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Mordukhovich Normal Cone of Optimization Problems with Switching Constraints

Arash Farhadi, Sharifeh Rezagholi

Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-4697, Tehran, Iran.

Correspondence:
Sharifeh Rezagholi
E-mail:
rezagholi@pnu.ac.ir

Abstract. This paper examines normal cones of the feasible set for mathematical programming problems with switching constraints (MPSC). Functions involved are assumed to be continuously differentiable. The primary focus is on providing the upper estimate of the Mordukhovich normal cone for the feasible set of MPSCs. First, a constraint qualification, called the "MPSC-No Nonzero Abnormal Multiplier Constraint Qualification", is considered for the problem. Based on this qualification, the main result of the paper is presented. Finally, an optimality condition, called the "necessary M-stationarity condition" is proposed for optimal solutions of the considered problems. Since other optimization problems with multiplicative constraints can be rewritten in the form of MPSCs, results obtained in this paper can be extended to a wider class of problems involving multiplicative constraints.

Keywords. Constraint qualification, Stationary conditions, Optimality conditions, Switching constraints.

MSC. 90C30; 90C33; 90C46.

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1 Introduction

In 2020, Mehlitz [13] introduced a class of complex optimization problems called "mathematical programming with switching constraints" (MPSC). After defining the MPSC problem and establishing the optimality conditions, known as stationary conditions, this topic has been of interest to several researchers in both the smooth [7, 10, 11, 12, 13, 15, 17] and nonsmooth [3, 4] cases.

In this paper, we consider the following MPSC:

$$(\Delta): \quad \begin{array}{l} Minimize \quad f(x) \\ \text{s.t.} \\ G_i(x)H_i(x) = 0, \quad i \in I = \{1, \dots, p\}, \end{array}$$

where functions $f, H_i, G_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are continuously differentiable for all $i \in I$. It is important to note that MPSC generalizes two well-known classes of optimization problems: called "Mathematical programming with equilibrium constraints" (MPEC [2, 14]) and "Mathematical programming with vanishing constraints" (MPVC [1, 8, 9]). The product function $G_i(x)H_i(x)$ is typically nonconvex, even when $G_i(x)$ and $H_i(x)$ are convex. Consequently, the feasible set of MPSC is nonconvex and, unlike convex sets, has multiple normal cones. The Fréchet and Mordukhovich normal cones are two of the most important normal cones for nonconvex sets, and they are used to define the "strong stationarity condition" and the "M-stationarity condition", respectively, as optimality conditions for MPSC. Estimating these normal cones of the feasible set of problem Δ is an important research topic in MPSC theory. The Fréchet normal cone of the feasible set of MPSCs has been estimated in [6], and this paper focuses on estimating the Mordukhovich normal cone of the feasible set of problem Δ. It should be noted that the strong stationarity condition and M-stationarity conditions have been presented in [3, 4, 13, 15] using some geometric constraint qualifications (in Abadie and Goignard types) for specific MPSC problems. Since the geometric constraint qualifications are based on tangent cones, which are often difficult to calculate, introduction of some algebraic constraint qualifications and the calculation of mentioned normal cones are of great practical importance, and this paper focuses on this important issue.

We organize the paper as follows. In Section 2, we provide the preliminary results to be used in the rest of the paper. Section 3 contains the main results, including MPSC-No Nonzero Abnormal Multiplier Constraint Qualification, upper approximations for the Mordukhovich normal cone of the feasible set of Δ, and necessary M-stationarity conditions for problem Δ. Section 4 includes a conclusion an overview of the content of the presented paper.

2 Notations and Preliminaries

In this section, we introduce some notations and preliminary results from [16] that will be used throughout the paper. Let $Y \neq \emptyset$ be a subset of $\mathbb{R}^p$ and $\hat{y} \in \overline{Y}$. The Bouligand tangent (or contingent) cone of Y at $\hat{y}$ is defined as:

$$\Gamma(Y, \hat{y}) := \Big\{ u \in \mathbb{R}^p \mid \exists t_\ell \downarrow 0, \ \exists u_\ell \to u \text{ such that } \hat{y} + t_\ell u_\ell \in Y, \ \ \forall \ell \in \mathbb{N} \Big\}.$$

The Fréchet (or regular) normal cone and the Mordukhovich (or limiting) normal cone of Y at $\hat{y}$ are respectively defined as:

$$N_F(Y, \hat{y}) := \Big\{ y \in \mathbb{R}^p \mid \langle y, u \rangle \leq 0, \quad \text{for all } u \in \Gamma(Y, \hat{y}) \Big\},$$
$$N_M(Y, \hat{y}) := \limsup_{y \to \hat{y}} N_F(Y, y),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^p$, and the $limsup$ in the latter definition represents the outer set-limit, i.e.,

$$\limsup_{y \to \hat{y}} N_F(Y, y) := \Big\{ v \in \mathbb{R}^p \mid \exists y_\ell \to \hat{y},\ \exists v_\ell \to v, \text{ with } v_\ell \in N_F(Y, y_\ell) \text{ as } \ell \to \infty \Big\}.$$

Theorem 1. [16, Theorem 6.12] If the continuously differentiable function $\varphi : \mathbb{R}^p \to \mathbb{R}$ attains its minimum on $Y \subseteq \mathbb{R}^p$ at $\hat{y} \in Y$, then

$$-\nabla \varphi(\hat{y}) \in N_F(Y, \hat{y}) \subseteq N_M(Y, \hat{y}).$$

Here, $\nabla \varphi(\hat{y})$ denotes the gradient of function $\varphi(\cdot)$ at $\hat{y}$. This inclusion can be written as:

$$0_p \in \{\nabla \varphi(\hat{y})\} + N_F(Y, \hat{y}) \subseteq \{\nabla \varphi(\hat{y})\} + N_M(Y, \hat{y}),$$

where 0_p denotes the zero vector in $\mathbb{R}^p$.

We recall from [16, Theorem 6.41] that if $Y \subseteq \mathbb{R}^p$, $Z \subseteq \mathbb{R}^q$, and $(\hat{y}, \hat{z}) \in Y \times Z$, then

$$N_M\Big(Y \times Z, (\hat{y}, \hat{z})\Big) = N_M\big(Y, \hat{y}\big) \times N_M\big(Z, \hat{z}\big). \tag{1}$$

Theorem 2. [5, Corollary 3.4] Assume that the set-valued mapping $M : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ is given by

$$M(y) := \{x \in \mathbb{R}^q \mid \Psi(x, y) \in E\},$$

where the function $\Psi : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^s$ is continuously differentiable and $E \subseteq \mathbb{R}^s$ is closed. If $x_0 \in M(y_0)$ and

$$\Big\{ \nabla_x \langle \nu, \Psi(\cdot, \cdot) \rangle (x_0, y_0) \mid 0_s \neq \nu \in N_M\left(E, \Psi(x_0, y_0)\right) \Big\} \cap \{0_q\} = \emptyset,$$

then M is calm at (y_0, x_0), i.e., there exist some $L > 0$ and neighborhoods U and V around x_0 and y_0, respectively, such that

$$d\big(M(y_0), x\big) \leq L\|y - y_0\|, \quad \text{for all } y \in V,\ x \in U \cap M(y),$$

where $d\big(M(y_0), x\big)$ denotes the distance between x and $M(y_0)$.

Theorem 3. [5, Theorem 4.1] Assume that the set-valued mapping $M : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ is defined by

$$M(y) := \{x \in \mathbb{R}^p \mid \Phi(x) + y \in E\},$$

where the function $\Phi : \mathbb{R}^p \to \mathbb{R}^q$ is continuously differentiable and $E \subseteq \mathbb{R}^q$ is closed. If $x_0 \in M(0_q)$ and M is calm at $(0_q, x_0)$, then

$$N_M\big(M(0_q), x_0\big) \subseteq \Big\{ \nabla \langle \nu, \Phi(\cdot) \rangle (x_0) \mid \nu \in N_M\big(E, \Phi(x_0)\big) \Big\}.$$

3 Main Results

Throughout this paper, we will suppose that the feasible set of (Δ), denoted by S, is nonempty, i.e.,

$$S = \{x \in \mathbb{R}^n \mid G_i(x)H_i(x) = 0,\ i \in I\} \neq \emptyset.$$

Considering a feasible point $\hat{x} \in S$ (this point will be fixed throughout this paper), we define following index sets:

$$\begin{aligned} I_G &:= \{i \in I \mid\ G_i(\hat{x}) = 0,\ H_i(\hat{x}) \neq 0\}, \\ I_H &:= \{i \in I \mid\ G_i(\hat{x}) \neq 0,\ H_i(\hat{x}) = 0\}, \\ I_{GH} &:= \{i \in I \mid G_i(\hat{x}) = 0,\ H_i(\hat{x}) = 0\}. \end{aligned}$$

Let us define the set:

$$D := \big\{(a,b) \in \mathbb{R}^2 \mid ab = 0\big\} = \big\{(0,b) \in \mathbb{R}^2 \mid b \neq 0\big\} \cup \big\{(a,0) \in \mathbb{R}^2 \mid a \neq 0\big\} \cup \{0_2\}. \tag{2}$$

Employing [13, Lemma 3.2], we deduce that:

$$\begin{cases} \Gamma\big(D,(0,b)\big) = \{0\} \times \mathbb{R}, & \text{for all } \ b \neq 0, \\ \Gamma\big(D,(a,0)\big) = \mathbb{R} \times \{0\}, & \text{for all } \ a \neq 0, \\ \Gamma\big(D,0_2\big) = D, \end{cases}$$

and hence,

$$\begin{cases} N_F\big(D,(0,b)\big) = \mathbb{R} \times \{0\}, & \text{for all } \ b \neq 0, \\ N_F\big(D,(a,0)\big) = \{0\} \times \mathbb{R}, & \text{for all } \ a \neq 0, \\ N_F\big(D,0_2\big) = \{0_2\}. \end{cases} \tag{3}$$

Consequently, we have:

$$N_F\big(D,(a,b)\big) \subseteq D, \quad \text{for all } \ (a,b) \in D. \tag{4}$$

For calculating the Mordukhovich normal cone of D at $(a,b) \in D$ the following two lemmas are required. Note that the counterpart of the following lemma for MPVC is presented in [1].

Lemma 1. Assume that D is defined as in (2). Then,

$$N_M\big(D,0_2\big) = D.$$

Proof. It is enough to prove the inclusions $\subseteq$ and $\supseteq$.

"⊆": Suppose that $w \in N_M\big(D, 0_2\big)$ is arbitrarily given. In view of the definition of the Mordukhovich normal cone, we can find some sequences $\{(a_k, b_k)\} \subseteq D$ converging to 0_2 and $\{w_k\} \subseteq \mathbb{R}^2$ converging to w such that $w_k \in N_F\big(D, (a_k, b_k)\big)$ for all $k \in \mathbb{N}$, and hence $w_k \in D$ for all $k \in \mathbb{N}$ by (4). Since D is a closed set, the limiting element w also belongs to D. This gives the desired inclusion.

"⊇": Let $w \in D$. Thus, (2) implies that $w = (0, b)$ for some $b \neq 0$ or $w = (a, 0)$ for some $a \neq 0$ or $w = 0_2$.

- If $w = (0, b)$ for some $b \neq 0$, then there exists a sequence $\{(0, b_k)\}$ converging to w such that $b_k \neq 0$ for all $k \in \mathbb{N}$. Since $(0, b_k) \in N_F\big(D, (\frac{1}{k}, 0)\big)$ for all $k \in \mathbb{N}$ by (3), and since $\{(\frac{1}{k}, 0)\}$ converges to 0_2 as $k \to \infty$, we have $w \in N_M\big(D, 0_2\big)$, as required.
- If $w = (a, 0)$ for some $a \neq 0$, then there exists a sequence $\{(a_k, 0)\}$ converging to w such that $a_k \neq 0$ for all $k \in \mathbb{N}$. Since $(a_k, 0) \in N_F\big(D, (0, \frac{1}{k})\big)$ for all $k \in \mathbb{N}$ by (3), and since $\{(0, \frac{1}{k})\}$ converges to 0_2 as $k \to \infty$, we have $w \in N_M\big(D, 0_2\big)$, as required.
- Because of $N_M\big(D, 0_2\big)$ is a cone, we have $w \in N_M\big(D, 0_2\big)$ if $w = 0_2$.

The proof is complete. □

Lemma 2. Assume that D is defined as in (2). Then,

$$\begin{cases} N_M\big(D, (0, b)\big) = \mathbb{R} \times \{0\}, & \text{for all } \ b \neq 0, \\ \\ N_M\big(D, (a, 0)\big) = \{0\} \times \mathbb{R}, & \text{for all } \ a \neq 0, \end{cases}$$

Proof. Since the proofs of the both equalities are similar, we just prove the second one.

"⊆": Suppose that $w \in N_M\big(D, (a, 0)\big)$ is arbitrarily given. Thus, there exist some sequences $\{(a_k, b_k)\} \subseteq D$ converging to $(a, 0)$ and $\{w_k\} \subseteq \mathbb{R}^2$ converging to w such that $w_k \in N_F\big(D, (a_k, b_k)\big)$ for all $k \in \mathbb{N}$. We can suppose that $a_k \neq 0$ for all $k \in \mathbb{N}$, and hence $b_k = 0$ by $(a_k, b_k) \in D$. Consequently, owing to (3), we obtain that

$$w_k \in N_F\big(D, (a_k, 0)\big) = \{0\} \times \mathbb{R}, \quad \forall k \in \mathbb{N},$$

and hence, $w \in \{0\} \times \mathbb{R}$ by closedness of $\{0\} \times \mathbb{R}$.

"⊇": Let $w \in \{0\} \times \mathbb{R}$. So, $w = (0, a)$ for some $a \neq 0$ or $w = 0_2$.

- If $w = (0, a)$ for some $a \neq 0$, then there exists a sequence $\{(0, a_k)\}$ converging to w such that $a_k \neq 0$ for all $k \in \mathbb{N}$. Since

  $$(0, a_k) \in \{0\} \times \mathbb{R} = N_F\big(D, (a_k, 0)\big), \quad \forall k \in \mathbb{N},$$

  by (3), and since $\{(a_k, 0)\}$ converges to $(a, 0)$ as $k \to \infty$, then $w = (a, 0) \in N_M\big(D, (0, a)\big)$, as required.
- If $w = 0_2$, then $w \in N_M\big(D, (0, a)\big)$ by conicity of $N_M\big(D, (0, a)\big)$.

□

The following theorem is a direct consequence of Lemmas 1 and 2.

Theorem 4. Assume that D is defined as in (2). Then, the following statements hold:

$$\begin{cases} N_M\big(D,(0,b)\big) = \mathbb{R}\times\{0\}, & \text{for all } \ b\neq 0, \\ \\ N_M\big(D,(a,0)\big) = \{0\}\times\mathbb{R}, & \text{for all } \ a\neq 0, \\ \\ N_M\big(D,0_2\big) = D. \end{cases}$$

Now, we recall the following definition from [13, Definition 4.2].

Definition 1. We say that the "MPSC-No Nonzero Abnormal Multiplier Constraint Qualification" (MPSC-NNAMCQ) is satisfied at a given point $\hat{x}$ if the following implication holds:

$$\left.\begin{array}{l} \displaystyle\sum_{i\in I_G\cup I_{GH}} \alpha_i \nabla G_i(\hat{x}) + \sum_{I_H\cup I_{GH}} \beta_i \nabla H_i(\hat{x}) = 0_n, \\ \\ \alpha_i\beta_i = 0, \qquad \forall i\in I_{GH}, \end{array}\right\} \Longrightarrow \begin{cases} \alpha_i = 0, & \forall i\in I_G\cup I_{GH}, \\ \\ \beta_i = 0, & \forall i\in I_H\cup I_{GH}. \end{cases}$$

Equivalently, the MPSC-NNAMCQ holds at $\hat{x}$ if

$$\left.\begin{array}{ll} \displaystyle\sum_{i=1}^{p}\Big(\alpha_i\nabla G_i(\hat{x}) + \beta_i\nabla H_i(\hat{x})\Big) = 0_n, & \\ \\ \alpha_i = 0, & \forall i\in I_H, \\ \\ \beta_i = 0, & \forall i\in I_G, \\ \\ \alpha_i\beta_i = 0, & \forall i\in I_{GH}, \end{array}\right\} \Longrightarrow (\alpha_1,\ldots,\alpha_p) = (\beta_1,\ldots,\beta_p) = 0_p.$$

It should be noted that the MPSC-NNAMCQ is a generalization of the MPVC-NNAMCQ, which was introduced in [1], and is weaker than the MPSC-LICQ, which was defined in [3].

Example 1. Consider the following MPSC:

$$Minimize \ \ x_1^4 + x_2^2 \quad \text{s.t.} \quad x_1^2 x_2 = 0.$$

This problem can be formalized as Δ by the following data:

$$f(x_1,x_2) = x_1^4 + x_2^2, \quad G_1(x_1,x_2) = x_1^2, \quad H_1(x_1,x_2) = x_2.$$

Let $\hat{x} = 0_2$. Since

$$I_{GH} = \{1\}, \quad \nabla G_1(\hat{x}) = 0_2, \quad \nabla H_1(\hat{x}) = (0,1),$$

we have:

$$\left.\begin{array}{l} \alpha_1\nabla G_1(\hat{x}) + \beta_1\nabla H_1(\hat{x}) = 0_2, \\ \\ \alpha_1\beta_1 = 0, \end{array}\right\} \not\Rightarrow \ \ \alpha_1 = \beta_1 = 0.$$

Consequently, the MPSC-NNAMCQ is not satisfied at the point $\hat{x}$.

Example 2. Taking the following data in Δ:

$$f(x_1, x_2) = x_1^4 + x_2^2, \quad G_1(x_1, x_2) = x_1 - x_2^2, \quad H_1(x_1, x_2) = x_2 - x_1^2,$$

and considering $\hat{x} = 0_2$, we have

$$I_{GH} = \{1\}, \quad \nabla G_1(\hat{x}) = (1, 0), \quad \nabla H_1(\hat{x}) = (0, 1).$$

Since

$$\left.\begin{aligned} &\alpha_1 \nabla G_1(\hat{x}) + \beta_1 \nabla H_1(\hat{x}) = 0_2, \\ &\alpha_1 \beta_1 = 0, \end{aligned}\right\} \Rightarrow \quad \alpha_1 = \beta_1 = 0,$$

the MPSC-NNAMCQ is satisfied at the point $\hat{x}$.

Let the function $\varphi : \mathbb{R}^n \to \mathbb{R}^{2p}$ be defined by

$$\varphi(x) := \big(G_1(x), H_1(x), \ldots, G_p(x), H_p(x)\big).$$

Clearly, we can write $\varphi = (\varphi_1, \ldots, \varphi_p)$, where the functions $\varphi_i : \mathbb{R}^n \to \mathbb{R}^2$, for $i \in I$, are defined as:

$$\varphi_i(x) := \big(G_i(x), H_i(x)\big), \quad \text{for all } i \in I.$$

Therefore, we have:

$$x \in S \iff \varphi(x) \in D^p \iff \varphi_i(x) \in D, \quad \text{for all } i \in I.$$

Now, we consider the following parameterized problem, which has been parameterized regarding $w \in \mathbb{R}^{2p}$:

$$\begin{aligned} Q(w): \quad & Minimize \quad f(x) \\ & \qquad\quad \text{s.t.} \quad \varphi(x) + w \in D^p, \\ & \qquad\qquad\quad\; x \in \mathbb{R}^n, \end{aligned}$$

where D is defined as (2). If the feasible set of $Q(w)$ is denoted by $M(w)$, we can consider $M(\cdot)$ as a set-valued mapping from $\mathbb{R}^{2p}$ to $\mathbb{R}^n$, i.e.,

$$M : \mathbb{R}^{2p} \rightrightarrows \mathbb{R}^n, \quad M(w) := \{x \in \mathbb{R}^n \mid \varphi(x) + w \in D^p\}.$$

Obviously, $M(0_{2p}) = S$ and $Q(0_{2p})$ coincides with the problem Δ.

The following theorem presents an interrelation between the concepts of MPSC-NNAMCQ and calmness.

Theorem 5 (Necessary Conditions for Calmness)**.** If the MPSC-NNAMCQ is satisfies at $\hat{x} \in S$, then $M(\cdot)$ is calm at $(0_{2p}, \hat{x})$.

Proof. Suppose that $i \in I_G$ is considered. We can write $\varphi_i(\hat{x}) = (0, b_i)$ where $b_i := H_i(\hat{x}) \neq 0$, and so, $N_M\big(D, \varphi_i(\hat{x})\big) = \mathbb{R} \times \{0\}$ by Theorem 4. Repeating this process for $i \in I_H$ and $i \in I_{GH}$, we conclude that:

$$N_M\big(D, \varphi_i(\hat{x})\big) = \begin{cases} \mathbb{R} \times \{0\}, & \text{for } i \in I_G, \\ \\ \{0\} \times \mathbb{R}, & \text{for } i \in I_H, \\ \\ D, & \text{for } i \in I_{GH}. \end{cases}$$

Regarding (1) and the above equality, we deduce that:

$$N_M\big(D^p, \varphi(\hat{x})\big) = N_M\Big(\prod_{i=1}^{p} D, \big(\varphi_1(\hat{x}), \ldots, \varphi_p(\hat{x})\big)\Big) = \prod_{i=1}^{p} N_M\big(D, \varphi_i(\hat{x})\big) = \prod_{i=1}^{p} A_i,$$

where,

$$A_i := \begin{cases} \mathbb{R} \times \{0\}, & \text{for } i \in I_G, \\ \\ \{0\} \times \mathbb{R}, & \text{for } i \in I_H, \\ \\ \big\{(a,b) \in \mathbb{R}^2 \mid ab = 0\big\}, & \text{for } i \in I_{GH}. \end{cases}$$

Now, due to the above equality and the MPSC-NNAMCQ assumption at $\hat{x}$, we have

$$\left.\begin{aligned} & \sum_{i=1}^{p} \Big(\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x})\Big) = 0_n, \\ \\ & (\alpha_i, \beta_i) \in N_M\left(D, \varphi_i(\hat{x})\right), \quad i \in I, \end{aligned}\right\} \Longrightarrow (\alpha_i, \beta_i) = 0_2, \quad \text{for all } i \in I.$$

This implication can be rewritten as

$$\left.\begin{aligned} & \sum_{i=1}^{p} \Big(\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x})\Big) = 0_n, \\ \\ & (\alpha_1, \beta_1, \ldots, \alpha_p, \beta_p) \in N_M\big(D^p, \varphi(\hat{x})\big), \end{aligned}\right\} \Longrightarrow (\alpha_1, \beta_1 \ldots, \alpha_p, \beta_p) = 0_{2p}. \tag{5}$$

On the other hand, for all $w_i := (\dot{w}_i, \ddot{w}_i) \in \mathbb{R}^2$ and $i \in I$ we have

$$\nabla G_i(\hat{x}) = \nabla_x\big(G_i + \dot{w}_i\big)(\hat{x}, 0), \quad \text{and} \quad \nabla H_i(\hat{x}) = \nabla_x\big(H_i + \ddot{w}_i\big)(\hat{x}, 0).$$

Therefore, putting $\lambda_i = (\alpha_i, \beta_i)$ for $i \in I$, we obtain that

$$\begin{aligned} & \sum_{i=1}^{p} \Big(\alpha_i \nabla G_i(\hat{x}) + \beta_i \nabla H_i(\hat{x})\Big) \\ & \quad = \alpha_1 \nabla_x\big(G_1 + \dot{w}_1\big)(\hat{x}, 0) + \beta_1 \nabla_x\big(H_1 + \ddot{w}_1\big)(\hat{x}, 0) + \ldots \\ & \qquad + \alpha_p \nabla_x\big(G_p + \dot{w}_p\big)(\hat{x}, 0) + \beta_p \nabla_x\big(H_p + \ddot{w}_p\big)(\hat{x}, 0) \\ & \quad = \nabla_x\Big(\alpha_1\big(G_1 + \dot{w}_1\big) + \beta_1\big(H_1 + \ddot{w}_1\big) + \ldots + \alpha_p\big(G_p + \dot{w}_p\big) + \beta_p\big(H_p + \ddot{w}_p\big)\Big)(\hat{x}, 0_{2p}) \\ & \quad = \nabla_x\Big\langle(\alpha_1, \beta_1, \ldots, \alpha_p, \beta_p), (G_1 + \dot{w}_1, H_1 + \ddot{w}_1, \ldots, G_p + \dot{w}_p, H_p + \ddot{w}_p\Big\rangle(\hat{x}, 0_{2p}) \end{aligned}$$

$$\begin{aligned}&= \nabla_x \Big\langle (\lambda_1, \ldots, \lambda_p), (\varphi_1(x) + w_1, \ldots, \varphi_p(x) + w_p) \Big\rangle (\hat{x}, 0_{2p}) \\ &= \nabla_x \big\langle \lambda, \varphi(x) + w \big\rangle (\hat{x}, 0_{2p}),\end{aligned}$$

where $\lambda := (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^{2p}$ and $w := (w_1, \ldots, w_p) \in \mathbb{R}^{2p}$. This equality and (5) yield

$$\left.\begin{array}{l}\nabla_x \big\langle \lambda, \varphi(x) + w \big\rangle (\hat{x}, 0_{2p}) = 0_n, \\ \\ \lambda \in N_M\left(D^p, \varphi(\hat{x})\right),\end{array}\right\} \Longrightarrow \lambda = 0_{2p}.$$

This means that

$$0_n \notin \Big\{ \nabla_x \big\langle \lambda, \varphi(x) + w \big\rangle (\hat{x}, 0_{2p}) \mid 0_{2p} \neq \lambda \in N_M\left(D^p, \varphi(\hat{x})\right) \Big\},$$

and hence,

$$\Big\{ \nabla_x \big\langle \lambda, \varphi(x) + w \big\rangle (\hat{x}, 0_{2p}) \mid 0_{2p} \neq \lambda \in N_M\left(D^p, \varphi(\hat{x})\right) \Big\} \cap \{0_n\} = \emptyset.$$

Now, employing Theorem 2 with data $E = D^p$ and $\Psi(x, w) = \varphi(x) + w$, and observing that $\hat{x} \in S = M(0_{2p})$, we deduce $M(\cdot)$ is calm at $(0_{2p}, \hat{x})$. □

The following theorem, which is the main result, presents an upper estimate for the Mordukhovich normal cone of the set S at the point $\hat{x}$, under the MPSC-NNAMCQ assumption.

Theorem 6. (Mordukhovich Normal Cone Inclusion) If the MPSC-NNAMCQ is satisfied at $\hat{x} \in S$, then

$$N_M(S, \hat{x}) \subseteq \left\{ \sum_{i=1}^{p} \left(\lambda_i^G \nabla G_i(\hat{x}) + \lambda_i^H \nabla H_i(\hat{x}) \right) \;\middle|\; \begin{array}{ll} \lambda_i^G = 0, & \text{for all } i \in I_H \\ \\ \lambda_i^H = 0, & \text{for all } i \in I_G \\ \\ \lambda_i^G \lambda_i^H = 0, & \text{for all } i \in I_{GH} \end{array} \right\}.$$

Proof. According to Theorem 5, we deduce that $M(\cdot)$ is calm at $(0_{2p}, \hat{x})$. Therefore, by Theorem 3, we have:

$$N_M(S, \hat{x}) \subseteq \Big\{ \big\langle \lambda, \varphi(\cdot) \big\rangle (\hat{x}) \mid \lambda \in N_M(D^p, \varphi(\hat{x})) \Big\}, \tag{6}$$

where $\lambda := \left(\lambda_1^G, \lambda_1^H, \ldots, \lambda_p^G, \lambda_p^H\right)$. Regarding

$$\langle \lambda, \varphi(\cdot) \rangle (\hat{x}) = \sum_{i=1}^{p} \left(\lambda_i^G \nabla G_i(\hat{x}) + \lambda_i^H \nabla H_i(\hat{x}) \right),$$

and Theorem 4, we get

$$\begin{aligned}&\Big\{ \big\langle \lambda, \varphi(\cdot) \big\rangle (\hat{x}) \mid \lambda \in N_M(D^p, \varphi(\hat{x})) \Big\} \\ &= \left\{ \sum_{i=1}^{p} \left(\lambda_i^G \nabla G_i(\hat{x}) + \lambda_i^H \nabla H_i(\hat{x}) \right) \;\middle|\; \begin{array}{ll} \lambda_i^G = 0, & \text{for all } i \in I_H \\ \\ \lambda_i^H = 0, & \text{for all } i \in I_G \\ \\ \lambda_i^G \lambda_i^H = 0, & \text{for all } i \in I_{GH} \end{array} \right\}.\end{aligned}$$

The above equality and (6) conclude that

$$N_M(S,\hat{x}) \subseteq \left\{ \sum_{i=1}^{p} \left(\lambda_i^G \nabla G_i(\hat{x}) + \lambda_i^H \nabla H_i(\hat{x})\right) \;\middle|\; \begin{array}{ll} \lambda_i^G = 0, & \text{for all } i \in I_H \\ \lambda_i^H = 0, & \text{for all } i \in I_G \\ \lambda_i^G \lambda_i^H = 0, & \text{for all } i \in I_{GH} \end{array} \right\},$$

as required. □

Theorem 7 presents a necessary optimality condition for the problem Δ derived from the application of Theorem 6.

Theorem 7. Suppose that $\hat{x} \in S$ is an optimal solution for Δ and the MPSC-NNAMCQ is satisfied at $\hat{x} \in S$. Then, there exist some scalars λ_i^G and λ_i^H as $i \in I$ such that:

$$\begin{cases} \nabla f(\hat{x}) + \sum_{i=1}^{p} \Big(\lambda_i^G \nabla G_i(\hat{x}) + \lambda_i^H \nabla H_i(\hat{x})\Big) = 0_n, & \\ \lambda_i^G = 0, & i \in I_H, \\ \lambda_i^H = 0, & i \in I_G, \\ \lambda_i^G \lambda_i^H = 0, & i \in I_{GH}. \end{cases} \tag{7}$$

Proof. Since $\hat{x}$ is a minimizer of f on S, by applying Theorem 1, we deduce that

$$-\nabla f(\hat{x}) \in N_M(S,\hat{x}).$$

From this inclusion and Theorem 6, we have

$$-\nabla f(\hat{x}) \in \left\{ \sum_{i=1}^{p} \left(\lambda_i^G \nabla G_i(\hat{x}) + \lambda_i^H \nabla H_i(\hat{x})\right) \;\middle|\; \begin{array}{ll} \lambda_i^G = 0, & i \in I_H \\ \lambda_i^H = 0, & i \in I_G \\ \lambda_i^G \lambda_i^H = 0, & i \in I_{GH} \end{array} \right\}.$$

Therefore, there exist some λ_i^G and λ_i^H as $i \in I$ such that:

$$\begin{cases} -\nabla f(\hat{x}) = \sum_{i=1}^{p} \Big(\lambda_i^G \nabla G_i(\hat{x}) + \lambda_i^H \nabla H_i(\hat{x})\Big), & \\ \lambda_i^G = 0, & i \in I_H, \\ \lambda_i^H = 0, & i \in I_G, \\ \lambda_i^G \lambda_i^H = 0, & i \in I_{GH}. \end{cases}$$

The proof is now complete. □

It is worth mentioning that conditions (7) are referred to as the "*M-stationarity conditions*" for Δ at $\hat{x}$ in the literature [3, 13]. Additionally, Theorem 7 is proved in [13, Corollary 4.3] using a different method.

4 Conclusion

This paper has explored mathematical programming problems with switching constraints (abbreviated as MPSC), assuming that all involved functions are continuously differentiable. We have analyzed the properties of the Mordokhovich normal cone associated with MPSC. Specifically, we have introduced Mangasarian-Fromovitz type constraint qualifications for the MPSC, referred to as MPSC-NNAMCQ. Additionally, we have derived an upper estimate for the Mordukhovich normal cone of MPSC and established the M-stationarity conditions for optimal solutions in MPSC.

Declarations

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