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On Constraint Qualifications and Optimality Conditions in Nonsmooth Semi-infinite Optimization

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Abstract. The primary objective of this paper is to enhance several well-known geometric constraint qualifications and necessary optimality conditions for nonsmooth semi-infinite optimization problems (SIPs). We focus on defining novel algebraic Mangasarian-Fromovitz type constraint qualifications, and on presenting two Karush-Kuhn-Tucker type necessary optimality conditions for nonsmooth SIPs defined by locally Lipschitz functions. Then, by employing a new type of generalized invex functions, we present sufficient conditions for the optimality of a feasible point of the considered problems. It is noteworthy that the new class of invex functions we considered encompasses several classes of invex functions introduced previously. Our results are based on the Michel-Penot subdifferential.

Keywords. Semi-Infinite optimization, Constraint qualification, Optimality conditions, Michel-Penot subdifferential.

MSC. 90C34; 90C40; 49J52.

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1 Introduction

Inspired by [13], we consider the following semi-infinite programming problem (SIP):

$$\begin{array}{ll} Minimize & f(x), \\ \text{s.t.} & \\ f_t(x) \leq 0, & t \in T, \end{array}$$

where T is an arbitrary index set, and all the emerging functions f and f_t for $t \in T$ are locally Lipschitz from $\mathbb{R}^n$ to $\mathbb{R}$, not necessarily differentiable. If the index set T is finite, the SIP reduces to the nonsmooth standard optimization problem (OP).

The Karush-Kuhn-Tucker (KKT) necessary optimality conditions for OP can be derived using different constraint qualifications (CQs). For an investigation and comparison of these CQs in both smooth and non-smooth scenarios, refer to [5, 24]. We recall that the CQs are generally divided into two categories: geometric CQs and algebraic CQs. Geometric CQs are generally weaker than algebraic CQs, but their main drawback is that verifying their validity requires the calculation of a special tangent cone of the feasible set of OP, which is typically very difficult. In contrast, algebraic CQs have wider and simpler applications than geometric CQs.

The KKT necessary optimality conditions for SIP have been studied extensively in the literature, covering linear [6], convex and D.C. (difference of convex functions) [12], smooth [3, 4, 9], and nonsmooth cases [7, 11, 13, 14, 15, 16, 17, 18, 19, 28], as well as nonsmooth fractional and minimax cases [1, 25, 26].

In the existing works, almost all the CQs defined for SIP are geometric, and to the best of our knowledge, there are only a few references considering algebraic CQs for SIP. The first goal of this paper is to introduce two algebraic CQs for SIP in the Mangasarian-Fromovitz type. The second goal is to present necessary optimality conditions for SIP under these CQs and compare them with geometric CQs introduced in [13]. The third goal is to introduce a new category of generalized convex functions, where the necessary KKT condition for optimality becomes a sufficient condition.

The structure of the subsequent sections is as follows. In Section 2, we define the required definitions, theorems and relations of non-smooth analysis. In Section 3, we will introduce various constraint qualifications for nonsmooth SIP and present relations between the defined constraint qualifications and necessary optimality conditions. Section 4 focuses on sufficient optimality conditions for SIP, and Section 5 contains a short conclusion.

2 Notations and Preliminaries

In this section, we briefly introduce some notations, basic definitions, and standard preliminaries that will be used throughout the text.

The standard inner product of two vectors $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y\rangle$, and the zero vector of $\mathbb{R}^n$ is denoted by 0_n. Given a nonempty set $D \subseteq \mathbb{R}^n$, the notations $\overline{D}$, $conv(D)$, and $cone(D)$ represent the closure of D, the convex hull of D, and the convex cone generated by D (containing the origin), respectively.

Theorem 1. [5]. Let $D \subseteq \mathbb{R}^n$ be given:

i. If D is compact, then $conv(D)$ is also compact.

ii. If D is finite, then $cone(D)$ is closed.

It is established in [5] that if $\Pi := \{C_i \mid i \in I\}$ is a collection of convex sets in $\mathbb{R}^n$, then:

$$conv\Big(\bigcup_{i\in I} C_i\Big) = \Big\{ \sum_{r=1}^{n+1} \mu_r c_{i_r} \mid c_{i_r} \in C_{i_r},\ \mu_r \geq 0,\ \sum_{r=1}^{n+1} \mu_r = 1 \Big\}. \tag{1}$$

Let $\vartheta : \mathbb{R}^n \to \mathbb{R}$ be a convex function, i.e.,

$$\vartheta\big(\mu x + (1-\mu)y\big) \leq \mu\vartheta(x) + (1-\mu)\vartheta(y), \qquad \forall x, y \in \mathbb{R}^n,\ \mu \in [0,1],$$

the subdifferential of ϑ at $x_0 \in \mathbb{R}^n$ is defined as:

$$\partial\vartheta(x_0) := \{\xi \in \mathbb{R}^n \mid \vartheta(x) - \vartheta(x_0) \geq \langle \xi, x - x_0\rangle,\ \forall x \in \mathbb{R}^n\}.$$

Theorem 2. [5]. If the convex function $\vartheta : \mathbb{R}^n \to \mathbb{R}$ attains its minimum on $\mathbb{R}^n$ at $x_0 \in \mathbb{R}^n$, then $0_n \in \partial\vartheta(x_0)$.

The function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz if for each $x_0 \in \mathbb{R}^n$, there exist a neighborhood U_{x_0} of x_0 and a positive number $L^{U_{x_0}} > 0$ such that

$$|\varphi(x) - \varphi(y)| \leq L^{U_{x_0}} \|x - y\|, \quad \forall x, y \in U_{x_0}.$$

Let $x_0 \in \mathbb{R}^n$ and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Michel-Penot (M-P) directional derivative of φ at x_0 in the direction $\nu \in \mathbb{R}^n$ introduced in [22] is given by:

$$\varphi^{\text{MP}}(x_0; \nu) := \sup_{w \in \mathbb{R}^n} \limsup_{\alpha \downarrow 0} \frac{\varphi(x_0 + \alpha\nu + \alpha w) - \varphi(x_0 + \alpha w)}{\alpha},$$

and the M-P subdifferential of φ at x_0 is defined as:

$$\partial_{\text{MP}}\varphi(x_0) := \big\{\xi \in \mathbb{R}^n \mid \langle \xi, \nu\rangle \leq \varphi^{\text{MP}}(x_0; \nu) \quad \text{for all } \nu \in \mathbb{R}^n\big\}.$$

The M-P subdifferential is a natural generalization of the derivative, as it is known (see [22]) that when the function ψ is differentiable at x_0, then $\partial_{\text{MP}}\psi(x_0) = \{\nabla\psi(x_0)\}$. Furthermore, $\partial_{\text{MP}}\vartheta(x_0) = \partial\vartheta(x_0)$ for the convex function $\vartheta : \mathbb{R}^n \to \mathbb{R}$.

The following theorem summarizes some important properties of the M-P directional derivative and the M-P subdifferential from [22, 23] which are widely used in the subsequent discussions.

Theorem 3. [22]. Let φ_1 and φ_2 be locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$, and $x_0 \in \mathbb{R}^n$. Then, the following assertions hold:

i. The following equalities and inclusions are valid:

$$\varphi_1^{\text{MP}}(x_0; \nu) = \max\big\{ \langle \xi, \nu\rangle \mid \xi \in \partial^{\text{MP}}\varphi_1(x_0)\big\}, \tag{2}$$

$$\partial_{\text{MP}}\big(\max\{\varphi_1, \varphi_2\}\big)(x_0) \subseteq conv\big(\partial_{\text{MP}}\varphi_1(x_0) \cup \partial_{\text{MP}}\varphi_2(x_0)\big), \tag{3}$$

$$\partial_{\text{MP}}(\alpha_1\varphi_1 + \alpha_2\varphi_2)(x_0) \subseteq \alpha_1\partial_{\text{MP}}\varphi_1(x_0) + \alpha_2\partial_{\text{MP}}\varphi_2(x_0), \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}. \tag{4}$$

ii. The function $\nu \to \varphi_1^{\text{MP}}(x_0; \nu)$ is finite, positively homogeneous, and subadditive on $\mathbb{R}^n$, and

$$\partial\big(\varphi_1^{\text{MP}}(x_0; \cdot)\big)(0_n) = \partial_{\text{MP}}\varphi_1(x_0). \tag{5}$$

iii. $\partial_{\text{MP}}\varphi_1(x_0)$ is a nonempty, convex, and compact subset of $\mathbb{R}^n$.

3 Qualification and Necessary Conditions

As a starting point, we assume that the feasible set of SIP is non-empty:

$$\Omega := \big\{x \in \mathbb{R} \mid f_t(x) \leq 0, \ \ \forall t \in T\big\} \neq \emptyset.$$

For a given $\hat{x} \in \Omega$, the index set of all active constraints at $\hat{x}$ is denoted by

$$T(\hat{x}) := \big\{t \in T \mid f_t(\hat{x}) = 0\big\},$$

Additionally, we define the function

$$\Psi(x) := \sup_{t\in T} f_t(x), \quad \forall x \in \Omega. \tag{6}$$

One reason for the difficulty in extending results from OP to the SIP is that if $T = \{1, \dots, m\}$, the sup-function $\Psi(\cdot)$ reduces to the max-function, and hence, it is locally Lipschitz. In this case, we have

$$\partial_{\text{MP}}\Psi(x) \subseteq conv\Big(\bigcup_{t\in T(x)} \partial_{\text{MP}} f_t(x)\Big), \quad \forall x \in \Omega,$$

but these properties do not necessarily hold for the SIP.

The following concept, known as the Pshenichnyi-Levin-Valadire (PLV) property, was introduced by Kanzi [11] for the first time:

Definition 1. We say that the SIP has the PLV property at $\hat{x} \in \Omega$, if $\Psi(\cdot)$ is Lipschitz around $\hat{x}$, and

$$\partial_{\text{MP}}\Psi(\hat{x}) \subseteq conv\Big(\bigcup_{t\in T(\hat{x})} \partial_{\text{MP}} f_t(\hat{x})\Big).$$

Remark 1. Reference [11] presents an interesting sufficient condition that ensures the satisfaction of the PLV property.

From [5], we recall that for a differentiable OP with $T = \{1, 2, \dots, m\}$, the Mangasarian-Fromovitz CQ (MFCQ) holds at $\hat{x}$ if there exists a vector $u^* \in \mathbb{R}$ such that

$$\langle \nabla f_t(\hat{x}), u^*\rangle < 0, \quad \forall t \in T(\hat{x}).$$

It can be shown that this condition is equivalent to the following implication (see, e.g., [5]):

$$\sum_{t\in T(\hat{x})} \mu_t \nabla f_t(\hat{x}) = 0_n, \quad \mu_t \geq 0 \ \ \forall t \in T(\hat{x}) \ \ \Rightarrow \ \mu_t = 0 \ \ \ \forall t \in T(\hat{x}).$$

We now extend the MFCQ to SIP in two different forms.

Definition 2. Suppose that the PLV property holds at a point $\hat{x} \in \Omega$. We say that SIP satisfies:

$i.$ the strong MFCQ (SMFCQ) at $\hat{x}$ if we can find a vector $u^* \in \mathbb{R}^n$ such that:

$$\langle \xi_t, u^*\rangle < 0, \quad \forall \xi_t \in \partial_{\text{MP}} f_t(\hat{x}), \ \forall t \in T(\hat{x}). \tag{7}$$

$ii.$ the weak MFCQ (WMFCQ) at $\hat{x}$ if for each finite index set $T^* \subseteq T(\hat{x})$, the following implication is true:

$$0_n \in \sum_{t\in T^*} \mu_t \partial_{\text{MP}} f_t(\hat{x}), \;\; \mu_t \geq 0, \;\; \forall t \in T^* \;\Rightarrow\; \mu_t = 0, \quad \forall t \in T^*. \tag{8}$$

We observe that there is no direct implication between the PLV property and the conditions (7) and (8). The PLV property may hold for any finite index set T, but SIP may still not satisfy (7) or (8). Conversely, the following example demonstrates the problem that satisfies both the (7) and (8) at $\hat{x} = 0$, but the PLV property does not hold at this point.

Example 1. Let $T = \mathbb{N}$, $\hat{x} = 0$, $f(x) = |x|$, and

$$f_t(x) = \begin{cases} 10x - \frac{5}{t+1}, & \text{if} \quad t \text{ is odd}, \\ 12x, & \text{if} \quad t = 2, \\ 14x - \frac{3}{t}, & \text{if} \quad t \geq 4 \text{ and } t \text{ is even}. \end{cases}$$

Then, $T(\hat{x}) = \{2\}$, $\partial_{\text{MP}} f_2(\hat{x}) = \{12\}$, and we have

$$\langle 12, u^* \rangle < 0, \quad \text{for all } u^* \in (-\infty, 0),$$

$$0 \in \mu_2 \partial_{\text{MP}} f_2(\hat{x}) \;\Rightarrow\; 12\mu_2 = 0 \;\Rightarrow\; \mu_2 = 0.$$

Therefore, SIP satisfies both (7) and (8) at $\hat{x}$. However, a simple calculation shows that:

$$\Psi(x) = \begin{cases} 14x, & \text{if} \quad x \geq 0, \\ 10x, & \text{if} \quad x < 0, \end{cases} \quad \Rightarrow$$

$$\partial_{\text{MP}} \Psi(\hat{x}) = [10, 14] \nsubseteq \{12\} = \partial_{\text{MP}} f_2(\hat{x}) = conv\Big(\bigcup_{t\in T(\hat{x})} \partial_{\text{MP}} f_t(\hat{x}) \Big).$$

Hence, the PLV property is not satisfied in this example.

The following theorem establishes the relationship between SMFCQ and WMFCQ at the given feasible point $\hat{x} \in \Omega$.

Theorem 4. Let $\hat{x} \in \Omega$ be given a feasible point. Then, the SMFCQ holds at $\hat{x}$ implies the WMFCQ also holds at $\hat{x}$.

Proof. Since the SMFCQ holds at $\hat{x}$, there exists a vector $u^* \in \mathbb{R}$ such that:

$$\langle \xi_t, u^* \rangle < 0, \qquad \forall \xi_t \in \partial_{\text{MP}} f_t(\hat{x}), \; \forall t \in T(\hat{x}). \tag{9}$$

Assume that there exists a finite index set $T^* \subseteq T$ and the non-negative scalars μ_t as $t \in T^*$ such that

$$0_n \in \sum_{t\in T^*} \mu_t \partial_{\text{MP}} f_t(\hat{x}).$$

Consequently, $\sum_{t\in T^*} \mu_t \xi_t = 0_n$ for some $\xi_t \in \partial_{\text{MP}} f_t(\hat{x})$ as $t \in T^*$. Multiplying both sides of this equation by u^* gives:

$$\sum_{t\in T^*} \mu_t \langle \xi_t, u^* \rangle = \langle 0_n, u^* \rangle = 0.$$

Due to the non-negativity of μ_t and (9), the last equality can only hold when $\mu_t = 0$ for all $t \in T^*$. Therefore , the WMFCQ holds at $\hat{x}$. □

The following definition, introduced by Kanzi and Nobakhtian [16], is required to establish the converse of Theorem 4.

Definition 3. Let $\Box \in \{ <, \le, >, \ge, = \}$, and consider the following inequality or equality system:

$$\Xi := \{\phi_\ell(x) \,\Box\, 0 \mid \ell \in \mathcal{L}\},$$

where $\mathcal{L}$ is an arbitrary index set, and $\phi_\ell : \mathbb{R}^n \to \mathbb{R}$ is a given function for each $\ell \in \mathcal{L}$. For each finite subset $\mathcal{L}_0$ with $\mathcal{L}_0 \subseteq \mathcal{L}$, we define the following subsystem:

$$\Xi_0 := \{\phi_\ell(x) \,\Box\, 0 \mid \ell \in \mathcal{L}_0\}.$$

We say that Ξ is compactable if the non-emptiness of the feasible set of all its finite subsystems Ξ_0 implies the non-emptiness of its own feasible set. In other words, Ξ is compactable if the following assertion holds: "*If Ξ_0 has a solution for each finite index set $\mathcal{L}_0 \subseteq \mathcal{L}$, then Ξ has a solution*."

Obviously, each finite system (i.e., the system Ξ with a finite index set $\mathcal{L}$) is automatically compactable. The following example from Kanzi and Nobakhtian [16] shows that there are some non-compactable systems with an infinite index set $\mathcal{L}$.

Example 2. Let $\mathcal{L} = \mathbb{N}$ and $\phi_\ell(x) = \ell + x \le 0$ for all $\ell \in \mathbb{N}$. It is easy to check that if $\mathcal{L}_0$ is a finite subset of $\mathcal{L}$, then $x^* = -\max\{\ell \mid \ell \in \mathbb{N}\}$ is a solution of Ξ_0, but Ξ has no solution. Thus, Ξ is a non-compactable system.

Theorem 5. Let $\hat{x} \in \Omega$ be given, and $\{f_t^{\text{MP}}(\hat{x}; d) < 0 \mid t \in T(\hat{x})\}$ be a compactable system with respect to d. Then, the WMFCQ implies the SMFCQ at $\hat{x}$.

Proof. Suppose that WMFCQ holds at $\hat{x}$. We first prove that for any given $t_1 \in T(\hat{x})$, there exists a vector $d^* \in \mathbb{R}^n$ such that

$$f_{t_1}^{\text{MP}}(\hat{x}; d^*) < 0. \tag{10}$$

Suppose, on the contrary, that $f_{t_1}^{\text{MP}}(\hat{x}; d) \ge 0$ for all $d \in \mathbb{R}^n$. From this and the fact that $f_{t_1}^{\text{MP}}(\hat{x}; 0_n) = 0$, we understand that $\hat{d} := 0_n$ is a solution for the following optimization problem:

$$\min_{d \in \mathbb{R}^n} f_{t_1}^{\text{MP}}(\hat{x}; d).$$

Since the objective function of the above problem is convex, Theorem 2 deduces that

$$0_n \in \partial\big(f_{t_1}^{\text{MP}}(\hat{x}, \cdot)\big)(0_n) = \partial_{\text{MP}} f_{t_1}(\hat{x}),$$

which contradicts the WMFCQ assumption. Hence, (10) holds for some $d^* \in \mathbb{R}^n$.

Now, we claim that for any two indexes $t_1, t_2 \in T(\hat{x})$, there exists a vector $d^* \in \mathbb{R}^n$ such that

$$\begin{cases} f_{t_1}^{\text{MP}}(\hat{x}; d^*) < 0, \\[2ex] f_{t_2}^{\text{MP}}(\hat{x}; d^*) < 0. \end{cases}$$

Suppose, on the contrary, that $f_{t_2}^{\text{MP}}(\hat{x}; d) \ge 0$ for all vector $d \in \mathbb{R}^n$ satisfying $f_{t_1}^{\text{MP}}(\hat{x}; d^*) < 0$. This implies that $\hat{d} := 0_n$ is a solution to the following convex optimization problem:

$$\begin{array}{rl} Minimize & f_{t_2}^{\text{MP}}(\hat{x};d) \\ \text{s.t.} & \\ & f_{t_1}^{\text{MP}}(\hat{x};d) \leq 0. \end{array}$$

From this problem and (5), we can find some non-negative scalars μ_{t_1} and μ_{t_2} such that $(\mu_{t_1}, \mu_{t_2}) \neq 0_2$ and

$$0_n \in \mu_2 \partial\big(f_{t_2}^{\text{MP}}(\hat{x},\cdot)\big)(0_n) + \mu_1 \partial\big(f_{t_1}^{\text{MP}}(\hat{x},\cdot)\big)(0_n) = \mu_{t_2}\partial^{\text{MP}} f_{t_2}(\hat{x}) + \mu_{t_1}\partial^{\text{MP}} f_{t_1}(\hat{x}),$$

which contradicts WMFCQ.

Now, the mathematical induction concludes that for each finite index set $T^* \subseteq T(\hat{x})$ there exists a vector $d^* \in \mathbb{R}^n$ such that

$$f_t^{\text{MP}}(\hat{x}, d^*) < 0, \quad \text{for all } \; t \in T^*.$$

Finally , the compactable assumption of $\big\{ f_t^{\text{MP}}(\hat{x}; d) < 0 \mid t \in T(\hat{x}) \big\}$ implies that there is a $u^* \in \mathbb{R}^n$ such that

$$f_t^{\text{MP}}(\hat{x}, u^*) < 0, \quad \text{for all } \; t \in T(\hat{x}).$$

Hence, owning to (2) we have:

$$\langle \xi_t, u^* \rangle \leq \max_{\zeta \in \partial_{\text{MP}} f_t(\hat{x})} \langle \zeta, u^* \rangle = f_t^{\text{MP}}(\hat{x}, u^*) < 0, \qquad \forall \xi_t \in \partial_{\text{MP}} f_t(\hat{x}), \; \forall t \in T(\hat{x}),$$

and the proof is complete. □

The following theorem gives a KKT necessary condition at optimal solution of SIP under the WM-FCQ.

Theorem 6. Let $\hat{x}$ be an optimal solution for SIP. If WMFCQ holds at $\hat{x}$, we can find a finite index set $T^* \subseteq T(\hat{x})$ and non-negative scalars μ_t for $t \in T^*$ such that

$$0_n \in \partial_{\text{MP}} f(\hat{x}) + \sum_{t \in T^*} \mu_t \partial_{\text{MP}} f_t(\hat{x}).$$

Proof. According to (6), the feasible set of the SIP can be written as

$$\Omega = \big\{ x \in \mathbb{R}^n \mid \Psi(x) \leq 0 \big\}.$$

Therefore, $\hat{x}$ is an optimal solution for the following optimization problem:

$$\begin{array}{rl} Minimize & f(x) \\ \text{s.t.} & \Psi(x) \leq 0. \end{array}$$

Since the objective function f and the constraint function F are Lipschitz near $\hat{x}$, the well-known Fritz-John necessaty optimality condition [5] concludes that:

$$0_n \in \beta_0 \partial_{\text{MP}} f(\hat{x}) + \beta_1 \partial_{\text{MP}} \Psi(\hat{x}),$$

for some non-negative scalars β_0, β_1 such that $\beta_0 + \beta_1 = 1$. This inclusion and the PLV property imply that:

$$0_n \in \beta_0 \partial_{\mathrm{MP}} f(\hat{x}) + \beta_1 conv\Big(\bigcup_{t\in T(\hat{x})} \partial_{\mathrm{MP}} f_t(\hat{x}) \Big).$$

Hence, we can find a finite index set $T^* \subseteq T(\hat{x})$ and $\gamma_t \geq 0$ for $t \in T^*$ such that $\sum_{t\in T^*} \gamma_t = 1$ and:

$$0_n \in \beta_0 \partial_{\mathrm{MP}} f(\hat{x}) + \beta_1 \sum_{t\in T^*} \gamma_t \partial_{\mathrm{MP}} f_t(\hat{x}).$$

We claim that $\beta_0 \neq 0$. Otherwise, if $\beta_0 = 0$, then $\beta_1 = 1$ by $\beta_0 + \beta_1 = 1$. Thus the above inclusion and WMFCQ assumption would imply $\gamma_t = 0$ for all $t \in \hat{T}$, which contradicts $\sum_{t\in T^*} \gamma_t = 1$, and our claimed is proved. The result is deduced by setting:

$$\mu_t := \frac{\beta_1 \gamma_t}{\beta_0}, \qquad \forall t \in \hat{T}^*.$$

□

As mentioned in the Introduction section, Kanzi [13, Theorem 5] has presented the KKT necessary optimality condition for the SIP under the *first regularity condition*(RC1), defined as follows:

Definition 4. [13, Definition 1] Let $\hat{x} \in \Omega$. We say that the first regularity condition (RC1) holds at $\hat{x}$ if:

$$\Big\{ \nu \in \mathbb{R}^n \mid \langle \nu, \xi \rangle \leq 0, \ \ \forall \xi \in \bigcup_{t\in T(\hat{x})} \partial_{\mathrm{MP}} f_t(\hat{x}) \Big\} \subseteq K_\Omega(\hat{x}),$$

where $K_\Omega(\hat{x})$ denotes the contingent cone of Ω at $\hat{x}$, defined by:

$$K_\Omega(\hat{x}) := \Big\{ \nu \in \mathbb{R}^n \mid \exists t_r \downarrow 0, \ \exists \nu_r \to \nu \text{ such that } \hat{x} + t_r \nu_r \in \Omega, \ \ \forall r \in \mathbb{N} \Big\}.$$

Theorem 7. [13, Theorem 5] Suppose that $\hat{x}$ is an optimal solution of SIP and the RC1 holds at $\hat{x}$. If the following set is closed,

$$cone\Big(\bigcup_{t\in T(\hat{x})} \partial_{\mathrm{MP}} f_t(\hat{x}) \Big), \tag{11}$$

then we can find a finite index set $T^* \subseteq T(\hat{x})$ and non-negative scalars μ_t as $t \in T^*$ such that:

$$0_n \in \partial_{\mathrm{MP}} f(\hat{x}) + \sum_{t\in T^*} \mu_t \partial_{\mathrm{MP}} f_t(\hat{x}).$$

We observe that the restrictive assumption in Theorem **??** is the closedness of the cone expressed in (11). If the active index set $T(\hat{x})$ is finite and the functions f_t for $t \in T(\hat{x})$ are continuously differentiable, then $\bigcup_{t\in T(\hat{x})} \partial_{\mathrm{MP}} f_t(\hat{x})$ is a finite subset of $\mathbb{R}^n$, and hence, $cone\Big(\bigcup_{t\in T(\hat{x})} \partial_{\mathrm{MP}} f_t(\hat{x}) \Big)$ is closed by Theorem 1. However, in the general case where the index set $T(\hat{x})$ is infinite or $\partial_{\mathrm{MP}} f_t(\hat{x})$ contains infinite many elements for some $t \in T(\hat{x})$, the closedness of the cone expressed in (11) is a very restrictive assumption, which reduces the effectiveness of Theorem 7. Another issue in using Theorem 7 is the necessity of calculating the contingent cone $K_\Omega(\hat{x})$, which is generally difficult, especially when Ω is non-convex. These two basic problems reduce the effectiveness of the above theorem and showcase the strength of Theorem 6. The following theorem demonstrates that the RC1 condition is weaker than SMFCQ at each feasible point of the SIP.

Theorem 8. If the SMFCQ holds at $\hat{x} \in \Omega$, then the RC1 condition is satisfied at $\hat{x}$.

Proof. Since the SMFCQ holds at $\hat{x}$, there exists a vector $u^* \in \mathbb{R}$ such that:

$$\langle \xi_t, u^* \rangle < 0, \qquad \forall \xi_t \in \partial_{\text{MP}} f_t(\hat{x}),\ \forall t \in T(\hat{x}). \tag{12}$$

Let $\xi \in conv\Big(\bigcup_{t\in T(\hat{x})} \partial_{\text{MP}} f_t(\hat{x})\Big)$ be given arbitrarily. According to (1), there exist $\xi_{t_1}, \ldots, \xi_{t_{n+1}} \in \bigcup_{t\in T(\hat{x})} \partial_{\text{MP}} f_t(\hat{x})$ and non-negative scalars $\alpha_{t_1}, \ldots, \alpha_{t_{n+1}}$ such that:

$$\xi = \sum_{\ell=1}^{n+1} \alpha_{t_\ell} \xi_{t_\ell}, \qquad \text{and} \qquad \sum_{\ell=1}^{n+1} \alpha_{t_\ell} = 1.$$

Using (12), we have:

$$\langle \xi, u^* \rangle = \Big\langle \sum_{\ell=1}^{n+1} \alpha_{t_\ell} \xi_{t_\ell}, u^* \Big\rangle = \sum_{\ell=1}^{n+1} \alpha_{t_\ell} \langle \xi_{t_\ell}, u^* \rangle < 0, \quad \forall \xi \in conv\Big(\bigcup_{t\in T(\hat{x})} \partial_{\text{MP}} f_t(\hat{x})\Big).$$

This inequality and the PLV property imply that $\langle \xi, u^* \rangle < 0$ for all $\xi \in \partial_{\text{MP}} \Psi(\hat{x})$, and hence, $\Psi^{\text{MP}}(\hat{x}; u^*) < 0$ by (2). Now, employing the definition of M-P subdifferential, we get:

$$\limsup_{\alpha \downarrow 0} \frac{\Psi(\hat{x} + \alpha u^*) - \Psi(\hat{x})}{\alpha} \leq \Psi^{\text{MP}}(\hat{x}; u^*) < 0.$$

Hence, there exists a scalar $\varepsilon > 0$ such that:

$$f_t(\hat{x} + \alpha u^*) \leq \Psi(\hat{x} + \alpha u^*) < \Psi(\hat{x}) \leq 0, \quad \forall\, \alpha \in (0, \varepsilon),\ \forall t \in T.$$

This means that $\hat{x} + \alpha u^* \in \Omega$ for all $\alpha \in (0, \varepsilon)$, and hence $u^* \in K_\Omega(\hat{x})$. Since u^* was arbitrarily chosen to satisfy (12), we have:

$$\Big\{\nu \in \mathbb{R}^n \mid \langle \nu, \xi \rangle < 0,\ \forall \xi \in \bigcup_{t\in T(\hat{x})} \partial_{\text{MP}} f_t(\hat{x})\Big\} \subseteq K_\Omega(\hat{x}).$$

After performing the closure on both sides of the above inclusion, considering the closedness of $K_\Omega(\hat{x})$, and recalling the following equality:

$$\Big\{\nu \in \mathbb{R}^n \mid \langle \nu, \xi \rangle \leq 0,\ \forall \xi \in \bigcup_{t\in T(\hat{x})} \partial_{\text{MP}} f_t(\hat{x})\Big\} = \overline{\Big\{\nu \in \mathbb{R}^n \mid \langle \nu, \xi \rangle < 0,\ \forall \xi \in \bigcup_{t\in T(\hat{x})} \partial_{\text{MP}} f_t(\hat{x})\Big\}},$$

we conclude that:

$$\Big\{\nu \in \mathbb{R}^n \mid \langle \nu, \xi \rangle \leq 0,\ \forall \xi \in \bigcup_{t\in T(\hat{x})} \partial_{\text{MP}} f_t(\hat{x})\Big\} \subseteq \overline{K_\Omega(\hat{x})} = K_\Omega(\hat{x}).$$

Hence, the RC1 condition holds at $\hat{x}$, and the proof is complete. □

4 Sufficient Conditions

To find sufficient conditions, we need the following definition.

Definition 5. Assume that $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ are given functions. A locally Lipschitz function $\hbar : \mathbb{R}^n \to \mathbb{R}$ is said to be (η, ψ)-invex at $x_0 \in \mathbb{R}^n$ if for each $x \in \mathbb{R}^n$ one has:

$$\hbar(x) - \hbar(x_0) \geq \langle \zeta, \eta(x, x_0) \rangle + \psi(x, x_0), \quad \forall \zeta \in \partial_{\text{MP}} \hbar(x_0).$$

Remark 2. From Definition 5, the following special cases arise:

- If $\psi(x, x_0) = 0$ and $\eta(x, x_0) = x - x_0$, we obtain the definition of a convex function.
- If $\psi(x, x_0) = 0$, we obtain the definition of a η-invex function (see [8] in differentiable case, and [20] in nonsmooth case).
- If $\psi(x, x_0) = 0$ and $\eta(x, x_0) = \frac{\vartheta(x,x_0)}{b(x,x_0)}$ for some $\vartheta : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ and $b : \mathbb{R}^n \times \mathbb{R}^n \to (0, +\infty)$, we obtain the definition of a b_ϑ-invex function, introduced in [21].
- If $\psi(x, x_0) = \rho \|x - x_0\|^2$ and $\eta(x, x_0) = x - x_0$ for some $\rho \in [0, +\infty)$, then (η, ψ)-invexity reduces to the definition of a nonsmooth ρ-convex function defined by Vial [27].
- If $\psi(x, x_0) = \rho \|\theta(x, x_0)\|^2$ for some $\rho \in [0, +\infty)$ and $\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \setminus \{0_n\}$, then (η, ψ)-invexity reduces to the definition of a nonsmooth ρ-invex function, introduced by Jeyakumar [10].

From the above notions, it follows that the following theorem extends the corresponding existing results in the literature, see, for instance, [28, Theorem 4.1] and [2, Theorems 4.3 and 4.4].

Theorem 9. Let $\hat{x} \in \Omega$ be a feasible point. Suppose that there exist a finite index set $T^* \subseteq T(\hat{x})$ and non-negative scalars μ_t as $t \in T^*$ such that

$$0_n \in \partial_{\text{MP}} f(\hat{x}) + \sum_{t \in T^*} \mu_t \partial_{\text{MP}} f_t(\hat{x}). \tag{13}$$

If f and f_t as $t \in T^*$ are (η, ψ)-invex functions at $\hat{x}$, then $\hat{x}$ is an optimal solution for SIP.

Proof. According to (13), there exist some $\xi \in \partial_{\text{MP}} f(\hat{x})$ and $\xi_t \in \partial_{\text{MP}} f_t(\hat{x})$ for $t \in T^*$ such that

$$\xi + \sum_{t \in T^*} \mu_t \xi_t = 0_n. \tag{14}$$

Since for each $t \in T^*$ and each $x \in \Omega$, we have $f_t(x) \leq 0 = f_t(\hat{x})$, the (η, ψ)-invexity of f_t at $\hat{x}$ implies that

$$0 \geq f_t(x) - f_t(\hat{x}) \geq \langle \xi_t, \eta(x, \hat{x}) \rangle + \psi(x, \hat{x}),$$

where the last inequality holds by (14). This means

$$\Big(\sum_{t \in T^*} \mu_t \Big) \psi(x, \hat{x}) \leq \langle \xi, \eta(x, \hat{x}) \rangle .$$

This inequality, along with the non-negativity of $\psi(x, \hat{x})$, and the (η, ψ)-invexity of f at $\hat{x}$ leads to the conclusion that

$$0 \leq \Big(1 + \sum_{t \in T^*} \mu_t\Big)\psi(x, \hat{x}) \leq \langle \xi, \eta(x, \hat{x})\rangle + \psi(x, \hat{x}) \leq f(x) - f(\hat{x}),$$

and hence, $f(\hat{x}) \leq f(x)$. Since x was an arbitrary feasible point for the SIP, the last inequality implies that $\hat{x}$ is an optimal solution for the SIP. □

At the final point, we note that similarly to Caristi and Kanzi [2], we can now define (η, ψ)-pseudo-invex and (η, ψ)-quasi-invex functions, receptively, and also obtain a generalization of Theorem 9 as follows.

Definition 6. Assume that $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ are given functions.

i. A locally Lipschitz function $\overline{h} : \mathbb{R}^n \to \mathbb{R}$ is said to be (η, ψ)-pseudo-invex at $x_0 \in \mathbb{R}^n$ if for each $x \in \mathbb{R}^n$ one has:

$$\overline{h}(x) - \overline{h}(x_0) < 0 \;\Rightarrow\; \langle \zeta, \eta(x, x_0)\rangle + \psi(x, x_0) < 0, \quad \forall \zeta \in \partial_{\text{MP}}\overline{h}(x_0).$$

ii. A locally Lipschitz function $\overline{h} : \mathbb{R}^n \to \mathbb{R}$ is said to be (η, ψ)-quasi-invex at $x_0 \in \mathbb{R}^n$ if for each $x \in \mathbb{R}^n$ one has:

$$\overline{h}(x) - \overline{h}(x_0) \leq 0 \;\Rightarrow\; \langle \zeta, \eta(x, x_0)\rangle + \psi(x, x_0) \leq 0, \quad \forall \zeta \in \partial_{\text{MP}}\overline{h}(x_0).$$

Since the proof of the following theorem is similar to Theorem 9, we omit it.

Theorem 10. Let $\hat{x} \in \Omega$ be a feasible point. Suppose that there exists a finite index set $T^* \subseteq T(\hat{x})$ and non-negative scalars μ_t as $t \in T^*$ such that:

$$0_n \in \partial_{\text{MP}} f(\hat{x}) + \sum_{t \in T^*} \mu_t \partial_{\text{MP}} f_t(\hat{x}).$$

If f is (η, ψ)-pseudo-invex and f_t for $t \in T^*$ are (η, ψ)-quasi-invex functions at $\hat{x}$, then $\hat{x}$ is an optimal solution for SIP.

5 Conclusion

In this paper, we have introduced several geometric constraint qualifications for nonsmooth semi-infinite programming problems (SIPs). We have presented the necessary optimality conditions for the SIPs under the proposed constraint qualifications. Furthermore, we have introduced the concept of (η, ψ)-invexity for locally Lipschitz functions, and leveraged this notion to establish a sufficient optimality condition for the SIP. The (η, ψ)-invexity property provides a generalization of convexity that allows for a broader class of functions to satisfy the sufficient optimality condition. This expanded class of functions can lead to improved modeling capabilities and solution methods for a variety of nonsmooth SIP problems encountered in practical applications.

Declarations

Availability of Supporting Data

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