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Multi-Objective Optimization Problem Involving Max-Product Fuzzy Relation Inequalities with Application in Wireless Communication

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Abstract. This paper explores a specific category of optimization management models tailored for wireless communication systems. To enhance the efficiency of managing these systems, we introduce a fuzzy relation multi-objective programming approach. We define the concept of a feasible index set and present a novel algorithm, termed the feasible index set algorithm, which is designed to determine the optimal lexicographic solution to the problem, demonstrating polynomial computational complexity. Previous studies have indicated that the emission base stations within wireless communication systems can be effectively modeled using a series of fuzzy relation inequalities through max-product composition. This topic is also addressed in our paper. Wireless communication is widely employed across various sectors, encompassing mobile communication and data transmission. In this framework, information is transmitted via electromagnetic waves generated by fixed emission base stations.

Keywords. Fuzzy relation inequality, Linear programming, Max-Product, Wireless communication.

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1 Introduction

The concept of Fuzzy Relation Equations (FREs) was initially introduced by Sanchez [25], who explored the characteristics of their solutions. In a subsequent study [5], the Fuzzy Relation System (FRS) was utilized for image and video compression and reconstruction, with encoding and decoding processes articulated through FREs. In this context, the author introduced the ideas of maximum and minimum solutions to delineate the solution set for Max-Min FREs, and proposed a variety of solution methodologies was suggested to deepen the theoretical understanding of the subject [8, 18]. Furthermore, the max-min composition framework was extended to include max-t, addition-min, and other operational forms [6, 14, 32].

Li and Yang [14] notably introduced the addition-min Fuzzy Relation Inequalities (FRIs) to effectively model a peer-to-peer file-sharing system, highlighting that the addition-min operator does not conform to the max-$*$ composition operator. Building on the concept of the pseudo-minimal index, Yang [28] developed a pseudo-minimal-index algorithm designed to minimize a linear objective function constrained by addition-min fuzzy relation inequalities. This work sought to enhance the findings presented in [9, 12, 28].

Fuzzy relation mathematical programming refers to optimization problems involving fuzzy relation equations or inequalities as constraints. Early research in this field was conducted by Wang who introduced fuzzy relation elasticized linear programming focusing on max-min composition. This approach optimizes variables and parameters constrained within a lattice framework defined by the interval $[0, 1]$, utilizing lattice operators such as $\vee, \wedge$. The researchers applied a conservative approach to derive a minimal solution set, allowing them to identify the optimal solution for the specified problem. Recent studies [21, 22] have continued to explore this issue.

In a separate investigation, researchers analyzed fuzzy relation linear programming, breaking down the primary problem into sub-problems for detailed examination. One sub-problem was solvable via straightforward methods, while the other was reformulated as a 0-1 integer programming problem, which was addressed using the branch-and-bound technique. This approach enabled the identification of optimal solutions without the need to compute all minimal solutions associated with the constraints. Further research has refined and popularized this solution method.

In the context of fuzzy relation nonlinear optimization problems, genetic algorithms are frequently employed to search for approximate optimal solutions. However, more direct methodologies have also been proposed for certain objective functions. For instance, Cao [7] introduced a solution approach to the fuzzy relation geometric programming problem associated with max-min composition. Abbasi Molai [1, 2] studied the quadratic programming problem with fuzzy relations under max-product composition, restructuring the original problem into a series of general quadratic programming problems by determining the minimal solutions of the FRIs, which were subsequently analyzed and solved.

Peeva et al. [20] presented an analytical approach for addressing fuzzy linear systems of equations utilizing max-product composition. Their method yields a universal algorithm for finding the greatest solution and the complete set of minimal solutions when the system is consistent (i.e., solvable). Additionally, Shieh proposed an innovative technique for deriving minimal solutions in max-product fuzzy relation systems, based on the principle of minimal covering [26]. This technique not only facilitates the extraction of the initial minimal solution, but also employs a backtracking procedure for identifying all subsequent minimal solutions. Our proposed algorithm improves upon this by negating the need to assess all minimal and maximal solutions, significantly reducing computational effort while offering enhanced solution accuracy and manageable computational complexity.

Wireless communication is extensively utilized across various sectors, including mobile communication and information transmission. This paper focuses on optimization management models for wireless communication. Our proposed model involves transmitting information via electromagnetic waves from fixed Emission Base Stations (EBSs). We aim to optimize the radiation intensity of these EBSs, recognizing that while higher radiation intensity can enhance communication quality; it also raises potential health concerns.

In a recent advancement building upon prior research in fuzzy relation geometric programming, Yang et al. [30] investigated the single-variable term semi-latticized geometric programming constrained by max-product FREs. This formulation was inspired by peer-to-peer network systems, aiming to minimize the maximum dissatisfaction levels experienced by terminals within the framework. The objective functions associated with these optimization problems are characterized as specific geometric functions, and due to their nonconvex nature and inherent complexity, general geometric objective functions have received limited exploration. Furthermore, it is essential to note that real-world data tends to be discrete rather than continuous, with statistical data often consisting of discrete values corresponding to tangible phenomena. This study specifically examined a monomial geometric programming objective function constrained by bipolar max-product fuzzy relation conditions [4].

In a recent study, researchers utilized FRIs in conjunction with max-product composition to develop a model for the Wireless Communication Terminal (WCT) system. The primary objective of this investigation was to minimize the maximum radiation intensity, intentionally avoiding the prioritization of the relative importance of the terminals, particularly the emission base stations.

The following model was formulated based on this objective:

$$\begin{array}{ll} \text{Min} & Z(x) = \min\ x_1 \vee x_2 \vee \ldots \vee x_n \\ \text{s.t.} & A o x^T \geq b^T. \end{array} \tag{1}$$

Here, the constraint $Aox^T \geq b^T$ governs the model, where $x = (x_1, x_2, \ldots, x_n)$ represents the intensities of electromagnetic radiation associated with the terminals.

For the purposes of this model, all terminals are considered to have equal importance. To address this optimization issue, the researchers initially compared the characteristics of max-product FRIs with those of max-product FREs, as described in prior studies [16, 23, 24, 31]. The concept of egalitarianism was elaborated upon in another study [10], highlighting that in practical management scenarios, it may be essential to assign distinct priority levels to terminals, reflecting their relative significance.

This paper presents a lexicographically optimal solution for a multi-objective programming problem constrained by max-product FRIs, thereby contributing to optimization management in wireless communication systems. To address this issue, we developed a Feasible Index Set (FIS) algorithm.

The organization of this paper is delineated as follows: Section 2 elaborates on the implementation of the model within the wireless communication system, introducing each variable and symbol utilized. Section 3 discusses the findings related to max-product FRIs, introduces the concept of a FIS, and provides a comprehensive examination of the sequential progression of the FIS algorithm, which is designed to find the unique lexicographic optimal solution for the specified problem. We aim to demonstrate both the feasibility and effectiveness of the algorithm in this context. Section 4 offers an in-depth discussion of the FIS algorithm, while Section 5 presents a numerical example to illustrate its application.

2 Wireless Communication System Model

This study examines a multi-objective programming problem constrained by max-product fuzzy relation inequalities:

$$\begin{array}{ll} \text{Min} & \{z_1(x), z_2(x), \ldots, z_n(x)\} = \{x_1, x_2, \ldots, x_n\} \\ \text{s.t.} & Aox^T \geq b^T, \end{array} \tag{2}$$

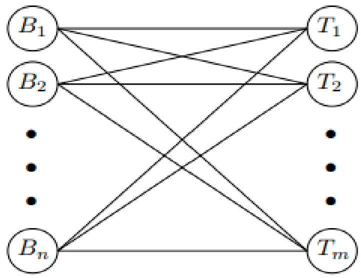

Figure 1: Wireless communication base-station system.

The optimization management framework for the Emission Base Station (EBS) in a wireless communication system is characterized by a set of fuzzy relational inequalities that employ max-product composition, as illustrated in Equation (2) and Figure 1. This system incorporates variables $x = (x_1, x_2, \ldots, x_n)$, matrices $A = (a_{ij})_{m\times n}$, vectors $b = (b_1, b_2, \ldots, b_m)$, and parameters x_j, a_{ij} and b_i all constrained to the range $[0, 1]$ with $b_i > 0$, for all $i \in I = \{1, 2, \ldots, m\}$ and $j \in J = \{1, 2, \ldots, n\}$. Here, I and J represent two distinct index sets.

In a specified geographical region, there are n Electronically Beam-Steerable (EBS) antennas labeled $E_1, E_2, \ldots, E_n$. These antennas are responsible for transmitting informational data through electromagnetic waves at uniform radiation intensity. To ensure effective communication, it is imperative that the radiation intensity of these electromagnetic waves meets specific established standards. The region contains m designated Testing Points (TPs) denoted as $T_1, T_2, \ldots, T_m$, each with a minimum requirement for the electromagnetic radiation intensity, denoted as b_i for $i = 1, 2, \ldots, m$. When a terminal device, such as a mobile phone, receives the electromagnetic signal at a TP, it connects to the EBS that exhibits provides the highest radiation intensity at that particular point.

Let the jth EBS emit electromagnetic waves with a radiation strength of x_j. When these electromagnetic waves reach the ith TP, their radiation intensity denoted as r_{ij}, decreases to a value that is less than or equal to x_j. Consequently, there exists a real number $a_{ij} \in [0, 1]$ such that $r_{ij} = a_{ij}x_j$.

We can therefore express the subsequent system:

$$\begin{cases} a_{11}x_1 \vee a_{12}x_2 \vee \ldots \vee a_{1n}x_n \geq b_1, \\ a_{21}x_1 \vee a_{22}x_2 \vee \ldots \vee a_{2n}x_n \geq b_2, \\ \vdots \\ a_{m1}x_1 \vee a_{m2}x_2 \vee \ldots \vee a_{mn}x_n \geq b_m. \end{cases} \tag{3}$$

The radiation intensity x_j and the communication quality requirement b_i are generally constrained for all $i \in I$ and $j \in J$, allowing for normalization within the unit interval $[0, 1]$. If there exists an index i_0 such that $b_{i_0} = 0$, removing the corresponding inequality involving i_0

from the system (3) will not affect its solution set. Consequently, it is typically assumed that a_{ij}, x_j and b_i are all within the range $[0, 1]$, with $b_i > 0$ in the System (3). Essentially, System (3) represents a set of max-product FRIs and can be expressed as follows:

$$Aox^T \geq b^T, \tag{4}$$

where

$$A = (a_{ij})_{m\times n} \in [0,1]^{m\times n}, x = (x_1, x_2, \ldots, x_n) \in [0,1]^n, b = (b_1, b_2, \ldots, b_m) \in (0,1]^m.$$

The aim of this study is to reduce the harmful effects of electromagnetic radiation on human health by minimizing the intensities of electromagnetic wave radiation, denoted as variables $x = (x_1, x_2, \ldots, x_n)$. This can be formulated as a vector optimization model, which addresses the issue of

$$\begin{aligned} \text{Min} \quad & Z(x) = \min\{x_1, x_2, \ldots, x_n\} \\ \text{s.t.} \quad & Aox^T \geq b^T. \end{aligned} \tag{5}$$

Simultaneously minimizing all variables is often infeasible, making it necessary to prioritize them based on a specific hierarchy. In this study, we establish a priority order of the variables: $x_1 \to x_2 \to \cdots \to x_n$. This indicates that the primary objective is to minimize x_1 first, followed by x_2, and so on, with x_n being the last variable to minimize. The primary focus of this research is to determine the lexicographic optimal solution for the Problem (5). The set X is characterized as the n-dimensional Cartesian product of the closed interval $[0, 1]$, which can be formally represented as follows:

$$X = [0,1]^n = \{x = (x_1, x_2, \ldots, x_n) \,|0 \leq x_j \leq 1, j = 1, 2, \ldots, n\}. \tag{6}$$

3 Max-Product Fuzzy Relations

It is essential to recognize that some equations related to fuzzy relations may not align with the corresponding maximum and minimum solutions. Consequently, it is crucial to discuss relevant findings regarding the resolution of max-product FRIs before addressing the issue outlined in Equation (2).

Definition 1. [20] States that for any two vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ belonging to the set X, $x \prec y$ if and only if

$$\begin{aligned} &(1)\ x_1 < y_1 \text{ or} \\ &(2)\ x_1 = y_1 \text{ and } x_2 < y_2 \text{ or} \\ &\vdots \\ &(n)\ x_1 = y_1, \ldots, x_{n-1} = y_{n-1} \text{ and } x_n < y_n. \end{aligned} \tag{7}$$

Remark 1. Based on Definition 1, it is straightforward to verify that for any $x, y \in X$, the relationship $x \preceq y$ holds if and only if

$$\begin{aligned}&(1)\ x_1 \le y_1 \text{ and}\\&(2)\ \text{If } x_1 = y_1 \text{ then } x_2 \le y_2 \text{ and}\\&\vdots\\&(n)\ \text{If } x_1 = y_1, \dots, x_{n-1} = y_{n-1} \text{ then } x_n \le y_n.\end{aligned} \tag{8}$$

Denote $X = [0,1]^n$ and $X(A,b) = \{x \in X \mid Aox^T \ge b^T\}$.

Definition 2. [3, 25] A solution $\hat{x} \in X\,(A,b)$ is referred to as the maximum solution of $Aox^T \ge b^T$ if $x \le \hat{x}$ for all $x \in X(A,b)$. A solution $\breve{x} \in X\,(A,b)$ is considered a minimal solution to $Aox^T \ge b^T$ when $x \le \breve{x}$ implies that $x = \breve{x}$ for any $x \in X\,(A,b)$.

Obviously, if the maximum solution of (3) exists, it is unique, with $\hat{x} = (1, 1, \dots, 1)$.

Theorem 1. [31] Let $Aox^T \ge b^T$ be a system of max-product FRIs. This system is consistent if and only if $\hat{x} = (1, 1, \dots, 1) \in X\,(A,b)$. Furthermore, when the system is consistent, the complete solution set $X\,(A,b)$ can be determined by identifying a singular maximum solution along with a finite quantity of minimal solutions, specifically

$$X\,(A,b) = \bigcup_{S \in F(S)} \{x \in [0,1]^n \mid \breve{x} \le x \le \hat{x}\}, \tag{9}$$

where $F(S)$ represents the set of all minimal solutions of (3).

The feasible domain defined by the max-product FRIs (3) can be thoroughly described by the solution set $X\,(A,b)$. Indeed, possessing knowledge of all minimal solutions allows for expressing the solution set $X\,(A,b)$. The primary challenge lies in computing all minimal solutions for the system represented by (3). However, this task is classified as NP-hard. Consequently, we propose a methodology aimed at identifying all potential minimal solutions, rather than exclusively focusing on minimal solutions.

Proposition 1. Let $x_1, x_2 \in X$. If $x_1 \le x_2$, then $x_1 \preceq x_2$.

Definition 3. [20] A feasible solution $x^* \in X\,(A,b)$ is considered a lexicographically optimal solution to Problem (2) if and only if $x^* \preceq x$ for every $x \in X\,(A,b)$.

Proposition 2. [10] If x is a solution of the System (3) and $x' \ge x$, with $x' \in X$, then x' is also a solution of System (3).

Corollary 1. Let $\hat{x} = (1, 1, \dots, 1) \in X$. If $x^1, x^2 \in X\,(A,b)$, then $x^1 \vee x^2 \in X\,(A,b)$.

Lemma 1. If both x^1 and x^2 are solutions of system (3), and $x^1 \le x^2$, then for every $y \in [x^1, x^2]$, y is also a solution of system (3).

Proof. The proof is straightforward. □

Definition 4. [22] For any two vectors $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T \in [0,1]^n$, we define $x \geq y$ (or $x \leq y$) if $x_j \geq y_j$ (or $x_j \leq y_j$) for each $j = 1, 2, \ldots, n$.

Lemma 2. [22] Let $x, y \in [0,1]^n$. If $x \leq y$, then $Z_1(x) \leq Z_1(y)$.

Theorem 2. [13, 17, 27] If System (3) is considered consistent, then it has a solution to equation (3), i.e. $\hat{x} \in X(A, b)$ and conversely.

The initial theorem asserts that the consistency of System (3) can be assessed by analyzing its potential maximum solution $\hat{x} = (1, 1, \ldots, 1)$. Following this, alternative approaches for evaluating the consistency of System (2) will be presented. The terms "discrimination index set" and "discrimination matrix" are derived from definitions outlined in previous research [6, 15]. These concepts have been adapted for use in the relevant fuzzy relation inequality systems.

The following statement provides a concise expression:

$$J_i = \{j \in J \mid a_{ij} \geq b_i\}, \tag{10}$$

for every $i \in I$. The sets $J_1, J_2, \ldots, J_m$ are referred to as the discrimination index sets of System (3). Additionally, this is denoted as $\pi = J_1 \times J_2 \times \ldots \times J_m$.

Theorem 3. [11, 29] If System (3) is consistent, then $J_i \neq \emptyset$ for every $i \in I$ and conversely.

From Theorem 3, we can derive the following:

Corollary 2. If System (3) is consistent, then $\pi \neq \emptyset$ and conversely.

Definition 5. [31] A matrix $D = (d_{ij})_{m \times n}$ is called the discrimination matrix of System (3) if and only if for every $i \in I$, $j \in J$

$$d_{ij} = \begin{cases} \frac{b_i}{a_{ij}}, & \text{if } j \in J_i, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Theorem 4. [31] The consistency of System (3) is defined by the condition that each row in the matrix D must contain at least one non-zero entry.

Definition 6. [10] Consider the matrix D, and $S = (s_{ij})_{m \times n}$, where $s_{ij} \in \{0, d_{ij}\}$. If the matrix S is regarded as a solution matrix for System (3), then every row of S contains a distinct non-zero element, and conversely.

Let $F(S)$ represent the collection of all matrices that satisfy System (3). If S is an element of the solution set $F(S)$ for System (3), we define the vector $x^S = (x_1^S, x_2^S, \ldots, x_n^S)$ as follows:

$$x_j^S = \bigvee_{i \in I} s_{ij}, \qquad j \in J. \tag{12}$$

Theorem 5. [10] If S represents a solution for System (3), then the vector x^S defined in Equation (12) serves as a solution to System (3).

The vector x^S, as delineated by Equation (12), is referred to as the solution associated with the solution matrix S in the context of System (3).

Theorem 6. If Equation (3) is considered consistent, then the resulting set of solutions is as follows:

$$X(A,b) = \bigcup_{S \in F(S)} \left\{ x^S \leq x \leq \hat{x} \right\}. \tag{13}$$

The solution denoted as x^S for a given set S, and $\hat{x} = (1, 1, \ldots, 1)$ represents the optimal solution for the System (3).

Proof. Considering the propositions: (Let S be a solution matrix for the discrimination matrix C, and let x^S be defined as (12). Then x^S is a solution of System (3)) and (Let x^0 be an arbitrary solution of System (3). Then, there exists a solution matrix S and its corresponding solution x^S such that $x^S \leq x^0$). As indicated in [20], this theorem can be readily demonstrated. □

In the next section, we introduce the concept of a FIS, which serves as the foundation for developing an algorithm known as the FIS algorithm. This algorithm is intended to identify the unique lexicographically optimal solution for the Problem (2). The following section has been adapted from reference [10], with some modifications in symbols and variables.

It is assumed that the system represented by Equation (3) is aligned with the discrimination matrix:

$$D = \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ d_{m1} & d_{m2} & \ldots & d_{mn} \end{bmatrix}. \tag{14}$$

Let

$$j_i^* = \max \left\{ j \in J \mid d_{ij} > 0 \right\}, \qquad i \in I, \tag{15}$$

and

$$I_j = \left\{ i \in I \mid j_i^* = j \right\}, \qquad j \in J. \tag{16}$$

then, we arrive at the index set $\{I_j \mid j \in J\}$.

Definition 7. In Equation (3), let the variable $x = (x_1, x_2, \ldots, x_n)$ be an element of the set X. The set

$$\left\{ i \in I \mid a_{i1}x_1 \vee a_{i2}x_2 \vee \ldots \vee a_{in}x_n \geq b_i \right\}, \tag{17}$$

is termed the FIS of x and is denoted by I_x.

Theorem 7. Let $x, y \in X$ with $x \leq y$, then $I_x \subseteq I_y$.

Proof. If $i \in I_x$, then $a_{i1}x_1 \vee a_{i2}x_2 \vee \cdots \vee a_{in}x_n \geq b_i$. Furthermore, the inequality $x \leq y$ indicates that each component of vector y is greater than or equal to the corresponding component of vector x for all indices $j \in J$. Since all elements in vector a are non-negative for all indices j in set J, we can conclude that

$$a_{i1}y_1 \vee a_{i2}y_2 \vee \cdots \vee a_{in}y_n \geq a_{i1}x_1 \vee a_{i2}x_2 \vee \cdots \vee a_{in}x_n.$$

From the inequalities established above, we deduce that $a_{i1}y_1 \vee a_{i2}y_2 \vee \cdots \vee a_{in}y_n \geq b_i$. Therefore, $I_x \subseteq I_y$. □

Definition 8. Let $x^1, x^2, \ldots, x^t \in X$. We define $y = x^1 \vee x^2 \vee \cdots \vee x^t = (y_1, y_2, \ldots, y_n)$ as follows:

$$y_j = x_j^1 \vee x_j^2 \vee \cdots \vee x_j^t. \tag{18}$$

Lemma 3. Let $x^1, x^2, \ldots, x^t \in X$, then $x^k \leq x^1 \vee x^2 \vee \cdots \vee x^t$ applies to each $k \in \{1, 2, \ldots, t\}$.

Lemma 4. Let $x^1, x^2, \ldots, x^t \in X$, then

i. $I_{x^1 \vee x^2 \vee \cdots \vee x^t} = I_{x^1} \cup I_{x^2} \cup \cdots \cup I_{x^t}$

ii. $x^1 \vee x^2 \vee \cdots \vee x^t$ is a solution of System (3) if and only if $I_{x^1} \cup I_{x^2} \cup \cdots \cup I_{x^t} = I$.

Proof. (i) By Lemmas 3 and associated theorem it is evident that $I_{x^k} \subseteq I_{x^1 \vee x^2 \vee \cdots \vee x^t}$ is applicable to any $k \in \{1, 2, \ldots, t\}$. As a result, we have $I_{x^1} \cup I_{x^2} \cup \cdots \cup I_{x^t} \subseteq I_{x^1 \vee x^2 \vee \cdots \vee x^t}$. To complete the proof, we must verify that $I_{x^1 \vee x^2 \vee \cdots \vee x^t} \subseteq I_{x^1} \cup I_{x^2} \cup \cdots \cup I_{x^t}$. Let $i \in I_{x^1 \vee x^2 \vee \cdots \vee x^t}$. Denote $x^1 \vee x^2 \vee \cdots \vee x^t = y = (y_1, y_2, \ldots, y_n)$, then $a_{i1}y_1 \vee a_{i2}y_2 \vee \cdots \vee a_{in}y_n \geq b_i$. There exists a $j_0 \in J$ such that $a_{ij_0}y_{j_0} \geq b_i$. Additionally, there is a $k_0 \in \{1, 2, \ldots, t\}$ such that $x_{j_0}^{k_0} = x_{j_0}^1 \vee x_{j_0}^2 \vee \cdots \vee x_{j_0}^t$. Therefore, we conclude that $a_{i1}x_1^{k_0} \vee a_{i2}x_2^{k_0} \vee \cdots \vee a_{in}x_n^{k_0} \geq a_{ij_0}x_{j_0}^{k_0} \geq b_i$. i.e., $i \in I_{x^{k_0}}$. In conclusion, we assert that $i \in I_{x^{k_0}} \subseteq I_{x^1} \cup I_{x^2} \cup \cdots \cup I_{x^t}$ and $I_{x^1 \vee x^2 \vee \cdots \vee x^t} \subseteq I_{x^1} \cup I_{x^2} \cup \cdots \cup I_{x^t}$.

(ii) The supporting evidence can be found in Theorem 2 and part (i).

□

Definition 9. [20] Let $x = (x_1, x_2, \ldots, x_n) \in X$. For any $j \in J$, let $x_j^\theta = \left(x_1^{j\theta}, x_2^{j\theta}, \ldots, x_n^{j\theta}\right)$, where

$$x_k^{j\theta} = \begin{cases} x_j, & k = j, \\ 0, & k \neq j. \end{cases}$$

The FIS of x_j^θ, denoted as $I_{x_j^\theta}$, is referred to as the feasible index set of x_j, i.e., I_{x_j}.

Lemma 5. Let $x^1, x^2, \ldots, x^t \in X$. Then:

i. $I_x = I_{x_1} \cup I_{x_2} \cup \cdots \cup I_{x_n}$,

ii. If the variable x represents a solution to System (3), then $I_{x_1} \cup I_{x_2} \cup \cdots \cup I_{x_n} = I$ and conversely.

Proof. Considering

$$\begin{aligned} x &= (x_1, x_2, \ldots, x_n) \\ &= (x_1, 0, 0, \ldots, 0) \vee (0, x_2, 0, \ldots, 0) \vee \cdots \vee (0, 0, 0, \ldots, x_n) \\ &= x_1^\theta \vee x_2^\theta \vee \cdots \vee x_n^\theta, \end{aligned}$$

the conclusion of this proof follows from Lemma 3. □

Proposition 3. Let $x^1, x^2, \ldots, x^t \in X$ and $D = (d_{ij})_{m\times n}$ denote the discrimination matrix for System (3). Then, for every $j \in J$, it holds that

$$I_{x_j} = \{i \in I \mid x_j \geq d_{ij} > 0\} \tag{19}$$

Based on this information, we will now proceed to the FIS algorithm.

4 Proposed Algorithm

We will introduce a novel algorithm aimed at obtaining the lexicographically optimal solution for Problem (2), employing the discrimination matrix alongside the concept of the FIS. Since this algorithm is based on the feasible index set, it will be referred to as the FIS Algorithm.

Remark 2. In the FIS algorithm, we have:

$$\begin{cases} (i) \begin{cases} I'_1 = I_1, \\ I'_2 = I_2 - I_{x_1^*}, \\ I'_3 = I_3 - I_{x_1^*} - I_{x_2^*}, \\ \vdots \\ I'_n = I_n - I_{x_1^*} - I_{x_2^*} - \cdots - I_{x_{n-1}^*}. \end{cases} \\ (ii)\ x_k^* = \bigvee_{i \in I'_k} d_{ik}, \quad k \in J. \end{cases} \tag{20}$$

The primary objective of the FIS algorithm is to verify the optimality of the resulting vector x^*.

Algorithm 1 FIS Proposed Algorithm

Data: Input sets I, J, matrix a_{ij}, and vector b_i

Result: Lexicographically optimal solution x^* for Problem (2)

Step 1: For $i \in I$, compute $J_i = \{j \in J \mid a_{ij} \geq b_i\}$.

Step 2: If $J_i \neq \emptyset$ holds for any $i \in I$, then Problem (2) is feasible and go to Step 3. Otherwise, if there exists an index $i \in I$ such that $J_i = \emptyset$, then Problem (2) does not possess a lexicographic optimal solution, and the process terminates.

Step 3: For $i \in I$ and $j \in J$, if $j \in J_i$, compute $d_{ij} = \frac{b_i}{a_{ij}}$, otherwise $d_{ij} = 0$.

Step 4: For $i \in I$, calculate $J_i^* = \max\{j \in J \mid d_{ij} > 0\}$.

Step 5: For $j \in J$, calculate $I_j = \{i \in I \mid J_i^* = j\}$.

Step 6:

For $k = 1$, set $I'_k = I_k$, compute $x_k^* = \bigvee_{i \in I'_k} d_{ik}$.

For $k \in [2, n]$, compute $I_{x_{k-1}^*} = \{i \in I \mid x_{k-1}^* \geq d_{i(k-1)} > 0\}$,

Let $I'_k = I_k - \bigcup_{j=1}^{k-1} I_{x_j^*}$,

Compute $x_k^* = \bigvee_{i \in I'_k} d_{ik}$.

We evaluate the computational complexity of the FIS algorithm based on the programming outlined. It is important to note that the dimensions m and n are determined by the matrices $A_{m\times n}$ and $b_{1\times n}$, where n represents the number of variables, and m denotes the number of inequalities in the constraints. Therefore, the computational complexity of the FIS algorithm is characterized as $O(mn^2)$.

5 Case Study

Example 1. In this section, we present a numerical example to illustrate the practicality and effectiveness of the proposed Algorithm 1. This example focuses on optimal resource management in wireless communication systems. Specifically, we consider a scenario involving an EBS, with six potential base stations and seven specific testing locations. The optimization model for this system can be redefined as a multi-objective programming problem characterized by fuzzy relations.

$$\begin{aligned} \min \quad & \{x_1, x_2, \ldots, x_6\} \\ \text{s.t.} \quad & A o x^T \geq b^T, \end{aligned} \tag{21}$$

where $x = (x_1, x_2, \ldots, x_6)$, $b = (b_i) = (0.6, 0.7, 0.7, 0.6, 0.6, 0.65)$. Consider the values presented in Table 1, which can be organized into a matrix $A = (a_{ij})$. Each entry of this matrix represents the radiation intensity between six base stations and seven designated test points.

Table 1: Radiation intensity values

Base Station \Test Point	# 1	# 2	# 3	# 4	# 5	# 6	# 7
Base Station 1	0.65	0.6	0.65	0.7	0.3	0.2	0.15
Base Station 2	0.7	0.5	0.6	0.55	0.1	0.3	0.35
Base Station 3	0.8	0.7	0.75	0.7	0.5	0.6	0.6
Base Station 4	0.6	0.25	0.3	0.35	0.9	0.8	0.9
Base Station 5	0.8	0.75	0.7	0.7	0.3	0.35	0.2
Base Station 6	0.7	0.6	0.45	0.8	0.8	0.7	0.5

Solution Steps: Step 1: We establish the discrimination index, indicating that:

$$\begin{aligned} J_1 &= \{1, 2, 3, 4\}, \\ J_2 &= \{1\}, \\ J_3 &= \{1, 2, 3, 4\}, \\ J_4 &= \{1, 5, 6, 7\}, \\ J_5 &= \{1, 2, 3, 4\}, \\ J_6 &= \{1, 4, 5, 6\}. \end{aligned} \tag{22}$$

Step 2: It is clear that $J_i \neq \emptyset$ for each $i \in \{1, 2, \ldots, 6\}$ satisfies the necessary conditions. Following Theorem 2, the constraints present in this scenario are consistent, confirming the feasibility of the problem.

Step 3: According to Definition 5, we define the relevant discrimination matrix for this analysis:

$$D = (d_{ij}) = \begin{bmatrix} 0.92 & 1 & 0.92 & 0.85 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.87 & 1 & 0.93 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.66 & 0.75 & 0.66 \\ 0.75 & 0.8 & 0.85 & 0.85 & 0 & 0 & 0 \\ 0.92 & 0 & 0 & 0.81 & 0.81 & 0.92 & 0 \end{bmatrix}. \tag{23}$$

Step 4: Using (15), we derive the following results:

$$J_1^* = 4, J_2^* = 1, J_3^* = 4, J_4^* = 7, J_5^* = 4, J_6^* = 6.$$

Step 5: As per (16), the feasible index sets can be identified through the following computational methods:

$$I_1 = \{2\}, I_4 = \{1,3,5\}, I_6 = \{6\}$$

and

$$I_2 = I_3 = I_5 = \emptyset.$$

Step 6:

i. Let $I'_1 = I_1 = \{2\}$. Then $x_1^* = \bigvee_{i \in I'_1} d_{i1} = \bigvee_{i \in \{2\}} d_{21} = 1$.

ii. Determine the feasible indices set $I_{x_1^*} = \{i \mid x_1^* \geq d_{i1} > 0\} = \{i \mid 1 \geq d_{i1} > 0\} = \{1,2,3,5,6\}$.

Let $I'_2 = I_2 - I_{x_1^*} = \emptyset - \{1,2,3,5,6\} = \emptyset$. Then $x_2^* = \bigvee_{i \in I'_2} d_{i2} = \bigvee_{i \in \emptyset} d_{i2} = 0$.

iii. Determine $I_{x_2^*} = \{i \mid x_2^* \geq d_{i2} > 0\} = \{i \mid 0 \geq d_{i1} > 0\} = \emptyset$.

Let $I'_3 = I_3 - I_{x_1^*} - I_{x_2^*} = \emptyset - \{1,2,3,5,6\} - \emptyset = \emptyset$. Then $x_3^* = \bigvee_{i \in I'_3} d_{i3} = \bigvee_{i \in \emptyset} d_{i3} = 0$.

iv. Determine $I_{x_3^*} = \{i \mid x_3^* \geq d_{i3} > 0\} = \{i \mid 0 \geq d_{i3} > 0\} = \emptyset$.

Let $I'_4 = I_4 - I_{x_1^*} - I_{x_2^*} - I_{x_3^*} = \{1,3,5\} - \{1,2,3,5,6\} - \emptyset - \emptyset = \emptyset$. Then $x_4^* = \bigvee_{i \in I'_4} d_{i4} = \bigvee_{i \in \emptyset} d_{i4} = 0$.

v. Determine $I_{x_4^*} = \{i \mid x_4^* \geq d_{i4} > 0\} = \{i \mid 0 \geq d_{i4} > 0\} = \emptyset$.

Let $I'_5 = I_5 - I_{x_1^*} - I_{x_2^*} - I_{x_3^*} - I_{x_4^*} = \emptyset - \{1,2,3,5,6\} - \emptyset - \emptyset - \emptyset = \emptyset$. Then $x_5^* = \bigvee_{i \in I'_5} d_{i5} = \bigvee_{i \in \emptyset} d_{i5} = 0$.

vi. Determine $I_{x_5^*} = \{i \mid x_5^* \geq d_{i5} > 0\} = \{i \mid 0 \geq d_{i5} > 0\} = \emptyset$.

Let $I'_6 = I_6 - I_{x_1^*} - I_{x_2^*} - I_{x_3^*} - I_{x_4^*} - I_{x_5^*} = \{6\} - \{1,2,3,5,6\} - \emptyset - \emptyset - \emptyset - \emptyset = \emptyset$. Then $x_6^* = \bigvee_{i \in I'_6} d_{i6} = \bigvee_{i \in \emptyset} d_{i6} = 0$.

Step 7: We find the lexicographically optimal solution to the problem:

$$x^* = (1,0,0,0,0,0). \tag{24}$$

The values $x_1, x_2, \ldots, x_6$ indicate the intensity of electromagnetic radiation at the terminals, ranging from a minimum of zero to a maximum of one.

Due to the complexity of minimizing each variable in this context, we decided to focus on minimizing these variables according to a specified hierarchy of importance, i.e.,

$$x_1 \to x_2 \to \cdots \to x_6.$$

This decision led us to derive the optimal lexical solution to the problem.

Example 2. In this scenario, we analyze a wireless communication system featuring a configuration of EBS that consists of 10 potential base stations and 8 designated testing points. The corresponding optimization model can be reformulated as a fuzzy relational latticed linear programming problem.

$$\begin{aligned} Min \quad & \{x_1, x_2, \ldots, x_{10}\} \\ \text{s.t.} \quad & Aox^T \geq b^T, \end{aligned}$$

where $x = (x_1, x_2, \ldots, x_{10})$, $b = (b_i) = (0.6, 0.65, 0.6, 0.7, 0.7, 0.6, 0.6, 0.65)$, and A is defined in Table 2.

Solution Steps: Step 1: We establish the discrimination index:

$$\begin{aligned} J_1 &= \{1, 2, 3, 4, 5, 6, 7\}, \\ J_2 &= \{1, 2, 3, 4\}, \\ J_3 &= \{1, 2, 3, 4, 5, 6, 7, 9, 10\}, \\ J_4 &= \{8, 9, 10\}, \\ J_5 &= \{2, 4, 5, 6, 7\}, \\ J_6 &= \{4, 5, 7, 8, 9\}, \\ J_7 &= \{6, 7, 8, 9, 10\}, \\ J_8 &= \{4, 5, 6, 7, 9\} \end{aligned} \tag{25}$$

Step 2: It is evident that $J_i \neq \emptyset$ for each $i \in \{1, 2, \ldots, 8\}$. Following Theorem 2, the constraints of the problem are consistent, indicating that the problem is feasible.

Step 3: In line with Definition 5, the relevant discrimination matrix is defined as follows:

$$D = (d_{ij}) = \begin{bmatrix} 0.67 & 0.75 & 0.71 & 0.92 & 1 & 0.92 & 0.86 & 0 & 0 & 0 \\ 0.93 & 0.81 & 0.87 & 0.93 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.75 & 0.80 & 0.75 & 0.75 & 0.86 & 0.80 & 0.86 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.78 & 0.88 & 0.78 \\ 0 & 1 & 0 & 0.88 & 0.93 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.86 & 1 & 0 & 0.75 & 0.75 & 0.86 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.71 & 0.86 & 0.86 & 1 & 0.92 \\ 0 & 0 & 0 & 0.81 & 0.81 & 0.93 & 0.87 & 0 & 0.81 & 0 \end{bmatrix}. \tag{26}$$

Step 4: From (15), we derive:

$$J_1^* = 7, J_2^* = 4, J_3^* = 10, J_4^* = 10, J_5^* = 7, J_6^* = 9, J_7^* = 10, J_8^* = 9.$$

Step 5: Using (16), the feasible index sets can be determined through the following computational methods: $I_4 = \{2\}$, $I_7 = \{1, 5\}$, $I_9 = \{6, 8\}$, $I_{10} = \{3, 4, 7\}$ and $I_1 = I_2 = I_3 = I_5 = I_6 = I_8 = \emptyset$.

Step 6: For $k = 1$, let $I'_k = I_k$. Compute $x^*_k = \bigvee_{i \in I'_k} d_{ik}$. For $k = 2, 3, \ldots, n$, compute

$$I_{x^*_{k-1}} = \left\{ i \in I | x^*_{k-1} \geq d_{i(k-1)} > 0 \right\},$$

let $I'_k = I_k - \bigcup_{j=1}^{k-1} I_{x^*_j}$, and compute $x^*_k = \bigvee_{i \in I'_k} d_{ik}$.

$$\begin{aligned} &x^*_1 = 0.67, \ x^*_2 = 0.81, \ x^*_3 = 0, \ x^*_4 = 0.88, x^*_5 = 0, \\ &x^*_6 = 0.71, \ x^*_7 = 0, \ x^*_8 = 0.78, \ x^*_9 = 0, \ x^*_{10} = 0. \end{aligned} \tag{27}$$

Step 7: We identify the lexicographically optimal solution to the problem

$$x^* = (0.67, 0.81, 0, 0.88, 0, 0.71, 0, 0.78, 0, 0). \tag{28}$$

The values $x_1, x_2, \ldots, x_6$ indicate the intensity of electromagnetic radiation at the terminals, ranging from a minimum of zero to a maximum of 0.88.

Given the impracticality of minimizing each variable in this context, we opted to prioritize minimizing these variables according to a designated hierarchy of importance, i.e., $x_1 \to x_2 \to \cdots \to x_6$. Thus, we achieved the optimal lexical solution to the problem.

Table 2: Radiation intensity values

Base Station \Test Point	# 1	# 2	# 3	# 4	# 5	# 6	# 7	# 8	# 9	# 10
Base Station 1	0.9	0.8	0.85	0.65	0.6	0.65	0.7	0.3	0.2	0.15
Base Station 2	0.7	0.8	0.75	0.7	0.5	0.6	0.55	0.1	0.3	0.35
Base Station 3	0.8	0.75	0.8	0.8	0.7	0.75	0.7	0.5	0.6	0.6
Base Station 4	0.4	0.55	0.5	0.1	0.25	0.3	0.35	0.9	0.8	0.9
Base Station 5	0.6	0.7	0.65	0.8	0.75	0.7	0.7	0.3	0.35	0.2
Base Station 6	0.1	0.15	0.35	0.7	0.6	0.45	0.8	0.8	0.7	0.5
Base Station 7	0.2	0.1	0.1	0.4	0.5	0.85	0.7	0.7	0.6	0.65
Base Station 8	0.35	0.5	0.4	0.8	0.8	0.7	0.75	0.5	0.8	0.6

6 Conclusions

The study outlines a multi-objective programming problem that utilizes max-product fuzzy relational inequalities (FRIs) to develop an optimal management strategy for Emission Base Stations (EBSs) in wireless communication systems. In this framework, the concept of a lexicographic optimal solution is introduced, and shown to be unique under specific conditions. The research also clarifies significant findings regarding max-product FRIs in contrast to fuzzy relational equations (FREs) and discusses the feasible solution set along with their relevant properties to address the proposed optimization challenge. Notably, the lexicographic optimal solution is identified as one of the minimal solutions subject to the given constraints. Although a polynomial algorithm for addressing max-product fuzzy relation systems has yet to be developed, the Feasible Index Set (FIS) algorithm has been designed to effectively determine the unique lexicographic optimal solution without requiring a comprehensive identification of all minimal solutions. This algorithm demonstrates polynomial computational complexity and can be adapted to tackle multi-objective programming problems constrained by max-min fuzzy relation inequalities.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

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Competing Interests

The authors declare that they have no competing interests relevant to the content of this paper.

Authors' Contributions

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