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New Soliton Solutions to the Coupled Conformable Time-Fractional Boussinesq Equation

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Abstract. In this study, we explore soliton solutions for the conformable time-fractional Boussinesq equation utilizing the three-wave method. To validate the precision of our findings, we discuss specific special cases by adjusting certain potential parameters and also present the graphical representations of our results. The results achieved in this research align closely with those from previous studies, demonstrating enhanced accuracy and simplicity. Given the extensive applications of this equation in particle physics, understanding its dynamics is crucial. Consequently, employing methods that encompass a broad spectrum of solutions is imperative. The versatility of this method in yielding diverse solutions is evident in the results we have obtained. The solutions derived in this paper are novel and offer greater precision compared to previous works.

Keywords. Soliton solutions, Coupled fractional Boussinesq equation, Three wave method.

MSC. 65L05; 34K06; 34K28.

1 Introduction

The pursuit of exact or approximate solutions to nonlinear partial differential equations (NLPDEs) and the Schrödinger equation is paramount across various disciplines within physics and chemistry [6, ?]. Despite its introduction nearly a century ago, the analytical solution of the Schrödinger equation particularly in the context of quantum mechanics remains a formidable challenge. Theoretical physicists have dedicated considerable effort to deriving exact or approximate solutions for the Schrödinger equation across a range of physically relevant potentials [5, ?, ?]. The Schrödinger equation is one of the most extensively utilized equations in physics, mathematics and engineering sciences, capable of describing a myriad of phenomena, including the emergence of light and dark solitons in optical physics. The solutions obtained are not only theoretically significant but also pragmatic and computationally accessible. To date, a plethora of methodologies have been employed to solve these equations, as detailed in the literature [2, ?, ?, ?, ?, ?, ?, ?]

The significance of the three-wave method in the context of the research lies in its application to solving the nonlinear partial differential equations (NLPDEs) that are the focus of the study. This method is likely one of the tools used to derive exact or approximate solutions for the equations, which are difficult to solve analytically.

The three-wave method is particularly important as it allows researchers to:

1. Obtain solutions that are relevant to the physical phenomena being studied, such as the behavior of light and dark solitons in optical physics.

2. Provide a systematic approach to handling the complexities of NLPDEs, which are known for their nonlinearity and the challenges they present in terms of analytical solutions.

3. Contribute to the broader goal of the research, which is to advance the understanding of NLPDEs and the Schrödinger equation in various branches of physics and chemistry.

By applying the three-wave method, the researchers aim to add to the body of knowledge regarding the solutions of NLPDEs and to potentially uncover new insights into the physical processes governed by these equations. The main objectives of the research presented in the paper are:

1. To explore the structure of the time-fractional coupled Boussinesq equation (tfcBE) and understand how to apply wavelet transformation to this equation.

2. To introduce the three-wave method and demonstrate its application in the context of the research.

3. To present the results of the discussion, which likely include the solutions obtained for the NLPDEs and the implications of these solutions for the relevant fields of physics and chemistry.

This research is structured as follows: The second section assigned to the formulation of the time-fractional coupled Boussinesq equation (tfcBE) and elucidates the application of wavelet transform techniques. The third section introduces the three-wave method and demonstrates its application, and the final section devoted to a brief discussion.

2 Basic Structure of the Conformable Fractional Coupled Boussinesq Equation

We begin by considering the conformable time fractional coupled Boussinesq equation (tfcBE) which is represented as follows [10]:

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} + uu_x + v_x + qu_{xx} = 0, \\ \frac{\partial^\alpha v}{\partial t^\alpha} + (uv)_x + pu_{xxx} - qv_{xx} = 0, \end{cases} \quad 0 \leq \alpha < 1, \tag{1}$$

where $p, q \in R$. By employing the conformable time-fractional derivative [1] and applying the wavelet transformation $u(x,t) = U(\xi), v(x,t) = V(\xi)$ and $\xi = kx + \omega \frac{t^\alpha}{\alpha}$ plus once integrating respect to ξ, Equation (1) changes to the following ordinary differential equation:

$$\begin{aligned} &\omega U + \frac{k}{2}U^2 + kV + qk^2U' = R_1, \\ &\omega V + kUV + pk^3U'' - qk^2V' = R_2. \end{aligned} \tag{2}$$

In these equations R_1 and R_2 are the integration constants of first- and second-equation of system (2), respectively. From first-equation of system (2), we get

$$V = \frac{1}{k}\left(R_1 - \omega U - \frac{k}{2}U^2 - qkU'\right). \tag{3}$$

By substituting Equation (3) into the second-equation of system (2), while for simplifying we set $R_1 = 0$ and $R_2 = 0$, we get the following covering equation:

$$-\frac{\omega^2}{k}U - \frac{3}{2}\omega U^2 - \frac{k}{2}U^3 + k^3\left(p + q^2\right)U'' = 0. \tag{4}$$

In summary, this section provides the foundational structure and approach for analyzing the conformable time-fractional coupled Boussinesq equation, setting the stage for subsequent sections where the method is applied and the results are discussed. A concise application of the method to the Fractional Boussinesq equation (FBE) is detailed in the second section of this paper. In the third section, the graphical behavior of solutions is introduced and analyzed.

3 Three-Wave Approaches to the Conformable TFCBE

Initially, we assume that Equation (4) admits three-wave solutions of the form:

$$U(\xi) = \gamma_1 e^{\delta\xi} + \gamma_2 \cos(\lambda_1 \xi) + \gamma_3 e^{-\delta\xi} + 2\gamma_4 \cosh(\lambda_2 \xi), \tag{5}$$

where $\gamma_1, \ldots, \gamma_4, \delta, \lambda_1, \lambda_2$ are constants to be determined. Substituting (5) into (4) and collecting coefficients of $e^{i\delta\xi}$, $\cos(\lambda_1\xi)$, $\cosh(\lambda_2\xi)$, $\sin(\lambda_1\xi)$, and $\sinh(\lambda_2\xi)$ for $i = -2, -1, 0, 1, 2$ and setting them to zero, we obtain a system of algebraic equations. Solving these equations yields:

Set 1: $\gamma_1 = 0, \gamma_3 = 0, \gamma_2 \neq 0$, and $\gamma_4 \neq 0$. Solving the algebraic equations gives:

$$\gamma_2 = \frac{1}{2}, \gamma_4 = -\frac{1}{2}, w = -\frac{1}{2}\sqrt{q^2 + pk^2}, k = k. \tag{6}$$

Consequently, we arrive at the general solutions of Equation (1) as depicted in Figure 1:

$$u_1(x,t) = \frac{1}{2}\cos\left(\lambda_1\left(kx - \frac{1}{2}\sqrt{q^2+pk^2}\frac{t^\alpha}{\alpha}\right)\right) - \cosh\left(\lambda_2\left(kx - \frac{1}{2}\sqrt{q^2+pk^2}\frac{t^\alpha}{\alpha}\right)\right).$$

From (3), we directly obtain:

$$\begin{aligned} v_1(x,t) = \Bigg[& R_1 - \frac{w}{2}\cos\left(\lambda_1\left(kx - \frac{1}{2}\sqrt{q^2+pk^2}\frac{t^\alpha}{\alpha}\right)\right) \\ & + w\cosh\left(\lambda_2\left(kx - \frac{1}{2}\sqrt{q^2+pk^2}\frac{t^\alpha}{\alpha}\right)\right) \\ & - \frac{k}{2}\left(\frac{1}{2}\cos\left(\lambda_1\left(kx - \frac{1}{2}\sqrt{q^2+pk^2}\frac{t^\alpha}{\alpha}\right)\right)\right. \\ & \left. - \cosh\left(\lambda_2\left(kx - \frac{1}{2}\sqrt{q^2+pk^2}\frac{t^\alpha}{\alpha}\right)\right)\right)^2 \\ & + \frac{1}{2}qk\sin\left(\lambda_1\left(kx - \frac{1}{2}\sqrt{q^2+pk^2}\frac{t^\alpha}{\alpha}\right)\right) \\ & + qk\sinh\left(\lambda_2\left(kx - \frac{1}{2}\sqrt{q^2+pk^2}\frac{t^\alpha}{\alpha}\right)\right)\Bigg]. \end{aligned}$$

Set 2: $\gamma_2 = 0, \gamma_4 = 0, \gamma_1 \neq 0$, and $\gamma_3 \neq 0$. Solving the algebraic equations gives:

$$\gamma_1 = \frac{2}{3}\frac{\delta^2 k^4 q^2 + \delta^2 k^4 p - w^2}{k^2\gamma_3}, \gamma_3 = \gamma_3, w = \sqrt{q^2 + pk^2}\delta, k = k. \tag{7}$$

Thus, we have:

$$u_2(x,t) = \frac{2}{3}\frac{\delta^2 k^4 q^2 + \delta^2 k^4 p - w^2}{k^2\gamma_3} e^{\delta\left(kx + \sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)} + \gamma_3 e^{-\delta\left(kx + \sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)}.$$

From (3), we have:

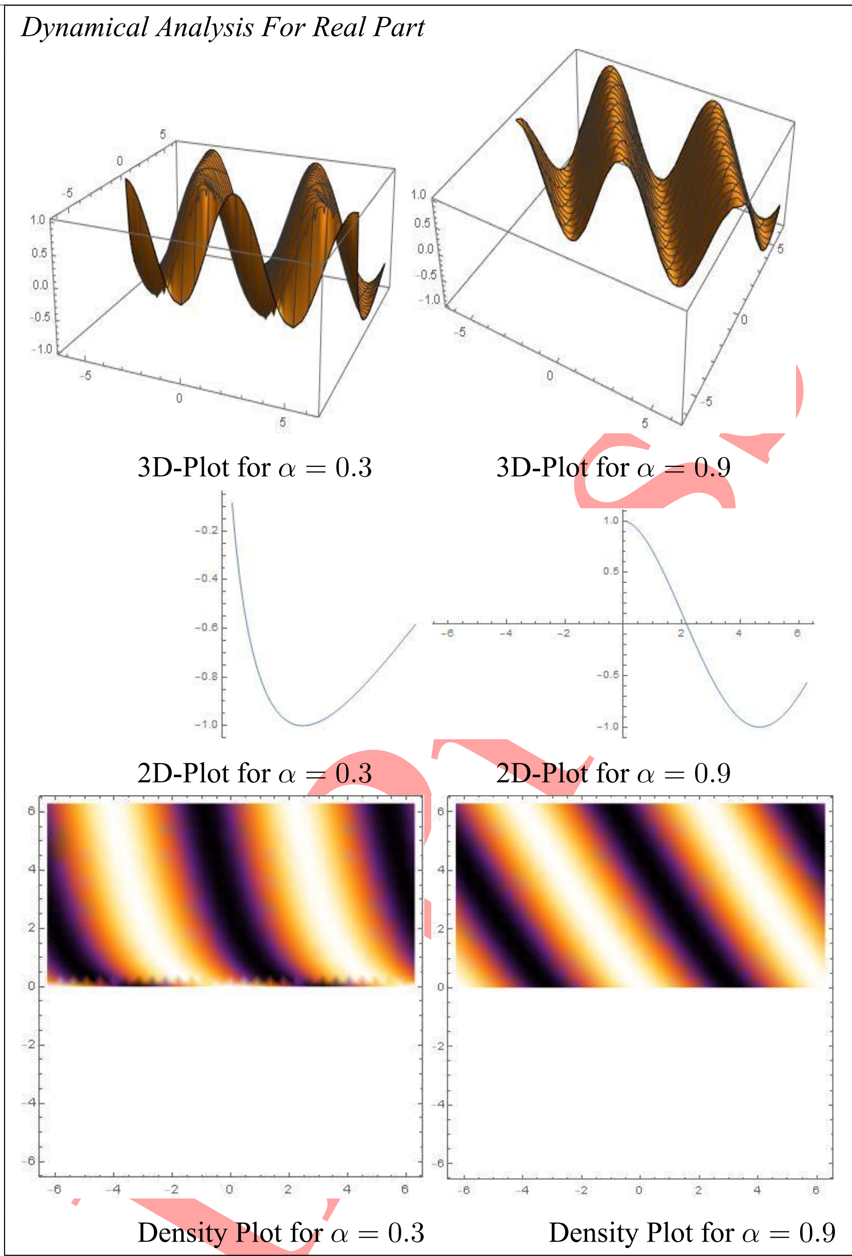

Figure 1: Geraphical representation of the behavior of $u_1(x,t)$, for $\alpha = 0.3$ (A, C and E) and $\alpha = 0.9$ (B, D and F) with $p = q = k = 1$.

$$v_2(x,t) = \frac{2}{3}\frac{\delta^2k^4q^2+\delta^2k^4p-w^2}{k^3\gamma_3}e^{\delta\left(kx+\sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)} + \frac{\gamma_3}{k}e^{-\delta\left(kx+\sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)}$$
$$-\frac{1}{2}\left(\frac{2}{3}\frac{\delta^2k^4q^2+\delta^2k^4p-w^2}{k^2\gamma_3}e^{\delta\left(kx+\sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)} + \gamma_3 e^{-\delta\left(kx+\sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)}\right)^2$$
$$-q\left(\frac{2}{3}\delta\frac{\delta^2k^4q^2+\delta^2k^4p-w^2}{k^2\gamma_3}e^{\delta\left(kx+\sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)} - \gamma_3\delta e^{-\delta\left(kx+\sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)}\right).$$

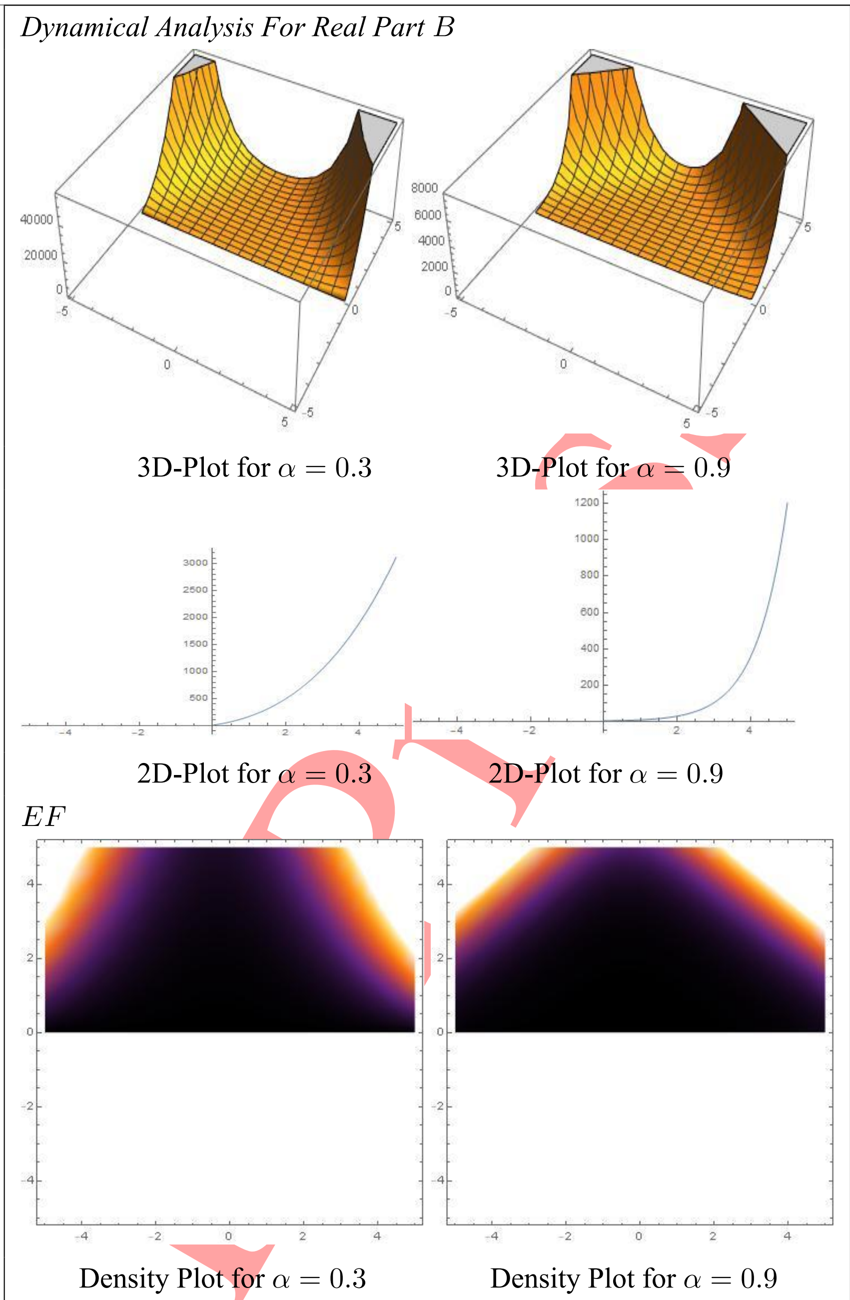

Figure 2: Geraphical representation of the behavior of $u_3(x,t)$, for $\alpha = 0.3$ (A, C and E) and $\alpha = 0.6$ (B, D and F) with $p = q = k = 1$

Set 3: $\gamma_1 = 0, \gamma_2 = 0, \gamma_3 \neq 0$, and $\gamma_4 \neq 0$. Solving the algebraic equations gives (as shown in Figure 2):

$$\gamma_4 = \frac{2}{3}\frac{\lambda_2^2 k^4 q^2 + \lambda_2^2 k^4 p - w^2}{kw}, \gamma_3 = \gamma_3, w = w, k = \frac{\sqrt{\sqrt{q^2+p}\delta w}}{\sqrt{q^2+p}\delta}. \tag{8}$$

Therefore,

$$u_3(x,t)=\gamma_3 e^{-\delta\left(kx+\omega\frac{t^\alpha}{\alpha}\right)}+\frac{4}{3}\frac{\lambda_2^2k^4q^2+\lambda_2^2k^4p-w^2}{kw}\cosh\left(\lambda_2\left(kx+\omega\frac{t^\alpha}{\alpha}\right)\right),$$

and

$$v_3(x,t)=\frac{1}{k}\left(R_1-\omega U-\frac{k}{2}U^2-qkU'\right)\times$$
$$\frac{R_1}{k}-\frac{\gamma_3}{k}e^{-\delta\left(kx+\omega\frac{t^\alpha}{\alpha}\right)}-\frac{4}{3}\frac{\lambda_2^2k^4q^2+\lambda_2^2k^4p-w^2}{k^2w}\cosh\left(\lambda_2\left(kx+\omega\frac{t^\alpha}{\alpha}\right)\right)$$
$$-w\gamma_3 e^{-\delta\left(kx+\omega\frac{t^\alpha}{\alpha}\right)}-\frac{4}{3}\frac{\lambda_2^2k^4q^2+\lambda_2^2k^4p-w^2}{k}\cosh\left(\lambda_2\left(kx+\omega\frac{t^\alpha}{\alpha}\right)\right)$$
$$-\frac{1}{2}\left(\gamma_3 e^{-\delta\left(kx+\omega\frac{t^\alpha}{\alpha}\right)}+\frac{4}{3}\frac{\lambda_2^2k^4q^2+\lambda_2^2k^4p-w^2}{kw}\cosh\left(\lambda_2\left(kx+\omega\frac{t^\alpha}{\alpha}\right)\right)\right)^2$$
$$+\frac{\gamma_3}{k}\delta e^{-\delta\left(kx+\omega\frac{t^\alpha}{\alpha}\right)}-\frac{4\lambda_2}{3}\frac{\lambda_2^2k^4q^2+\lambda_2^2k^4p-w^2}{k^2w}\sinh\left(\lambda_2\left(kx+\omega\frac{t^\alpha}{\alpha}\right)\right)$$

Set 4: $\gamma_1\neq 0,\gamma_2=0,\gamma_3\neq 0$, and $\gamma_4\neq 0$ then by solving algebraic equation we have

$$\gamma_4=\frac{4}{3}\frac{\delta^2k^4q^2+\delta^2k^4p-w^2}{k^2},\gamma_3=\frac{1}{2},\gamma_1=\gamma_1,w=w,$$
$$k=\frac{\sqrt{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}}{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}},\tag{9}$$

$$u_4(x,t)=\gamma_1 e^{\delta\left(\frac{\sqrt{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}}{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}x+w\frac{t^\alpha}{\alpha}\right)}+\frac{1}{2}e^{-\delta\left(\frac{\sqrt{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}}{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}x+w\frac{t^\alpha}{\alpha}\right)}$$
$$+\frac{8}{3}\frac{\delta^2k^4q^2+\delta^2k^4p-w^2}{k^2}\times$$
$$\cosh\left(\lambda_2\left(\frac{\sqrt{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}}{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}x+w\frac{t^\alpha}{\alpha}\right)\right)$$

Now, from (3) we have:

$$v_4(x,t)=\frac{1}{k}R_1-\frac{1}{k}\gamma_1 we^{\delta\left(\frac{\sqrt{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}}{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}x+w\frac{t^\alpha}{\alpha}\right)}$$
$$-\frac{1}{2k}we^{-\delta\left(\frac{\sqrt{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}}{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}x+w\frac{t^\alpha}{\alpha}\right)}-\frac{8w}{3}\frac{\delta^2k^4q^2+\delta^2k^4p-w^2}{k^3}$$
$$\times\cosh\left(\lambda_2\left(\frac{\sqrt{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}}{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}x+w\frac{t^\alpha}{\alpha}\right)\right)$$

$$
\begin{aligned}
&-\frac{1}{2}\left[\gamma_1 e^{\delta\left(\frac{\sqrt{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}}{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}x+w\frac{t^\alpha}{\alpha}\right)}\right.\\
&+\frac{1}{2}e^{-\delta\left(\frac{\sqrt{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}}{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}x+w\frac{t^\alpha}{\alpha}\right)}\\
&+\frac{8}{3}\frac{\delta^2k^4q^2+\delta^2k^4p-w^2}{k^2}\\
&\left.\times\cosh\left(\lambda_2\left(\frac{\sqrt{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}}{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}x+w\frac{t^\alpha}{\alpha}\right)\right)\right]^2\\
&-\gamma_1\delta q e^{\delta\left(\frac{\sqrt{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}}{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}x+w\frac{t^\alpha}{\alpha}\right)}\\
&+\frac{q\delta}{2}e^{-\delta\left(\frac{\sqrt{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}}{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}x+w\frac{t^\alpha}{\alpha}\right)}\\
&-\frac{8q\lambda_2}{3}\frac{\delta^2k^4q^2+\delta^2k^4p-w^2}{k^2}\\
&\times\cosh\left(\lambda_2\left(\frac{\sqrt{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}}{\sqrt{2\delta^2q^2-q^2\lambda_2^2+2\delta^2p-p\lambda_2^2w}}x+w\frac{t^\alpha}{\alpha}\right)\right).
\end{aligned}
$$

4 Conclusion

In this study, we have successfully derived innovative solitary soliton solutions of the coupled fractional Boussinesq equation by applying the three-wave method. The equation under consideration features a set of arbitrary parameters, which, when assigned specific values, can generate various well-established models. A significant advantage of this approach is its capacity to produce a wide array of solutions within a single unified framework, thereby offering a singular methodological tool for addressing numerous equations. Furthermore, as previously emphasized, the solutions obtained through this method demonstrate superior accuracy and computational efficiency compared to those derived from alternative techniques, such as the generalized Kudryashov method. This advancement not only broadens the scope of potential applications but also reinforces the robustness of the three-wave method as a powerful tool in nonlinear wave dynamics and analysis.

Declarations

Availability of supporting data

All data generated or analyzed during this study are included in this published paper.

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Competing interests

The authors have no competing interests to declare that are relevant to the content of this paper.

Authors' contributions

The main manuscript text was written collectively by the authors.

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