sections of this paper are organized as follows: In Section 2, we introduce the adaptive discretization of the Milstein method and investigate the details of the adaptive step-size algorithm. In Section 3, we describe some simulation experiments to verify the efficiency of the algorithm. At the end, in Section 4 a conclusion is drawn.

2 The Proposed Adaptive Algorithm

In general, any variable step size algorithm consists of the following three basic components [18]:

- *Numerical method*,
- *Step-size selection mechanism*,
- *Step-size control mechanism*.

Inspired by the fixed step-size algorithm proposed in [9] for (3), a variable time discretization of the Milstein method is used to drive the numerical solution of (3) in the proposed adaptive algorithm. To select and control the step-sizes in the proposed algorithm, a local error estimation is obtained using some reminder terms of the Stratonovic-Taylor expansion in the context of the Milstein method. In the following, we will describe the details of this procedure.

2.1 Milstein method with variable step-sizes

Consider the following general SDE in Itô form:

$$dX(t) = f(X,t)dt + g(X,t)dW, \tag{5}$$

where $f, g \in C^2(\mathbb{R})$ [4]. The Stratonovic form of (5) is:

$$dX(t) = \bar{f}(X,t)dt + g(X,t)odW, \tag{6}$$

in which $\bar{f} = f - \frac{1}{2}gg'$.

The Milstein method for the solution of (6) is defined as follows:

$$X_{n+1} = X_n + \Delta W g(X_n) + h\bar{f}(X_n) + \frac{1}{2}g(X_n)g'(X_n)(\Delta W)^2 + R, \tag{7}$$

where h and ΔW are the step-size and the corresponding Wiener increment, respectively, and R denotes the reminder terms, which is $O(h^{\frac{3}{2}})$.

Now, assuming a variable step-size partition of an interval $[0,\ T]$, such that $0 = t_0 < t_1 < \ldots < t_N = T$, the Milstein method for (3) is given by

$$x_1^i = x_1^{i-1} + x_1^{i-1}\left[\delta_1 - b_{11}x_1^{i-1} - b_{12}x_1^{i-1-\left[\frac{\tau_1((i-1)h_i)}{h_i}\right]} - b_{13}x_2^{i-1-\left[\frac{\tau_2((i-1)h_i)}{h_i}\right]}\right]h_i$$

$$+\mu_1 x_1^{i-1}(x_1^{i-1}-x_1^\star)\Delta W_1^i+\frac{\mu_1^2}{2}x_1^{i-1}(2x_1^{i-1}-x_1^\star)(x_1^{i-1}-x_1^\star)[(\Delta W_1^i)^2-h_i], \tag{8}$$

$$x_2^i=x_2^{i-1}+x_2^{i-1}\left[\delta_2-b_{21}x_2^{i-1}-b_{22}x_2^{i-1-[\frac{\tau_3((i-1)h_i)}{h_i}]}-b_{23}x_1^{i-1-[\frac{\tau_4((i-1)h_i)}{h_i}]}\right]h_i$$

$$+\mu_2 x_2^{i-1}(x_2^{i-1}-x_2^\star)\Delta W_2^i+\frac{\mu_2^2}{2}x_2^{i-1}(2x_2^{i-1}-x_2^\star)(x_2^{i-1}-x_2^\star)[(\Delta W_2^i)^2-h_i],$$

where $h_i=t_i-t_{i-1}$ for $i=1,\ldots,n$, and ΔW_1^i and ΔW_2^i are Wiener increments $\Delta W_j^i=\sqrt{h_i}N(0,1)$, for $i=1,\ldots,n$ and for $j=1,2$. Due to the time delayed nature of (3), the following initial solutions are also assumed:

$$x_1^i=\zeta_1(ih_0),\quad x_2^i=\zeta_2(ih_0),\quad i=-m,\ldots,-1,0, \tag{9}$$

with partition $-\overline{\tau}=t_{-m}<\ldots<t_{-1}<t_0=0$ of the interval $[-\overline{\tau},\ 0]$, where h_0 is a predetermined fixed value, $\overline{\tau}=\max\limits_{i=1,2,3,4}\sup_{t\geq 0}\{\tau_i(t)\}\geq 0$ and $m=\overline{\tau}/(h_0)$.

As illustrated in Figure 1, all step-sizes in the interval $[-\overline{\tau},\ 0]$ have the fixed value h_0, while the step sizes in the interval $[0,\ T]$ are variable.

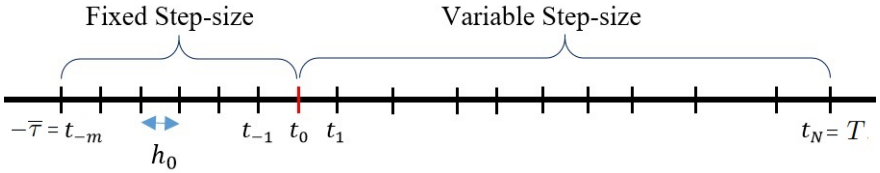

Figure 1: Time partition on the interval $[-\overline{\tau},\ T]=[-\overline{\tau},\ 0]\cup[0,\ T]$.

2.2 Mechanism for selecting step-size

In the proposed algorithm, the following generalization of the standard step-size selection mechanism from the literature on ordinary differential equations is used to select the next step-size:

$$h_{\text{new}}=h_{\text{old}}\times\min\left(\text{facmax},\max\left(\text{facmin},\text{fac}\times\left(\frac{\text{TOL}}{EST}\right)^{\frac{1}{p+\frac{1}{2}}}\right)\right), \tag{10}$$

where h_{old} is the accepted step-size in the previous step, TOL is the tolerance threshold calculated for the current step, EST is the local error estimate, p is the global order of the procedure, fac $\in(0,1)$ is a safety factor for the possibility of reducing the step-size in each step of the algorithm, and facmin and facmax are the minimum and maximum scale coefficients of the step-size, respectively [3].

2.2.1 Drift and diffusion based local error estimation

In each step of the presented algorithm, two different approximations of the local truncation error, EST, based on drift and diffusion coefficient functions, are calculated using the reminder term R of (7):

$$R = \underbrace{J_{10}\bar{f}'g + J_{01}g'\bar{f} + \frac{1}{6}(\Delta W)^3 g''^2 + \frac{1}{6}(\Delta W)^3 g'^2 g}_{O(h^{\frac{3}{2}})} + \underbrace{\frac{1}{2}h^2\bar{f}'\bar{f}}_{O(h^2)} + \dots, \tag{11}$$

where for simplicity (X_n) has been removed [7]. The multiple Stratonovic integrals J_{01} and J_{10} in (11) are defined by

$$\begin{aligned} J_{01} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} ds_1 \, o \, dW, \\ J_{10} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} o \, dW \, ds_1, \end{aligned} \tag{12}$$

and are calculated using numerical integration methods.

Neglecting the higher order terms in (11), we propose the following local error estimates concerning the diffusion and drift coefficient functions, respectively

$$\begin{aligned} E_{diff}(X_n, h, \Delta W) := \|&J_{10}\bar{f}'(X_n)g(X_n) + J_{01}g'(X_n)\bar{f}(X_n) \\ &+ \frac{1}{6}(\Delta W)^3 g''^2(X_n) + \frac{1}{6}\Delta W^3 g'(X_n)^2 g(X_n)\|, \end{aligned} \tag{13}$$

$$E_{drift}(X_n, h) := \frac{1}{2}h^2\|\bar{f}'(X_n)\bar{f}(X_n)\|, \tag{14}$$

where $\|\cdot\|$ denotes a vector norm.

Remark 1. Since the computation of $\bar{f}$ and $\bar{f}'$ requires the computation of the first and second derivatives of the diffusion coefficient, respectively, and the diffusion function in (3) is very simple, the computational cost of $g'(X_n)$ and $g''(X_n)$ is very low. Therefore, all the first four terms of R were used in the definition of E_{diff}.

In each step of the presented variable step size algorithm, four different choices of new step sizes are proposed, one of which is selected by comparing the values of E_{diff} and E_{drift}, to continue the numerical procedure. When E_{diff} is greater than E_{drift}, E_{drift} is used instead of EST to determine the new step-size in (10), and vice versa, when E_{diff} is less than E_{drift}, E_{diff} is used instead of EST. Algorithm 1 illustrates more details of the proposed new step-size.

2.3 Controlling the step size

In each step of the algorithm, the new step-size h_{new} and the corresponding Wiener expansion ΔW are accepted if

$$EST(t_n, h_{\text{new}}, \Delta W) \leq \sigma(TOL), \tag{15}$$

where TOL is a tolerance threshold, $\sigma(TOL)$ is an upper bound on the local error, and $EST(t_n, h_{\text{new}}, \Delta W)$ is an approximation of the local error obtained by the following proposition.

Proposition 1. Let $X_{n-1} = X_{t_{n-1}}$ and $X_n = X_{t_n}$ be two currently approximated solutions computed in $n-1$ and n^{th} steps, respectively, of a numerical method with the order of accuracy p. Then, for each function $f : \mathbb{R}^d \to \mathbb{R}^d$, the local error of the numerical method is as follows

$$EST(t_n, h_{\text{new}}, \Delta W) \approx \frac{h_{\text{new}}^{p+1}}{\left((h_{n-1}+h_{\text{new}})^{p+1} - h_{\text{new}}^{p+1}\right)} \|E[f({}^1X_{t_n+h_{\text{new}}})] - E[f({}^2X_{t_n+h_{\text{new}}})]\|. \tag{16}$$

where, ${}^1X_{t_n+h_{\text{new}}}$ and ${}^2X_{t_n+h_{\text{new}}}$ are two numerical approximations of the analytical solution $X(t_n + h_{new})$ calculated in two different ways. ${}^1X_{t_n+h_{\text{new}}}$ is calculated starting from X_n, with step-size h_{new}, and ${}^2X_{t_n+h_{\text{new}}}$ is calculated starting from X_{n-1} with step size $h_{n-1}+h_{\text{new}}$. Figure 2 demonstrates these different possibilities.

Proof. Based on the definitions of the numerical approximations ${}^1X_{t_n+h_{\text{new}}}$ and ${}^2X_{t_n+h_{\text{new}}}$, the following are satisfied

$$E[f(X_{t_n+h_{new}})] - E[f({}^1X_{t_n+h_{\text{new}}})] = h_{\text{new}}^{p+1} E[\Psi(t_n, X_n, h_{\text{new}}; f)], \tag{17}$$

$$E[f(X_{t_n+h_{\text{new}}})] - E[f({}^2X_{t_n+h_{\text{new}}})] = (h_{n-1}+h_{\text{new}})^{p+1} E[\Psi(t_{n-1}, X_{n-1}, h_{n-1}+h_{\text{new}}; f)], \tag{18}$$

where $E[\cdot]$ denotes the mathematical expectation and $\Psi : I \times \mathbb{R}^d \times \mathbb{R}_+ \longrightarrow \mathbb{R}^d$ is an expression containing elementary differentials that is of big-oh order one, $O(1)$, when h_{new} tends to zero [13].

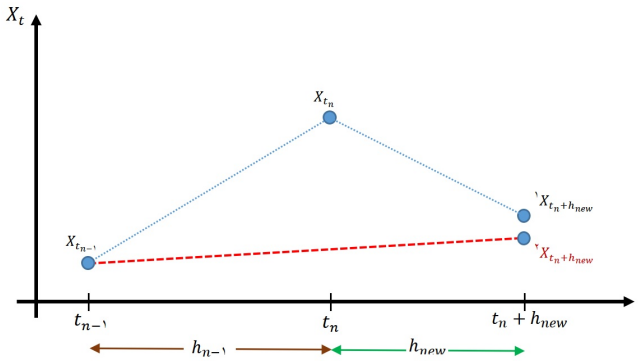

Figure 2: Two different numerical approximations of $X(t_n + h_{\text{new}})$.

Now, considering the continuity of the preliminary differentials and assuming that the behavior of Ψ for h_{new} is such that the expressions on the right-hand side of the relations (17) and (18) are asymptotically approximately equal, the difference of the relations (17) and (18) leads, by some algebraic calculations, to the following equality

$$\frac{h_{\text{new}}^{p+1}}{\left((h_{n-1}+h_{\text{new}})^{p+1} - h_{\text{new}}^{p+1}\right)} \|E[f({}^1X_{t_n+h_{\text{new}}})] - E[f({}^2X_{t_n+h_{\text{new}}})]\| = h_{\text{new}}^{p+1} \|E[\Psi(t_n, X_n, h_{\text{new}}; f)]\| + \mathcal{O}(h_{\text{new}}^{p+2}). \tag{19}$$

From (19) the proof is now complete. □

Remark 2. In addition to the main steps mentioned in the proposed variable step size algorithm, we set two predefined parameters $h_{\min}$ and $h_{\max}$ to control the upper and lower bounds of the step size in each step of Algorithm 1.

Remark 3. Based on the definition of the numerical solution ${}^1X_{t_n+h_{\text{new}}}$, the tolerance threshold TOL in (15) is calculated as follows:

$$TOL = \text{Atol} + \text{Rtol} \times \max\{\|E[{}^1X_{t_n+h_{\text{new}}}]\|, \|E[X_n]\|\}. \tag{20}$$

where Atol and Rtol are the given absolute and relative error tolerance thresholds, respectively. Algorithm 1 shows the main steps of the proposed adaptive strategy. In this algorithm, we define $\sigma(TOL) = 0.85\,TOL$.

In addition, there are two predefined parameters dtmin and dtmax which control the upper and lower bounds of the step size. That is, in each iteration of the proposed algorithm, a step size with a minimum length of dtmin is always selected to go to the next iteration, and therefore the algorithm does not get stuck at a specific time point. This guarantees that it will surely converge to the endpoint of the simulation domain.

2.4 Economical saving of values of the Wiener increments

If the inequality (15) in Algorithm 1 is not satisfied, the step-size h_{new}, the Wiener increment ΔW and the approximate solution ${}^1X_{t_n+h_{\text{new}}}$ should be discarded, and the calculations must be repeated for a smaller step-size. For new calculations, we may need intermediate values for the Wiener increments. To avoid repeating the calculations already performed, it is recommended to save the intermediate values of the Wiener increments and use them again in the future.

Assume that the Wiener increment $\Delta W_h^t = W(t+h) - W(t)$, related to the two points t and $t+h$, has been calculated for $t > 0$, and $h > 0$. Let $h = h_1 + h_2$ for $h_1 > 0$ and $h_2 > 0$, then the intermediate values of Wiener increments $\Delta W_{h_1}^t = W(t+h_1) - W(t)$ and $\Delta W_{h_2}^{t+h_1} = W(t+h_1+h_2) - W(t+h_1)$, can be simulated as follows:

$$\Delta W_{h_1}^t = \frac{h_1}{h}\Delta W_h^t + \sqrt{\frac{h_1 h_2}{h}}\xi,$$

and

$$\Delta W_{h_2}^t = \frac{h_2}{h}\Delta W_h^t - \sqrt{\frac{h_1 h_2}{h}}\xi,$$

where ξ is a normal vector random variable [12]. As the above formulas show, to compute $\Delta W_{h_1}^t$, $\Delta W_{h_2}^t$ and $W(t+h_1)$, it is not necessary to recalculate $W(t)$ and $W(t+h)$. Using the value of $\Delta W_{h_1}^t$, we obtain the value of $W(t+h_1)$ as $W(t+h_1) = \Delta W_{h_1}^t + W(t)$. In the implementation of Algorithm 1, we define a two-dimensional array to save the pair $(t, W(t))$ at each time t.

3 Numerical Results

Here, some test problems are examined to check the applicability of the proposed algorithm. In the absence of an analytical solution for the system (3), we compute an approximation using a very small fixed step-size h_{ref} by the numerical method (8), which we call "*Reference*" solution. Moreover, starting

Algorithm 1 Main steps of the proposed adaptive step-size algorithm

Inputs: start time t_0; initial stepsize h_0; error tolerance $TOL > 0$; absolute error $Atol$ and relative error $Rtol$.

Output: Numerical solution of (3) at discrete points on interval $[0, T]$.

Step 0: Compute the initial solution by (9).

Step 1: Set $n = 1$ and $t_n = t_{n-1} + h_{n-1}$.

Step 2: Compute X_n using (8) as an estimation of $X(t_n)$.

Step 3: Compute $EST(t_n, h_{new}, \Delta W)$ using (16) and TOL using (20).

Step 4: If $EST(t_n, h_{new}, \Delta W) \leq 0.85 \times TOL$ then

$h_n = h_{new}$,

$X_{n+1} = {}^1X_{t_n+h_{new}}$,

If $E_{drift}(X_n, h_{new}) \leq E_{diff}(X_n, h_{new}, \Delta W)$ then

$$h_{\text{new}} = h_n \times \min\left(1.9, \max\left(0.6, 0.8 \times \left(\frac{\text{TOL}}{E_{\text{drift}}(X_n, h_{\text{new}})}\right)^{\frac{1}{2}}\right)\right),$$

else

$$h_{\text{new}} = h_n \times \min\left(1.65, \max\left(0.6, 0.8 \times \left(\frac{\text{TOL}}{E_{\text{diff}}(X_n, h_{\text{new}})}\right)^{\frac{2}{3}}\right)\right).$$

end

else

If $E_{drift}(X_n, h_{new}) \leq E_{diff}(X_n, h_{new}, \Delta W)$ then

$$h_{\text{new}} = h_{n-1} \times \min\left(0.99, \max\left(0.1, 0.8 \times \left(\frac{\text{TOL}}{E_{\text{drift}}(X_n, h_{\text{new}})}\right)^{\frac{1}{2}}\right)\right),$$

else

$$h_{\text{new}} = h_{n-1} \times \min\left(0.99, \max\left(0.1, 0.8 \times \left(\frac{\text{TOL}}{E_{\text{diff}}(X_n, h_{\text{new}})}\right)^{\frac{2}{3}}\right)\right).$$

end

If $n = 1$ then

Set $h_0 = h_{new}$ and go to Step 1,

else

Set $h_n = h_{new}$ and go to Step 3.

end

end

Step 5: Set $h_{n+1} = \min\{T - t_n, h_{new}\}$, $n = n + 1$ and $t_n = t_{n-1} + h_{n-1}$.

Step 6: Compute X_n using (8) as an estimation of $X(t_n)$.

Step 7: If $t_n < T$ then go to Step 3, else **Stop**.

from a predefined step size h_0, we compute another numerical solution of (3) by applying Algorithm 1, which we name it "*Adaptive*" solution.

The total number of accepted and rejected steps of this scenario is used to calculate the following fixed step size

$$h_{\text{fixed}} = \frac{T - t_0}{S_{\text{tried}}}, \tag{21}$$

where the symbol S_{tried} indicates the total number of steps, including rejected and accepted steps during the execution of the adaptive algorithm. The calculated value of h_{fixed}, is now used to create another numerical solution, which we refer to as the "*Fixed*" solution.

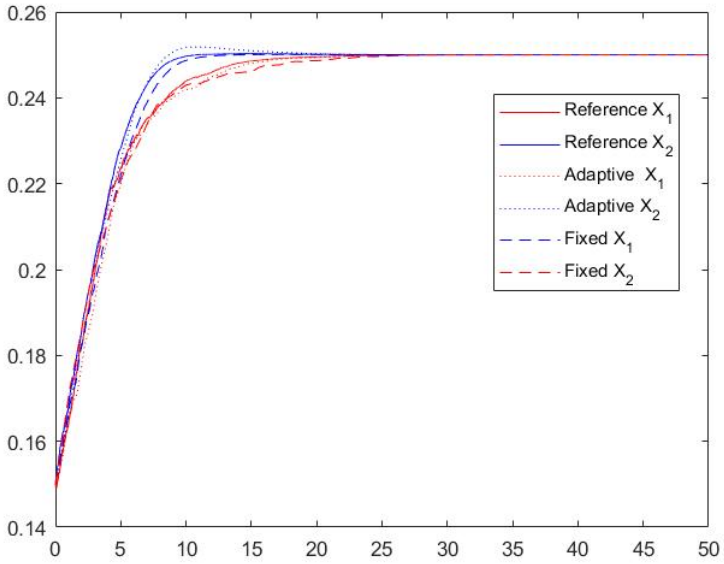

Figure 3: Dotted line: solution with adaptive step-size, dashed line: solution with fixed step-size, solid-line: reference solution with small step-size, for Example 1.

Example 1. In this example, we use the simulation parameters $t_0 = 0$, $h_0 = 10^{-3}$, Atol $= 0.001$, Rtol $= 0.0005$, dtmin $= h_0$, dtmax $= 0.1$, $\delta_1 = \delta_2 = 0.3$, $b_{11} = 0.95$, $b_{12} = 0.15$, $b_{13} = 0.1$, $b_{21} = 0.9$, $b_{22} = 0.2$, $b_{23} = 0.1$, $\tau_1 = \frac{1}{3}(5 + \sin t)$, $\tau_2 = \frac{1}{4}(4 + \sin \frac{t}{2})$, $\tau_3 = \frac{1}{2}(6 + 4 \sin \frac{t}{3})$, $\tau_4 = \frac{1}{5}(8 + \sin \frac{t}{4})$, $\zeta_1(\theta) = \zeta_2(\theta) = 0.15$ for $\theta \in [-5, 0]$, $x_1^* = x_2^* = 0.25$, $\mu_1^2 = 0.8$ and $\mu_2^2 = 0.2$. The resulted values of S_{tried} and h_{fixed} are 65 and $\frac{T}{65}$ respectively. As Figure 3 shows, the solution (x_1, x_2) tends to the equilibrium state $x^* = (x_1^*, x_2^*)$, and the proposed adaptive algorithm is more accurate than the fixed step size to the reference solution. In addition, the number of iterations of the function calculations in the variable step size algorithm is lower compared to the fixed step size one, which leads to a reduction in the computational cost and, as a result, an increase in the algorithm execution speed.

Example 2. To demonstrate the influence of user-adjusted parameters on the performance of our proposed algorithm, in this example, numerical results obtained by choosing different values of the maximum step size control parameter, dtmax, and accuracy parameter, *Rtol* are reported. Other parameters are set as in Example 1, except for Atol.

In the literature on adaptive step-size methods, one usually tends to satisfy the following inequality

$$\|\text{Exact solution} - \text{calculated solution}\| < Rtol \times |\text{exact solution}| + Atol.$$

Using this fact, in the first iteration of Algorithm 1, starting with an initial guess, a suitable value for the *Atol* parameter is calculated. In this order, if the inequality $Atol < Rtol * (\text{calculated solution})$ is

established, *Atol* is accepted; otherwise new value $Atol = Rtol * (\text{calculated solution})$ is computed and the first step of Algorithm 1 is re-run. This process is iterated until convergence.

As Table 1 shows, by decreasing the value of the dtmax, the S_{tried} increases but the number of step-size rejection cases decreases. Also, by decreasing the value of the *Rtol*, the S_{tried} and the number of step-size rejection cases increases. In all cases, the proposed adaptive algorithm is more accurate and faster than the fixed step size.

Table 1: Numerical results for different values of *Rtol* and dtmax.

dtmax	*Rtol*	*Atol*	S_{tried}	S_{reject}
$\frac{T}{75}$	0.00001	$1.125e-06$	265	21
	0.0001	$1.125e-05$	135	6
	0.001	$1.126e-04$	106	1
$\frac{T}{100}$	0.00001	$1.125e-06$	279	14
	0.0001	$1.125e-05$	151	3
	0.001	$1.126e-04$	128	1
$\frac{T}{200}$	0.00001	$1.125e-06$	344	19
	0.0001	$1.125e-05$	228	2
	0.001	$1.126e-04$	223	2
$\frac{T}{500}$	0.00001	$1.125e-06$	569	13
	0.0001	$1.125e-05$	514	4
	0.001	$1.126e-04$	512	2

4 Conclusion

This paper presented a variable step size strategy for a stochastic Lotka-Volterra competitive system with time delays. The strategy utilizes the Milstein technique for numerical solution and employs two local error estimators based on the diffusion and drift components of the system to determine and control the step sizes. The numerical results exhibit higher accuracy of the proposed strategy compared to a fixed step size mechanism when compared to a reference solution. It would be interesting for future investigations to explore the combination of the proposed adaptive strategy with other well-known finite difference methods for solving stochastic models. Additionally, the method presented in this work

has potential applicability to other biological systems, such as the HIV/AIDS system, the Gilpin-Ayala system, cancer metastasis models, etc.

Declarations

Availability of Supporting Data

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Competing Interests

The authors declare that they have no competing interests relevant to the content of this paper.

Authors' Contributions

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