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Abadie-Type Constraint Qualifications and Optimality Conditions for Nonsmooth Multi-Objective Semi-Infinite Problems

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Abstract. This paper introduces several Abadie-type constraint qualifications and we derives necessary optimality conditions in the Karush-Kuhn-Tucker, for both weakly efficient solutions and efficient solutions of a nonsmooth multi-objective semi-infinite programming problem characterized by locally Lipschitz data. The findings are expressed in term of the Micheal-Penot subdifferential.

Keywords. Semi-infinite optimization, Constraint qualification, Micheal-Penot subdifferential, Optimality condition.

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1 Introduction

This paper explores the intersection of three key research areas: semi-infinite optimization, multi-objective programming, and nonsmooth analysis. Our focus is on nonsmooth multi-objective semi-infinite optimization problems characterized by locally Lipschitz functions. For an in-depth understanding of the properties and significance of locally Lipschitz functions in nonsmooth analysis, we refer the reader to [1, 3, 4, 19].

Multi-objective semi-infinite programming problems, which entail the simultaneous minimization or maximization of multiple conflicting objective functions, constitute a vital class of optimization problems. Due to their extensive range of applications, these problems have garnered considerable researchers attention from various perspectives. Notable contributions include studies on differentiable cases (see [2, 7], linear cases [14], convex cases [5, 15, 21], as well as nonsmooth cases [6, 10, 11, 12, 13, 20]. It is important to note that the results reported in [6, 10, 11, 12] rely on the Clarke subdifferential.

In this paper, we specifically investigate nondifferentiable non convex multi-objective semi-infinite optimization problems with both locally Lipschitz objective and constraint functions. It has been established that the Michel-Penot (M-P) subdifferential can be strictly contained within the Clarke subdifferential for locally Lipschitz functions (as noted in [1]). Consequently, for optimization problems incorporating locally Lipschitz functions, the necessary optimality conditions articulated through the M-P subdifferential are sharper than those derived from the Clarke subdifferential. Thus, our objective is to present a multiplier rule based on the M-P subdifferential, with subsequent implications for larger subdifferentials.

The structure of paper is as follows. Section 2 introduces the necessary notations, foundational definitions, and preliminary concepts that will be utilized throughout the paper. In Section 3, we establish several constraint qualifications and derive Karush-Kuhn-Tucker type necessary optimality conditions for nonsmooth multi-objective semi-infinite optimization problems. Finally, Section 4 concludes the paper with a summary of our findings.

2 Notations and Preliminaries

In this paper, we denote the standard inner product of two vectors x and y in $\mathbb{R}^n$ will be denoted by $\langle x, y\rangle$, and the zero vector in $\mathbb{R}^n$ is represented by $\mathbf{0}$.

For a subset $A \subseteq \mathbb{R}^n$, we use $\overline{A}$ to denote the closure of A and $ri(A)$ to represent the relative interior of A. Additionally, we define the convex hull, convex cone, closed convex hull, and closed convex cone generated by A using the following notations: $\text{conv}(A)$, $\text{cone}(A)$, $\overline{\text{conv}}(A)$ and $\overline{\text{cone}}(A)$, respectively. These are formally defined as follows:

$$\operatorname{conv}(A) := \bigcap \{B \mid B \text{ is convex and } A \subseteq B\}, \quad \text{if } A \neq \emptyset, \text{ and } \operatorname{conv}(\emptyset) := \emptyset,$$

$$\operatorname{cone}(A) := \bigcup \{r \operatorname{conv}(A) \mid r \geq 0\}, \quad \text{if } A \neq \emptyset, \quad \text{and} \quad \operatorname{cone}(\emptyset) := \{\mathbf{0}\},$$

$$\overline{\operatorname{conv}}(A) := \overline{\operatorname{conv}(A)}, \quad \text{and} \quad \overline{\operatorname{cone}}(A) := \overline{\operatorname{cone}(A)}.$$

The following theorem, as established by [8, 18], provides important properties regarding the convex hull and cone associated with a nonempty compact set in $\mathbb{R}^n$:

Theorem 1. Let A be a nonempty compact subset of $\mathbb{R}^n$. Then, we have:

i. $\operatorname{conv}(A)$ is a closed set.

ii. If $\mathbf{0} \notin \operatorname{conv}(A)$, then $\operatorname{cone}(A)$ is a closed cone.

The negative polar and the strictly negative polar of a set $A \subseteq \mathbb{R}^n$ are defined respectively, as follows:

$$A^0 := \{u \in \mathbb{R}^n \mid \langle u, x\rangle \leq 0, \ \forall x \in A\}, \quad \text{if } A \neq \emptyset \quad \text{and} \quad \emptyset^0 := \{\mathbf{0}\},$$

$$A^- := \{u \in \mathbb{R}^n \mid \langle u, x\rangle < 0, \ \forall x \in A\}, \quad \text{if } A \neq \emptyset \quad \text{and} \quad \emptyset^- := \emptyset.$$

It is established in [8] that the negative polar A^0 is always a closed convex cone in $\mathbb{R}^n$, with the following relationships holding true:

$$A^0 = (\overline{A})^0 = \big(\operatorname{conv}(A)\big)^0 = \big(\operatorname{cone}(A)\big)^0, \tag{1}$$

and

$$A^- = \big(\operatorname{conv}(A)\big)^- = \big(\operatorname{cone}(A)\big)^-. \tag{2}$$

Furthermore, if $A^- \neq \emptyset$, then it follows that $A^0 = \overline{A^-}$.

For an arbitrary index set Ω, let $B_\gamma \subseteq \mathbb{R}^n$ be a nonempty convex set for each $\gamma \in \Omega$, and let $B := \bigcup_{\gamma\in\Omega} B_\gamma$. According to findings presented in [8, 18], the convex hull and the cone of B can be expressed as:

$$\operatorname{conv}(B) = \left\{ \sum_{\gamma\in\Omega_*} \alpha_\gamma B_\gamma \mid \alpha_\gamma \geq 0, \ \sum_{\gamma\in\Omega_*} \alpha_\gamma = 1, \ \Omega_* \subseteq \Omega, \ |\Omega_*| < \infty \right\}, \tag{3}$$

$$\operatorname{cone}(B) = \left\{ \sum_{\gamma\in\Omega_*} \alpha_\gamma B_\gamma \mid \alpha_\gamma \geq 0, \ \Omega_* \subseteq \Omega, \ |\Omega_*| < \infty \right\}. \tag{4}$$

We also define the tangent cone of a set $A \subseteq \mathbb{R}^n$, at a point $\hat{x} \in \overline{A}$ as follows:

$$\Gamma_A(\hat{x}) := \Big\{ d \in \mathbb{R}^n \mid \exists \{(t_k, d_k)\} \to (0^+, d);\;\; \hat{x} + t_k d_k \in A,\;\; \forall k \in \mathbb{N} \Big\}.$$

It is noteworthy that $\Gamma_A(\hat{x})$ is closed cone, which may not necessarily be convex, in $\mathbb{R}^n$.

Consider a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$. The Micheal-Penot (M-P) directional derivative of f at the point $\hat{x} \in \mathbb{R}^n$ in the direction of $v \in \mathbb{R}^n$, as introduced in [16], is defined by

$$f^{\diamond}(\hat{x}; v) := \sup_{w \in \mathbb{R}^n} \limsup_{\alpha \downarrow 0} \frac{f(\hat{x} + \alpha v + \alpha w) - f(\hat{x} + \alpha w)}{\alpha}.$$

The M-P subdifferential of f at the point $\hat{x}$ is then defined as

$$\partial^{\diamond} f(\hat{x}) := \{\xi \in \mathbb{R}^n \mid \langle\langle \xi, v \rangle\rangle \leq f^{\diamond}(\hat{x}; v), \quad \forall v \in \mathbb{R}^n\}.$$

The M-P subdifferential serves as a natural generalization of the standard derivative. Notably, it is known (see [16, Proposition 1.3]) that if the function f is differentiable at $\hat{x}$, then $\partial^{\diamond} f(\hat{x}) = \nabla f(\hat{x})$. Moreover, when $g : \mathbb{R}^n \to \mathbb{R}$ is convex function, the M-P subdifferential coincides with the subdifferential as defined in convex analysis:

$$\partial^{\diamond} g(\hat{x}) = \partial g(\hat{x}) := \{\xi \in \mathbb{R}^n \mid g(x) - g(\hat{x}) \geq \langle \xi, x - \hat{x} \rangle, \quad \forall x \in \mathbb{R}^n\}.$$

The following theorem summarizes essential properties of the M-P directional derivative and the M-P subdifferential, as delineated in [16, 17]. These properties will serve as a critical foundation for the analyses that follow.

Theorem 2. Let f and h be locally Lipschitz functions mapping from $\mathbb{R}^n$ to $\mathbb{R}$, with a given point $\hat{x} \in \mathbb{R}^n$. Then, the following assertions are established:

i. The following relationships hold:

$$f^{\diamond}(\hat{x}; v) = \max \big\{ \langle \xi, v \rangle \mid \xi \in \partial^{\diamond} f(\hat{x}) \big\},$$
$$\partial^{\diamond} \big(\max\{f, g\} \big)(\hat{x}) \subseteq \operatorname{conv}\big(\partial^{\diamond} f(\hat{x}) \cup \partial^{\diamond} h(\hat{x}) \big),$$
$$\partial^{\diamond}(\lambda f + \mu h)(\hat{x}) \subseteq \lambda \partial^{\diamond} f(\hat{x}) + \mu \partial^{\diamond} h(\hat{x}), \quad \forall \lambda, \mu \in \mathbb{R}.$$

ii. The mapping $v \to f^{\diamond}(\hat{x}; v)$ is finite, positively homogeneous, and subadditive on $\mathbb{R}^n$. Moreover, it holds that

$$\partial\big(f^{\diamond}(\hat{x}; \cdot) \big)(\mathbf{0}) = \partial^{\diamond} f(\hat{x}).$$

iii. The set $\partial^{\diamond} f(\hat{x})$ is nonempty, convex , and compact within $\mathbb{R}^n$.

Theorem 3. (**Mean Value Theorem**) [3]. Let $x, y \in \mathbb{R}^n$, and assume that f is a locally Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}$. Then, there exist a point u located within the open line segment (x, y), such that

$$f(y) - f(x) \in \big\langle \partial^{\diamond} f(u), y - x \big\rangle.$$

3 Main Results

In the subsequent sections of this paper, we will analyze the following multi-objective semi-infinite optimization problem:

$$\begin{aligned} \text{(P)} : \inf \quad & \big(\varphi_1(x), \ldots, \varphi_m(x)\big) \\ \text{s.t.} \quad & \vartheta_j(x) \leq 0, \quad i \in J, \\ & x \in \mathbb{R}^n, \end{aligned}$$

where the functions $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ and $\vartheta_j : \mathbb{R}^n \to \mathbb{R}$, for $i \in I := \{1, \ldots, m\}$ and $j \in J$, are assumed to be locally Lipschitz. The index set $J \neq \emptyset$ is arbitrary and may not necessarily be finite. We define the feasible region of problem (P) as follows:

$$\mathcal{F} = \big\{x \in \mathbb{R}^n \mid \vartheta_j(x) \leq 0, \quad \forall j \in J\big\}.$$

Let $\hat{x} \in \mathcal{F}$ be a specified point. The index set of all active constraints at $\hat{x}$ is defined as

$$J(\hat{x}) := \big\{j \in J \mid \vartheta_j(\hat{x}) = 0\big\}.$$

A feasible point $\hat{x} \in \mathcal{F}$ is classified as a weakly efficient solution to problem (P) if there is no $x \in \mathcal{F}$ such that

$$\varphi_i(x) < \varphi_i(\hat{x}), \quad \forall i \in I.$$

Furthermore, a point $\hat{x} \in \mathcal{F}$ is termed an efficient solution to (P) if there is no $x \in \mathcal{F}$ that satisfies $\varphi_i(x) \leq \varphi_i(\hat{x})$ for all $i \in I$ with

$$\big(\varphi_1(x), \ldots, \varphi_m(x)\big) \neq \big(\varphi_1(\hat{x}), \ldots, \varphi_m(\hat{x})\big).$$

We denote the set of all weakly efficient solutions as $\mathcal{E}$ and the set of all efficient solutions as $\mathcal{W}$. It is evident that $\mathcal{E} \subseteq \mathcal{W}$.

Following the framework established in [5], for each $\hat{x} \in \mathcal{F}$ and $i_0 \in I$, we define:

$$\begin{aligned} &\mathcal{Q}^{i_0}(\hat{x}) := \Big\{x \in \mathcal{F} \mid \varphi_i(x) \leq \varphi_i(\hat{x}), \quad \forall i \in I \setminus \{i_0\}\Big\}, \\ &\mathcal{Q}^{i_0}(\hat{x}) := \mathcal{F}, \quad \text{if } m = 1. \end{aligned}$$

For simplicity, we will denote $\mathcal{Q}^i(\hat{x})$ by $\mathcal{Q}^i$ throughout this paper. We also introduce the following notations:

$$\Omega^{\hat{x}} := \bigcup_{i \in I} \partial^{\diamond} \varphi_i(\hat{x}), \quad \text{and} \quad \Upsilon^{\hat{x}} := \bigcup_{j \in J(\hat{x})} \partial^{\diamond} \vartheta_j(\hat{x}).$$

Now, we will consider following types of Abadie constraint qualifications, which include weak, generalized, and refined Abadie constraint qualifications:

$$
\begin{aligned}
&\text{(WACQ):} && (\Omega^{\hat{x}})^- \cap (\Upsilon^{\hat{x}})^0 \subseteq \Gamma_{\mathcal{F}}(\hat{x}),\\
&\text{(GACQ):} && (\Upsilon^{\hat{x}})^0 \subseteq \Gamma_{\mathcal{F}}(\hat{x}),\\
&\text{(RACQ):} && (\Omega^{\hat{x}})^0 \cap (\Gamma^{\hat{x}})^0 \subseteq \textstyle\bigcap_{i\in I} \Gamma_{\mathcal{Q}^i}(\hat{x}).
\end{aligned}
$$

The following theorem elucidates the interrelation among the aforementioned constraint qualifications.

Theorem 4. The subsequent implications hold true at the feasible point $\hat{x} \in \mathcal{F}$.

$$
\begin{array}{ccc}
\text{GACQ} & \overset{\star}{\longrightarrow} & \text{RACQ} \\
& \searrow \quad \swarrow & \\
& \text{WACQ} &
\end{array}, \tag{5}
$$

where the implication " $\overset{\star}{\longrightarrow}$ " is valid when $m = 1$.

Proof. **GACQ⟶WACQ:** This implication follows from the inclusion

$$(\Omega^{\hat{x}})^- \cap (\Upsilon^{\hat{x}})^0 \subseteq (\Upsilon^{\hat{x}})^0.$$

RACQ⟶WACQ: This relationship is a direct corollary of the following inclusions:

$$(\Omega^{\hat{x}})^- \cap (\Upsilon^{\hat{x}})^0 \subseteq (\Omega^{\hat{x}})^0 \cap (\Upsilon^{\hat{x}})^0 \quad \text{and} \quad \bigcap_{i\in I} \Gamma_{\mathcal{Q}^i}(\hat{x}) \subseteq \Gamma_{\mathcal{F}}(\hat{x}).$$

GACQ$\overset{\star}{\longrightarrow}$RACQ: In the case where $m = 1$,

$$\Gamma_{\mathcal{F}}(\hat{x}) = \bigcap_{i\in I} \Gamma_{\mathcal{Q}^i}(\hat{x}) \quad \text{and} \quad \big(\partial^\diamond \varphi_1(\hat{x})\big)^0 \cap (\Upsilon^{\hat{x}})^0 \subseteq (\Upsilon^{\hat{x}})^0,$$

leading to an immediate result. □

The following lemma is significant for the subsequent discussion.

Lemma 1. If $\hat{x} \in \mathcal{W}$, then

$$(\Omega^{\hat{x}})^- \cap \Gamma_{\mathcal{F}}(\hat{x}) = \emptyset.$$

Proof. Assume, for the sake of contradiction, that there exists a vector

$$d \in (\Omega^{\hat{x}})^- \cap \Gamma_{\mathcal{F}}(\hat{x}).$$

By the definition of the tangent cone, there exist sequences $\{t_k\} \to 0^+$ and $\{d_k\} \to d$ such that $\hat{x} + t_k d_k \in \mathcal{F}$ for all $k \in \mathbb{N}$. Moreover, since $d \in (\Omega^{\hat{x}})^-$, it follows that

$$\langle \xi, d\rangle < 0, \quad \forall \xi \in \partial^\diamond \varphi_i(\hat{x}), \quad \forall i \in I. \tag{6}$$

The Mean Value Theorem 3 implies that for each $k \in \mathbb{N}$, there exist some $u_k \in (\hat{x}\ , \hat{x}+t_k d_k)$ and $\xi_k \in \partial^\diamond \varphi_1(u_k)$ such that

$$\varphi_1(\hat{x}+t_k d_k) - \varphi_1(\hat{x}) = t_k \langle \xi_k, d_k \rangle. \tag{7}$$

The upper semicontinuity of the set-valued mapping $x \mapsto \partial^\diamond \varphi_1(x)$ and the convergence

$$u_k \to \hat{x},$$

indicate that we can extract a subsequence ξ_{k_p} of ξ_k such that $\xi_{k_p} \to \hat{\xi} \in \partial^\diamond \varphi_1(\hat{x})$. From equations (6) and (7), we derive that

$$\varphi_1(\hat{x}+t_{k_p} d_{k_p}) - \varphi_1(\hat{x}) = t_{k_p} \langle \xi_{k_p}, d_{k_p} \rangle \longrightarrow \langle \hat{\xi}, d \rangle < 0.$$

Thus, there exists a positive number $M_1 > 0$ such that

$$\varphi_1(\hat{x}+t_{k_p} d_{k_p}) < \varphi_1(\hat{x}), \quad \forall\, p > M_1.$$

This demonstrates that there exists a subsequence $\{\hat{x}+t_k^{(1)} d_k^{(1)}\}$ of $\{\hat{x}+t_k d_k\}$ satisfying

$$\varphi_1(\hat{x}+t_k^{(1)} d_k^{(1)}) < \varphi_1(\hat{x}).$$

Applying the same reasoning to $\{\hat{x}+t_k^{(1)} d_k^{(1)}\}$ and φ_2, we deduce from (7) that there exist a subsequence $\{\hat{x}+t_k^{(2)} d_k^{(2)}\}$ of $\{\hat{x}+t_k^{(1)} d_k^{(1)}\}$ such that for sufficiently large indices, we have

$$\varphi_1(\hat{x}+t_k^{(2)} d_k^{(2)}) < \varphi_1(\hat{x}) \quad \text{and} \quad \varphi_2(\hat{x}+t_k^{(2)} d_k^{(2)}) < \varphi_2(\hat{x}).$$

By repeating this argument, we can construct a subsequence $\{\hat{x}+t_k^{(m)} d_k^{(m)}\}$ of $\{\hat{x}+t_k d_k\}$ such that

$$\begin{cases} \varphi_1(\hat{x}+t_k^{(m)} d_k^{(m)}) < \varphi_1(\hat{x}), \\ \varphi_2(\hat{x}+t_k^{(m)} d_k^{(m)}) < \varphi_2(\hat{x}), \\ \vdots \\ \varphi_m(\hat{x}+t_k^{(m)} d_k^{(m)}) < \varphi_m(\hat{x}). \end{cases}$$

The derived inequalities along with the fact that $\{\hat{x}+t_k^{(m)} d_k^{(m)}\} \subset \mathcal{F}$, contradict the assumption that $\hat{x} \in \mathcal{W}$. This contradiction supports the validity of the lemma. □

Theorem 5. If the WACQ holds at $\hat{x} \in \mathcal{W}$, then the following relationship holds:

$$\mathbf{0} \in \operatorname{conv}(\Omega^{\hat{x}}) + \overline{\operatorname{cone}}(\Upsilon^{\hat{x}}). \tag{8}$$

Proof. By virtue of the WACQ and Lemma 1, we can conclude that

$$(\Omega^{\hat{x}})^{-} \cap (\Upsilon^{\hat{x}})^{0} = \emptyset. \tag{9}$$

Utilizing Equations (1) and (2), we determine the equalities

$$\Big(\text{conv}(\Omega^{\hat{x}})\Big)^{-} = (\Omega^{\hat{x}})^{-}, \quad \text{and} \quad \Big(\overline{\text{cone}}(\Upsilon^{\hat{x}})\Big)^{0} = (\Upsilon^{\hat{x}})^{0},$$

which, in conjunction with equation (9), leads us to derive:

$$\Big(\text{conv}(\Omega^{\hat{x}})\Big)^{-} \cap \Big(\overline{\text{cone}}(\Upsilon^{\hat{x}})\Big)^{0} = \emptyset. \tag{10}$$

We further assert that:

$$\Big(\text{conv}(\Omega^{\hat{x}})\Big) \cap \Big(-\overline{\text{cone}}(\Upsilon^{\hat{x}})\Big) \neq \emptyset. \tag{11}$$

Assuming the contrary, if relation (11) does not hold, we would have:

$$\Big(\text{conv}(\Omega^{\hat{x}})\Big) \cap \Big(-\overline{\text{cone}}(\Upsilon^{\hat{x}})\Big) = \emptyset.$$

Given that $\text{conv}(\Omega^{\hat{x}})$ is a non-empty compact convex set, while $\overline{\text{cone}}(\Upsilon^{\hat{x}})$ represents a closed convex cone, the Strong Separation Theorem (refer to e.g., [8]) and the last equality imply that there exists a vector $q \in \mathbb{R}^n$ satisfying:

$$\begin{cases} \langle q, y\rangle < 0, & \forall y \in \text{conv}(\Omega^{\hat{x}}) \\ \langle q, y\rangle \leq 0, & \forall y \in \overline{\text{cone}} \end{cases}$$

Consequently, we have:

$$q \in \Big(\text{conv}(\Omega^{\hat{x}})\Big)^{-} \cap \Big(\overline{\text{cone}}(\Upsilon^{\hat{x}})\Big)^{0},$$

which contradicts (10). This contradiction confirms the validity of (11). Since (11) implies (8), the proof is complete. □

As demonstrated, the Abadie type constraint qualifications introduced herein are effective for deriving optimality conditions for the problem (P). The primary challenge lies in verifying the establishment of these qualification conditions, which often proves difficult due to their dependence on the computation of tangent cones. Therefore, developing an algebraic condition to verify these qualifications, independent of the tangent cone calculations, holds considerable practical significance. This discussion will be elaborated upon in the remainder of the paper, beginning with the introduction of an appropriate definition.

Definition 1. The problem (P) is considered to be perfect at $\hat{x} \in \mathcal{F}$ if the following conditions are satisfied:

A1. The set J is a compact set within a certain metric space, and the set-valued function $j \to \vartheta_j(\hat{x})$ is upper-semicontinuous on J.

A2. The condition $(\Upsilon^{\hat{x}})^- \neq \emptyset$ holds.

It is important to note that the condition $(A1)$ is frequently assumed in many references (see e.g., [5, 7, 12]). Additionally, condition $(A2)$ is referred to as the Cottle constraint qualification in some literature, while in others, it is known as the Mangasarian-Fromovitz constraint qualification (see, e.g., [7] and [5], respectively).

Lemma 2. If the problem (P) is perfect at $\hat{x} \in \mathcal{F}$, then both $\text{conv}\big(\Upsilon^{\hat{x}}\big)$ and $\text{cone}\big(\Upsilon^{\hat{x}}\big)$ are closed convex sets.

Proof. By condition $A1$ and compactness of $\partial^\diamond \vartheta_j(\hat{x})$ as $j \in J(\hat{x})$, it is evident that $\Upsilon^{\hat{x}}$ is a compact set (refer to, e.g., [9, 20]). Consequently, $\text{conv}\big(\Upsilon^{\hat{x}}\big)$ is closed by Theorem 1(i). Moreover, given condition $A2$ and (2), we can assert that:

$$\Big(\text{conv}\big(\Upsilon^{\hat{x}}\big)\Big)^- = \big(\Upsilon^{\hat{x}}\big)^- \neq \emptyset,$$

which leads us to conclude that $\mathbf{0} \notin \text{conv}\big(\Upsilon^{\hat{x}}\big)$. Therefore, $\text{cone}\big(\Upsilon^{\hat{x}}\big)$ is also a closed set by Theorem 1(ii). □

Assuming that condition $(A1)$ is hold, we define:

$$\Psi(x) := \max_{j \in J} \vartheta_j(x), \quad \forall x \in \mathcal{F}.$$

It follows straightforwardly that $\Psi(\cdot)$ is locally Lipschitz, since each ϑ_j possesses this property. The proof of the estimate:

$$\Psi^\diamond(\hat{x}; d) \leq \max_{j \in J(\hat{x})} \vartheta_j^\diamond(\hat{x}; d), \quad \forall d \in \mathbb{R}^n, \tag{12}$$

is completely analogous to the proof of step 1 in [3, Theorem 2.8.2]. It is noteworthy that the function $j \to \vartheta_j^\diamond(\hat{x}; d)$ is upper-semicontinuous and $J(\hat{x})$ is compact, thus, the use of the notation "max" is justified in (12).

Lemma 3. If the problem (P) is perfect at $\hat{x} \in \mathcal{F}$, then the following inclusion holds:

$$\partial^\diamond \Psi(\hat{x}) \subseteq \text{conv}\big(\Upsilon^{\hat{x}}\big).$$

Proof. Let $\xi \in \partial^\diamond \Psi(\hat{x})$ be an arbitrary element. From inequality (12), we can derive that:

$$\max_{j \in J(\hat{x})} \hat{\vartheta}_j(d) \geq \langle \xi, d \rangle, \quad \forall d \in \mathbb{R}^n,$$

where $\hat{\vartheta}_j(d) := \vartheta_j^\diamond(\hat{x}; d)$. Given that the functions $\hat{\vartheta}_j(\cdot)$ are convex and satisfy $\hat{\vartheta}_j(\mathbf{0}) = 0$ for each $j \in J$, the above inequality implies that:

$$\hat{\Psi}(d) - \hat{\Psi}(\mathbf{0}) \geq \langle \xi, d - \mathbf{0} \rangle, \quad \forall d \in \mathbb{R}^n,$$

where $\hat{\Psi}$ is defined as $\hat{\Psi}(d) := \max_{j \in J(\hat{x})} \hat{\vartheta}_j(d)$. Consequently, we find that:

$$\xi \in \partial^\diamond \hat{\Psi}(\mathbf{0}).$$

Furthermore, the function $\hat{\vartheta}_j$ is continuous at $\hat{d} := \mathbf{0}$ for all $j \in J$, and the mapping $j \to \hat{\vartheta}_j(d)$ is upper-semicontinuous for every $d \in \mathbb{R}^n$. Thus, the well-known Pshenichnyi-Levin-Valadire Theorem ([8], pp. 267) can be utilized, yielding:

$$\partial \hat{\Psi}(\mathbf{0}) = \overline{\text{conv}}\Big(\bigcup_{j \in \hat{J}(\mathbf{0})} \partial \hat{\vartheta}_j(\mathbf{0}) \Big),$$

where $\hat{J}(\mathbf{0}) := \{ j \in J(\hat{x}) \mid \hat{\vartheta}_j(\mathbf{0}) = \hat{\Psi}(\mathbf{0}) = 0 \}$. This confirms the desired result, since $\text{conv}(\Upsilon^{\hat{x}})$ is closed by Lemma 2. Furthermore, the following equalities hold trivially:

$$\hat{J}(\mathbf{0}) = J(\hat{x}), \quad \text{and} \quad \partial \hat{\vartheta}_j(\mathbf{0}) = \partial^\diamond \vartheta_j(\hat{x}).$$

□

We can now present a significant result that provides a sufficient condition for establishing of WACQ.

Theorem 6. If the problem (P) is perfect at $\hat{x} \in \mathcal{F}$, then the WACQ holds at $\hat{x}$.

Proof. Let $d \in \big(\Upsilon^{\hat{x}}\big)^-$ be an arbitrarily vector. From the equality

$$\big(\Upsilon^{\hat{x}}\big)^- = \Big(\text{conv}\big(\Upsilon^{\hat{x}}\big)\Big)^-,$$

holds by (2), it follows that

$$d \in \Big(\text{conv}\big(\Upsilon^{\hat{x}}\big)\Big)^- \subseteq \big(\partial^\diamond \Psi(\hat{x})\big)^-,$$

by Lemma 3. Therefore, we have $\langle \xi, d \rangle < 0$ for all $\xi \in \partial^\diamond \Psi(\hat{x})$, which implies

$$\Psi^\diamond(\hat{x}; d) = \max\{\langle \xi, d \rangle \mid \xi \in \partial^\diamond \Psi(\hat{x})\} < 0.$$

Consequently, there exists a scalar $\delta > 0$ such that

$$\Psi(\hat{x}+\beta d) < \Psi(\hat{x}) \leq 0, \quad \forall\, \beta \in (0,\delta].$$

Thus, for all $j \in J$, we conclude that

$$\vartheta_j(\hat{x}+\beta d) \leq \Psi(\hat{x}+\beta d) < 0, \quad \forall \beta \in (0,\delta].$$

As a result, for all $\beta \in (0,\delta]$ we have $\hat{x}+\beta d \in \mathcal{F}$, leading to the conclusion

$$d \in \Gamma_{\mathcal{F}}(\hat{x}).$$

Since d was chosen arbitrary from $\left(\Upsilon^{\hat{x}}\right)^-$, we have thus proved

$$\left(\Upsilon^{\hat{x}}\right)^- \subseteq \Gamma_{\mathcal{F}}(\hat{x}).$$

Given that $\left(\Upsilon^{\hat{x}}\right)^- \neq \emptyset$ by condition A2, the closedness of $\Gamma_{\mathcal{F}}(\hat{x})$ entails that

$$big(\Upsilon^{\hat{x}})^0 = \overline{\left(\Upsilon^{\hat{x}}\right)^-} \subseteq \overline{\Gamma_{\mathcal{F}}(\hat{x})} = \Gamma_{\mathcal{F}}(\hat{x}),$$

Thus, the proof is complete. □

We can now present the Karush-Kuhn-Tucker optimality condition for the problem (P) at $\hat{x} \in \mathcal{W}$.

Theorem 7. Suppose that $\hat{x} \in \mathcal{W}$ and that at least one of the following assertions holds:

i. WACQ is satisfied at $\hat{x}$ and $cone\left(\Upsilon^{\hat{x}}\right)$ is closed.

ii. GACQ is satisfied at $\hat{x}$ and $cone\left(\Upsilon^{\hat{x}}\right)$ is closed.

iii. RACQ is satisfied at $\hat{x}$ and $cone\left(\Upsilon^{\hat{x}}\right)$ is closed.

iv. The problem (P) is perfect at $\hat{x}$.

Then, there exist non-negative scalars $\alpha_i \geq 0, \;\; i \in I$ such that $\sum_{i=1}^{m} \alpha_i = 1$, and non-negative scalars $\beta_j \geq 0, \;\; j \in J(\hat{x})$ with $\beta_j \neq 0$ for only finitely many indices, satisfying the relation

$$\mathbf{0} \in \sum_{i=1}^{m} \alpha_i \partial^{\diamond} \varphi_i(\hat{x}) + \sum_{j\in J(\hat{x})} \beta_j \partial^{\diamond} \vartheta_j(\hat{x}). \tag{13}$$

Proof. The result follows immediately from the application of (5), along with Lemma 2, and Theorems 5 and 6. □

It is important to note that assumption of closedness for cone$\left(\Upsilon^{\hat{x}}\right)$ is specified in first three conditions of the theorem above. The subsequent example illustrates that this assumption cannot be omitted, even in the convex case.

Example 1. Consider the following optimization problem:

$$\begin{aligned} \inf \quad & \big(\varphi_1(x_1,x_2), \varphi_2(x_1,x_2)\big) := (x_1, x_1) \\ \text{s.t.} \quad & \vartheta_j(x_1,x_2) \leq 0 \quad j \in J := \mathbb{N} \setminus \{1\}, \\ & (x_1, x_2) \in \mathbb{R}^2, \end{aligned}$$

where ϑ_j denotes the support function of the set

$$D_j := \text{conv}\big\{(-\sqrt[j]{\alpha}, -\alpha) \mid 0 \leq \alpha \leq 1\big\},$$

given by

$$\vartheta_j(x_1,x_2) := \sup_{(a_1,a_2)\in D_j} \big\{\big\langle (a_1,a_2), (x_1,x_2)\big\rangle\big\}, \quad \forall j \in J.$$

A brief analysis yields the following results:

- The feasible set is given by $\mathcal{F} = \big\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_1 + x_2 \geq 0\big\}$,
- The candidate optimal solution is $\hat{x} := (0,0) \in \mathcal{W}$,
- The directional derivatives are $\partial^{\diamond}\varphi_1(\hat{x}) = \partial^{\diamond}\varphi_2(\hat{x}) = \{(1,0)\}$,
- The index set is $J(\hat{x}) = J$,
- The directional derivatives of the support function are $\partial^{\diamond}\vartheta_j(\hat{x}) = D_j, \quad \forall j \in J$.

It is straightforward to verify that the WACQ holds at $\hat{x}$. Moreover, we can characterize the set of sums of weighted directional derivatives as follows:

$$\Big\{\sum_{i\in I} \beta_j \partial^{\diamond}\vartheta_j(\hat{x}) \mid \beta_j \geq 0, \text{and } \beta_j \neq 0 \text{ for finitely many } j \in J(\hat{x})\Big\}$$
$$= \Big\{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 \geq x_1,\ x_1 < 0,\ x_2 < 0\Big\} \cup \big\{(0,0)\big\}.$$

Therefore, there exists no sequence of scalars as in Theorem 7 satisfying (13). Importantly, the conclusions of Theorem 5 apply at the point $\hat{x}$; however, it can be noted that the optimization problem is not perfect at $\hat{x}$, and the cone generated by $cone\big(\Upsilon^{\hat{x}}\big)$ is not closed in this problem.

In nearly all examples considered, we have been unable to obtain positive multipliers α_is associated with the vector-valued objective function, specifically, some of the α_is values may equal to zero. This indicates that certain components of the vector-valued objective function do not influence the necessary conditions for weakly efficiency. To address the situation wherein some multipliers α_is associated with the objective function become zero in finite vector optimization problems, various approaches have developed in recent years, leading to the establishment of strong Karush-Kuhn-Tucker necessary optimality conditions. We define the strong Karush-Kuhn-Tucker condition for problem (P), as the requirement that all multipliers α_i remain positive for each component of the objective function.

Lemma 4. If $\hat{x} \in \mathcal{E}$, then the following holds:

$$\big(\partial^{\diamond}\varphi_{\ell}(\hat{x})\big)^{-} \cap \Gamma_{\mathcal{Q}^{\ell}}(\hat{x}) = \emptyset, \qquad \ell \in I.$$

Proof. Assume for the sake of contradiction that there exists a vector d such that for some $\ell \in I$,

$$d \in \big(\partial^{\diamond}\varphi_{\ell}(\hat{x})\big)^{-} \cap \Gamma_{\mathcal{Q}^{\ell}}(\hat{x}). \tag{14}$$

By the definition of the tangent cone, there exists a sequence $(t_k, d_k) \to (0^+, d)$ such that $\hat{x} + t_k d_k \in \mathcal{Q}^{\ell}$ for each $k \in \mathbb{N}$. Consequently, the definition of $\mathcal{Q}^{\ell}$ implies that

$$\begin{cases} \varphi_i(\hat{x} + t_k d_k) \leq \varphi_i(\hat{x}), \qquad i \in I \setminus \{\ell\}, \quad k \in \mathbb{N}, \\ \\ \hat{x} + t_k d_k \in \mathcal{F}, \qquad k \in \mathbb{N}. \end{cases} \tag{15}$$

Moreover, the condition represented by (14) leads to $\langle \xi, d \rangle < 0$ for all $\xi \in \partial^{\diamond}\varphi_{\ell}(\hat{x})$. Following the argument in the proof of Lemma 1, we can conclude that for some positive constant M,

$$\varphi_l(\hat{x} + t_k d_k) - \varphi_l(\hat{x}) < 0, \qquad k \geq M. \tag{16}$$

However, the statements in (15) combined with (16) contradicts the condition that $\hat{x} \in \mathcal{E}$. Therefore, the initial assumption is false, and the proof is complete. □

We now present the strong Karush-Kuhn-Tucker optimality condition for the problem (P) at $\hat{x} \in \mathcal{E}$.

Theorem 8. Let $\hat{x} \in \mathcal{E}$ and assume that the RACQ holds at $\hat{x}$. If

$$\big(\Omega^{\hat{x}}\big)^{0} \setminus \{\mathbf{0}\} \subseteq \bigcup_{i=i}^{m} \big(\partial^{\diamond}\varphi_i(\hat{x})\big)^{-},$$

then there exist scalars $\alpha_i > 0$ (for $i \in I$) and $\beta_j \geq 0$ (for $j \in J(\hat{x})$), with $\beta_j \neq 0$ for only finitely many indices, such that

$$\mathbf{0} \in \sum_{i=1}^{m} \alpha_i \partial^{\diamond}\varphi_i(\hat{x}) + \sum_{j \in J(\hat{x})} \beta_j \partial^{\diamond}\vartheta_j(\hat{x}).$$

Proof. From [18, Theorem 6.9], we have that

$$ri\Big(\operatorname{conv}\big(\Omega^{\hat{x}}\big)\Big) \subseteq \Big\{ \sum_{i=1}^{m} \alpha_i \xi_i \mid \xi_i \in \partial^{\diamond}\varphi_i(\hat{x}),\ \alpha_i > 0,\ \sum_{i=1}^{m} \alpha_i = 1 \Big\}.$$

Thus, it suffices to demonstrate that

$$\mathbf{0} \in ri\Big(\operatorname{conv}\big(\Omega^{\hat{x}}\big)\Big) + \operatorname{cone}\big(\Upsilon^{\hat{x}}\big). \tag{17}$$

Assuming for the sake of contradiction that the relation (17) does not hold, we then have

$$ri\Big(\text{conv}\big(\Omega^{\hat{x}}\big)\Big) \cap \Big(-\text{cone}\big(\Upsilon^{\hat{x}}\big)\Big) = \emptyset.$$

Applying the strong convex separation theorem ([18, Theorem 11.3]), there exists a hyperplane $H := \Big\{x \mid \langle x, d\rangle = 0 \ \text{ for some } d \in \mathbb{R}^n \setminus \{\mathbf{0}\}\Big\}$ that properly separates $\text{conv}\big(\Omega^{\hat{x}}\big)$ and $\Big(-\text{cone}\big(\Upsilon^{\hat{x}}\big)\Big)$.

Consequently, there exists a vector $d \in \mathbb{R}^n$ such that

$$\mathbf{0} \neq d \in \Big(\text{conv}\big(\Omega^{\hat{x}}\big)\Big)^0 \cap \Big(\text{cone}\big(\Upsilon^{\hat{x}}\big)\Big)^0 = \big(\Omega^{\hat{x}}\big)^0 \cap \big(\Upsilon^{\hat{x}}\big)^0.$$

Given the RACQ condition and the theorem assumption, we conclude that

$$d \in \Big(\bigcup_{i=1}^{m} \big(\partial^{\diamond}\varphi_i(\hat{x})\big)^-\Big) \cap \Big(\bigcap_{i=1}^{m} \Gamma_{Q^i}(\hat{x})\Big).$$

Thus, for all $\ell \in I$, we find

$$\big(\partial^{\diamond}\varphi_\ell(\hat{x})\big)^- \cap \Gamma_{Q^\ell}(\hat{x}) = \emptyset,$$

which contradicts the result established in Lemma 4. This contradiction verifies the theorem. □

4 Conclusion

This paper presents several extensions of Abadie constraint qualification tailored for nonsmooth multi-objective semi-infinite optimization problems. Utilizing these constraint qualifications, we establish various necessary optimality conditions in Karush-Kuhn-Tucker type for both weakly efficient and efficient solutions. The results are expressed in term of the Micheal-Penot subdifferential, which offers greater accuracy and nuanced framework compared to the Clarke subdifferential, albeit with increased complexity in application.

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