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Bell's Degree Variance and Degree Deviation in Graphs: Analyzing Optimal Graphs Based on These Irregularity Measures

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Abstract. An irregularity measure (IM) of a connected graph G is defined as a non-negative graph invariant that satisfies the condition $IM(G) = 0$ if and only if G is a regular graph. Among the prominent degree-based irregularity measures are Bell's degree variance, denoted as $Var_B(G)$, and degree deviation, represented as $S(G)$. Specifically, they are defined by the equations $Var_B(G) = \frac{1}{n}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2$ and $S(G) = \sum_{i=1}^{n}\left|d_i - \frac{2m}{n}\right|$, where m is the number of edges and n is the number of vertices in G. This paper studies the properties of Bell's degree-variance and degree deviation for acyclic, unicyclic, and cactus graphs. Our analysis shows how these measures relate to graph topology and structure, influencing the overall irregularity. Additionally, we identify and analyze optimal graphs that minimize both irregularity measures, providing insights into their implications for network design, data structure optimization, and real-world applications. This study contributes to the understanding of graph irregularity and offers a framework for future research into irregularity measures across different classes of graphs.

Keywords. Irregularity, Degree-variance, Degree deviation, Optimal graphs, Tree, Unicyclic, Cactus graphs.

MSC. 92C40; 62H30.

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1 Introduction

Let G be a connected simple graph with vertex set $V(G)$ and edge set $E(G)$, comprising n vertices and m edges, respectively. The maximum degree and the minimum degree of vertices in G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. For any vertex $u_i \in V(G)$, we denote its degree by $d_G(u_i)$ and throughout this paper, we will use the simpler notations $d(u_i)$ or d_i interchangeably. A vertex that is adjacent to all other vertices in G is referred to as a *universal vertex*. The number of vertices with degree i is represented as N_i. A graph G is defined to be R-regular if all its vertices have the same degree R; otherwise, it is classified as an irregular graph. If the degree sequence $D_s(G)$ of an irregular graph G consists of exactly k distinct degrees, then G is termed a k-degreed graph. Specifically, an irregular graph featuring precisely two different degrees is called a bidegreed graph, denoted as $G(\Delta, \delta)$.

According to Bell's definition [3], an irregularity measure (IM) for a (connected) graph G is a non-negative graph invariant such that $IM(G) = 0$ if and only if G is regular. The earliest irregularity measure was proposed by Collatz and Sinogowitz [5] in 1957. Other notable measures include the Collatz-Sinogowitz index [3], Albertson index [1], total irregularity [2], and the sigma index [8, 12]. Among these, the most widely used measures of irregularity are *degree deviation* $S(G)$ and *degree variance* $Var(G)$. The degree deviation was introduced by Nikiforov [17], and is defined for a connected graph G of order n and size m as:

$$S(G) = \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right|,$$

while Bell [3] defined degree variance as:

$$Var_B(G) = \frac{1}{n} \sum_{i=1}^{n} \left(d_i - \frac{2m}{n} \right)^2.$$

Here, $\frac{2m}{n}$ represents the average degree of the vertices in graph G. For clarity, we denote $Var_B(G)$ simply as $Var(G)$.

Additionally, we introduce two novel graph irregularity measures $IRD(G)$ and $IRR(G)$, which involve coefficients of the harmonic mean. These measures are defined as follows (see [9]):

$$IRD(G) = \frac{2N_\Delta N_\delta}{N_\Delta + N_\delta}(\Delta - \delta),$$

$$IRR(G) = \frac{n}{2}(\Delta - \delta).$$

The study of optimal graphs based on irregularity focuses on identifying graphs within a specific class that maximize or minimize particular measures of irregularity. Degree variance and degree deviation, stand as prevalent irregularity measures in this context. Consequently, exploring optimal graphs based on these measures often entails solving optimization problems where the objective function is an irregularity measure.

Cactus graphs, initially referred to as Husimi trees, were first introduced in scientific literature approximately sixty years ago in works of Husimi and Riddell, which examined cluster integrals related to condensation in statistical mechanics [14, 16, 18]. These graphs find applications in fields such as chemistry [15, 22] as well as in electrical and communication networks [21]. For further insights into cactus graphs, we direct the reader to the works of [4, 10, 11]. A connected graph is categorized as a cactus graph if every two distinct cycles share at most one edge. A cactus graph G is termed m-uniform if all of its blocks, which may be edges or a cycles, are cycles of the same size m.

In the Section 2, we study the degree deviation $S(G)$ and the degree variance $Var(G)$ of acyclic and unicyclic graphs. In Section 3, we consider several cactus chains to derive the degree deviation $S(G)$ and the degree variance $Var(G)$ of these types of graphs. The final conclusions are presented in Section 4.

2 Results for Acyclic and Unicyclic Graphs

In this section, we analyze the degree deviation and degree variance of acyclic and unicyclic graphs. We begin by presenting the following lemma and theorems, which will be referenced in subsequent discussions. Additionally, we identify optimal graphs based on these two measures of irregularity.

Lemma 1. [9] Let G be a connected bidegreed graph with n vertices. Then,

i. $IRD(G) = S(G) \leq IRR(G) = \frac{n}{2}(\Delta - \delta)$, with equality holding if and only if G is a balanced bidegreed graph.

ii. $Var(G) = \frac{1}{n^2} IRR(G) IRD(G) \leq \frac{1}{n^2}(IRR(G))^2 = \frac{(\Delta-\delta)^2}{4}$, with equality holding if and only if G is a balanced bidegreed graph.

iii. $Var(G) = \frac{\Delta-\delta}{2n} S(G)$.

Theorem 1. [9] Let T denote a tree with n vertices. Then,

i. $S(T) = \frac{4(n-2)}{n} + \frac{2(n-2)}{n} \sum_{i=3}^{\Delta} N_i(i-2)$.

ii. $Var(T) = \frac{2(n-2)}{n^2} + \frac{1}{n} \sum_{i=3}^{\Delta} N_i(i-1)(i-2)$.

According to Theorem 1, since star graphs S_n exhibit the maximum degree deviation and degree variance while and path graphs P_n show the minimum, we derive the following corollary:

Corollary 1. Let T be any tree of order n. Then:

i. $\frac{4(n-2)}{n} \leq S(T) \leq \frac{2(n-1)(n-2)}{n}$.

ii. $\frac{2(n-2)}{n^2} \leq Var(T) \leq \frac{(n-1)(n-2)^2}{n^2}$.

Theorem 2. [9] Let G be a connected unicyclic graph with n vertices. Then,

i. $S(G) = 2\sum_{i=3}^{\Delta} N_i(i-2)$.

ii. $Var(G) = \frac{1}{n}\sum_{i=3}^{\Delta} N_i(i-1)(i-2)$.

iii. $nVar(G) - S(G) = \sum_{i=4}^{\Delta} N_i(i-2)(i-3)$.

Theorem 3. If T is a tree with n vertices and t pendant vertices, then

$$S(T) = (2 - \frac{4}{n})t.$$

Proof. Let $D = \{v \in V(T) \mid deg(v) = 1\}$. In any tree of order n, we have $m = n - 1$, thus $\frac{2m}{n} = 2 - \frac{2}{n}$. Therefore, we can express the degree deviation as follows: $S(T) = \sum_{i=1}^{n} \left|d_i - \frac{2m}{n}\right| = \sum_{i=1}^{n} |d_i - 2 + \frac{2}{n}|$. This leads to:

$$S(T) = t(1 - \frac{2}{n}) + \sum_{v \notin D}(d_v - 2 + \frac{2}{n}) = t(1 - \frac{2}{n}) + \sum_{v \notin D} d_v - (n-t)(2 - \frac{2}{n}).$$

Thus,

$$= t(1 - \frac{2}{n}) + (2n - 2 - t) - (n-t)(2 - \frac{2}{n}) = (2 - \frac{4}{n})t.$$

□

We denote the path graph with n vertices as P_n and the star graph as S_n. Additionally, the number of pendant vertices in a tree T is represented by t_T. It is evident that the average degree of the vertices in a tree T with n vertices is given by

$$\overline{d_T} = \frac{2n-2}{n}.$$

Furthermore, it holds that $2 = t_{P_n} \leq t_T \leq t_{S_n} = n - 1$. Consequently, according to Theorem 1, we can derive the following results:

Corollary 2. For any two trees T_1 and T_2 with n vertices, $S(T_1) = S(T_2)$ if and only if $t_{T_1} = t_{T_2}$.

The next corollary offers optimal trees based on degree deviation:

Corollary 3. For any tree T that contains n vertices, the following inequality holds:

$$S(P_n) \leq S(T) \leq S(S_n).$$

Equality in the left inequality occurs if and only if T is a path, while equality in the right inequality occurs if and only if T is a star.

Edge contraction is an operation that removes an edge from a graph G and merges the two vertices connected by that edge into a single vertex. The resulting graph is denoted as G/e. The following theorem compares the degree deviation of T with that of T/e:

Theorem 4. Let T be a tree with n vertices and $e \in E(T)$. Then, $S(T) \geq S(T/e)$ with equality holding only when $T = K_2$.

Proof. Let $D_T = \{v \in V(T) \mid \deg(v) = 1\}$ denote the set of pendant vertices, where $|D_T| = t_T$ and let $T_1 = T/e$. It can be shown that

$$|D_{T_1}| - 1 \leq |D_T| \leq |D_{T_1}|.$$

By Theorem 3, we have:

$$S(T) = (2 - \frac{4}{n})t_T \geq (2 - \frac{4}{n})t_{T_1} \geq (2 - \frac{4}{n-1})t_{T_1} = S(T_1) = S(T/e),$$

with equality holding only if $T = K_2$. □

Proposition 1. If T is a tree with n vertices, then the value of $S(T) - S(T/e)$ is either 0 or 2 for sufficiently large n.

Proof. By Theorem 3, we establish that

$$S(P_n) - S(P_n/e) = \left(4(\frac{2(n-1)}{n} - 1) - 4(\frac{2(n-2)}{n-1} - 1)\right) = 4\left(\frac{2}{n^2 - n}\right),$$

which approaches 0 for sufficiently large n. Similarly, for the star graph S_n:

$$S(S_n) - S(S_n/e)$$
$$= \left(2(n-1)(\frac{2(n-1)}{n} - 1) - 2(n-2)(\frac{2(n-2)}{n-1} - 1)\right),$$

which approaches 2 for sufficiently large n.

Let T be a tree with n vertices and t_n pendant vertices. By Theorem 4, we have $0 \leq S(T) - S(T/e)$.

We examine the following cases:

- Case 1. The trees T and T/e have the same number of pendant vertices. By Theorem 3, we get
$$S(T) - S(T/e) = 2t_n(\frac{2}{n(n-1)}) \leq 2(n-1)(\frac{2}{n(n-1)} = \frac{4}{n}.$$
Therefore, $S(T) - S(T/e)$ approaches 0 for sufficiently large n.

- Case 2. The number of pendant vertices in tree T is one more than the number of pendant vertices in tree T/e. In this case, we have
$$\big(S(T) - S(T/e)\big) = \big(\frac{2t_n(n-2)}{n} - \frac{2(t_n-1)(n-3)}{n-1}\big).$$
This simplifies to
$$= \big(2t_n(\frac{2}{n(n-1)}) + \frac{2(n-3)}{n-1}\big) \leq \big(2(n-1)(\frac{2}{n(n-1)}) + \frac{2(n-3)}{n-1}\big),$$
yielding that $S(T) - S(T/e)$ approaches 2 for sufficiently large n.

□

Theorem 5. Let T be a tree of order n, and let $e = uv \in E(T)$.

i. If $\deg(u), \deg(v) \geq 2$ or $\deg(u), \deg(v) \leq 2$, then $Var(T) < Var(T/e)$.

ii. If $\deg(u) \geq 3$ and $\deg(v) = 1$ (or $\deg(u) = 1$ and $\deg(v) \geq 3$), then $nVar(T) \geq (n-1)Var(T/e)$.

Proof. Assume $\deg(u) = s$ and $\deg(v) = t$.

i. By Theorem 1, if $s, t \leq 2$, the terms $\sum_{i=3}^{\Delta} N_i(i-1)(i-2)$ in $Var(T)$ and $Var(T/e)$ are equal, therefore, given that $\frac{(n-2)}{n^2} < \frac{(n-3)}{(n-1)^2}$ and $\frac{1}{n} < \frac{1}{n-1}$, we conclude $Var(T) < Var(T/e)$.

ii. If $s \geq 3$ and $t = 1$ and $N_s = x$ and $N_{s-1} = y$, we have:
$$\sum_{i=3}^{\Delta'} N'_i(i-1)(i-2) =$$
$$\sum_{i=3,\, i\neq s,s-1}^{\Delta'} N'_i(i-2) + N'_{s-1}(n-2)(s-3) + N'_s(n-1)(s-2) =$$
$$\sum_{i=3,\, i\neq s,s-1}^{\Delta'} N'_i(i-2) + (y+1)(s-2)(s-3) + (x-1)(s-1)(s-2),$$
since for each $i = 1, 2, ..., \Delta'$ $i \neq s, s-1$, $N_i = N'_i$ and
$$y(s-2)(s-3) + x(s-1)(s-2) > (y+1)(s-2)(s-3)(x-1)(s-1)(s-2),$$
we have:

$$\sum_{i=3}^{\Delta} N_i(i-1)(i-2)$$
$$= \sum_{i=3,\, i\neq s,s-1}^{\Delta} N_i(i-1)(i-2) + y(s-2)(s-3) + x(s-1)(s-2)$$
$$> \sum_{i=3,\, i\neq s,s-1}^{\Delta'} N_i'(i-1)(i-2) + (y+1)(s-2)(s-3) + (x-1)(s-1)(s-2)$$
$$= \sum_{i=3}^{\Delta'} N_i'(i-1)(i-2).$$

which implies $nVar(T) > (n-1)Var(T/e)$ since $(\frac{(n-2)}{n} > \frac{(n-3)}{(n-1)})$.

□

Here, we study the degree deviation and the degree variance of several types of unicyclic connected graphs, which are utilized in Theorem 6.

Definition 1. Let C_n denote a cycle with n-vertices. The graph C_nT_k is formed by identifying a vertex from a tree T_k of order k with a vertex from the cycle C_n.

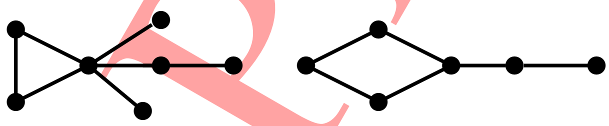

Figure 1: Graphs C_3T_5 (left) and C_4T_3 (right).

The following simple lemma can proven by induction:

Lemma 2. Let G be a connected unicyclic graph with a cycle C_n, formed by identifying trees $T_{k_1}, T_{k_2}, \ldots, T_{k_r}$ to the vertices $u_1, u_2, \ldots, u_r$ (where $r \leq n$ of the cycle C_n. Then,

$$S(G) = S(C_nT_{k_1}) + S(C_nT_{k_2}) + \cdots + S(C_nT_{k_r}).$$

Let $T_{m,n}$ denote the graph produced by joining a cycle C_m to a path P_n; this is referred to as the (m,n)-tadpole graph. The tadpole graphs $T_{3,1}$ and $T_{4,1}$ are illustrated in Figure 2.

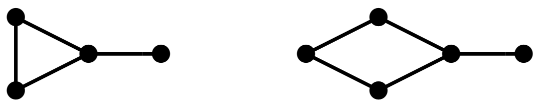

Figure 2: Tadpole graphs $T_{3,1}$ (left) and $T_{4,1}$ (right).

Lemma 3. For the unicyclic graph C_nT_k, it holds that $S(C_nT_k) = 2t$, where t denotes the number of pendant vertices in C_nT_k.

Proof. We will prove this lemma by induction on t. For the base case where $t = 1$, the graph C_nT_k represents a tadpole graph, and it is clear that $S(C_nT_k) = 2$. Now, assume that for any graph C_nT_k with t pendant vertices, the relationship $S(C_nT_k) = 2t$ holds. We will demonstrate that for every graph C_nT_k with $t + 1$ pendant vertices, it follows that $S(C_nT_k) = 2(t + 1)$. Consider a pendant vertex u_1 from the graph C_nT_k. If u_1 is adjacent to a vertex of degree 2, then upon removing u_1, the number of pendant vertices in both $S(C_nT_k)$ and $S(C_nT_{k-1})$ will equal $t + 1$. Consequently, we have $S(C_nT_k) = S(C_nT_{k-1})$.

Continuing this process, we eventually arrive at a point where u_i is not adjacent to a vertex of degree 2. In this case, upon removing the vertex u_i, the value of $S(C_nT_k)$ will decrease by 2, resulting in the new graph $S(C_nT_{k-j})$ containing t pendant vertices. Thus, we can conclude that $S(C_nT_k) = S(C_nT_{k-j} - \{u_i\}) + 2 = 2t + 2$. □

Note that a tree with at most one vertex of degree ≥ 3 is referred to as a starlike tree. The path P_n and the star graph S_n are two special examples of starlike tree.

Corollary 4. It follows that $S(C_nT_k) \geq 2(\Delta_{T_k} - 1)$ and $S(C_nT_k) = 2(\Delta_{T_k} - 1)$, with equality occurring if and only if T_k is a starlike tree connected to C_n by a pendant vertex.

Using Lemmas 2 and 3 we can establish the following theorem:

Theorem 6. If G is a unicyclic graph with t pendant vertices, then $S(G) = 2t$.

The following corollary provides a lower bound for the degree deviation of an irregular unicycle graph of order n, as well as identifying the optimal graphs of this category:

Corollary 5. If G_n is an irregular unicycle graph of order n, then $S(G_n) \leq 2(n - 3)$ and the equality holds if and only if G_n has a universal vertex. It is noteworthy that, in the equality, the cycle of G must be a triangle.

Applying the Cauchy-Schwarz inequality, we have $S(G) \leq n\sqrt{Var(G)}$. Additionally, for any unicyclic graph G of order n with $\Delta_G \leq 3$, it follows that $Var(G) \leq 1$, with equality holding if and only if G is a sun-graph (denoted as $(C_n \circ K_1)$, where $\circ$ denotes the corona operation). The following theorem offers an alternative version of the previous inequality applicable to unicyclic graphs:

Theorem 7. If G is a unicycle graph with n vertices, then $S(G) \leq nVar(G)$ and equality holds if and only if $\Delta_G \leq 3$.

Proof. For each vertex v_i with degree $\deg(v_i) = d_i$, we observe that $|d_i - 2| \leq (d_i - 2)^2$. This relationship allows us to derive the desired result. □

We aim to establish a lower bound for Bell's degree variance $Var(G)$ for unicycle graph of order n with t pendant vertices. We will also identify the optimal graphs that achieve these optimal values. Utilizing Theorems 6 and 7, we conclude the following corollary:

Corollary 6. If G is a unicycle graph of order n and t pendant vertices, then $Var(G) \geq \frac{2t}{n}$ with equality holding if and only if $\Delta_G \leq 3$.

Theorem 8. If graph G is formed by identifying the vertices $v_1, v_2, \cdots v_r$ of trees $T_{k_1}, T_{k_2}, \ldots, T_{k_r}$ with different vertices $u_1, u_2, \ldots, u_r$ of the cycle C_n, then the variance of (G) can be expressed as

$$Var(G) = \frac{\sum_{i=1}^{r} Var(C_n T_{k_i})(k_i + n - 1)}{n - r + \sum k_i}.$$

Proof. The total number of vertices in the graph G is given by $n - r + \sum_{i=1}^{r} k_i$. Thus, we can express the variance as follows:

$$Var(G) = \frac{(\sum_{i=1}^{k_1}(d_{1i}-2)^2) + \ldots + (\sum_{i=1}^{k_r}(d_{ri}-2)^2)}{n - r + \sum_{i=1}^{r} k_i} = \frac{\sum_{i=1}^{r} Var(C_n T_{k_i})(k_i + n - 1)}{n - r + \sum k_i}.$$

□

Theorem 9. If G_n is an irregular unicyclic graph of order n, then $Var(G_n) \leq \frac{(n-2)(n-3)}{n}$, with equality holding if and only if G_n contains a universal vertex.

Proof. We will prove the assertion by induction on n. For the base case $n = 4$, we find that $Var(G_4) = \frac{1}{2}$, hence $nVar(G_4) = 2 \leq (n-2)(n-3)$.

Assume that for the graph G_n, the inequality $nVar(G_n) \leq (n-2)(n-3)$ holds. We now consider an irregular unicycle graph G_{n+1} with $n+1$ vertices. Let u be a pendant vertex of graph G_{n+1}. By removing the vertex u, we analyze two cases.

Case 1. If removing u, does not change the number of pendant vertices, then we have

$$\sum_{i=1}^{n+1}(d_i - 2)^2 = \sum_{i=1}^{n}(d_i - 2)^2.$$

Consequently,

$$(n+1)Var(G_{n+1}) = nVar(G_n) \leq (n-2)(n-3) < (n-1)(n-2).$$

Case 2. If the removal of u reduces the number of pendant vertices, the maximum reduction in the value of $\sum_{i=1}^{n+1}(d_i - 2)^2$ occurs when u is a leaf adjacent to a universal vertex. The reduction value in this scenario is given by $(n-2)^2 - (n-3)^2 + 1$, yielding:

$$\begin{aligned}(n+1)Var(G_{n+1}) &\leq nVar(G_n) + (n-2)^2 - (n-3)^2 + 1 \\ &\leq (n-2)(n-3) + (n-2)^2 - (n-3)^2 + 1 = (n-1)(n-2).\end{aligned}$$

For the case of equality, let $nVar(G_n) = (n-2)(n-3)$ and assume the graph G does not contain a universal vertex. Suppose that vertex v is a vertex in the cycle of the graph with degree greater than 3. Since v is not universal vertex, there are vertices that are not adjacent to v. Two subcases arise:

Case A. The graph G has a support vertex (a vertex adjucent to a pendant vertex) that is not on the cycle. Let u_0 be a support vertex in tree T, where k pendant vertices are adjacent to u_0, and tree T is connected to the cycle at the vertex v. If we isolate all k pendant vertices from u_0 and connect them to v, we obtain a new graph G':

$$\begin{aligned}(n-2)(n-3) &= nVar(G) = (d_v-2)^2 + (d-u_0-2)^2 + k(1-2)^2 + B \\ &= (d_v-2)^2 + (k-1)^2 + k + B,\end{aligned}$$

where B represents the sum of terms of the form $(d-2)^2$ for the remaining vertices. Conversely, for the modified graph (G'):

$$nVar(G') = (d_v + k - 2)^2 + (1-2)^2 + k(1-2)^2 + B.$$

A straightforward calculation reveals that

$$(n-2)(n-3) = nVar(G) \lneqq nVar(G') \leq (n-2)(n-3),$$

resulting in a contradiction.

Case B. The graph G contains a support vertex that is part of the cycle. We consider two subcases:

Subcase 1. The cycle has length $n_1 \geq 4$. In this instance, the graph G is formed by identifying the central vertices of $1 \leq r \leq n_1$ stars K_{1,k_i} with vertices of C_{n_1}. Then, we have $n = n_1 + k_1 + \ldots + k_r$ and consequently,

$$\begin{aligned}nVar(G) &= (k_1^2 + k_1) + \ldots (k_r^2 + k_r) \lneqq (k_1 + \ldots + k_r)^2 \\ &\lneqq (k_1 + \ldots + k_r + 2)(k_1 + \ldots + k_r + 1) \\ &\leq (n_1 + k_1 + \ldots + k_r - 2)(n_1 + k_1 + \ldots + k_r - 3) = (n-2)(n-3).\end{aligned}$$

Subcase 2. When the cycle has a length of $n_1 = 3$, and the graph G does not have a universal vertex, it can be formed by connecting $2 \leq r \leq 3$ stars to C_3, where each star S_i comprises k_i pendant vertices connected to the cycle via their central vertex. Thus, $n = 3 + k_1 + k_2 + k_3$ and the proof follows similarly to that of the previous subcase. □

Theorem 10. Let G be a unicyclic graph and let $e = uv \in E(G)$ such that G/e is also a unicyclic graph. Then $S(G/e) \leq S(G)$, with equality holding if and only if $\deg(u) \geq 2$ and $\deg(v) \geq 2$ or $\deg(u) \leq 2$ and $\deg(v) \leq 2$.

Proof. Let $\deg(u) = s$ and $\deg(v) = t$. We consider the following cases:

1. If $s, t \geq 2$ or $s, t \leq 2$, then

$$|s-2|+|t-2|=|s+t-4|.$$

Consequently, we find $S(G)=\sum_{i=1}^{n}|d_i-2|=\sum_{i=1\&d_i\neq s,t}^{n}|d_i-2|+|s+t-4|=S(G/e)$.

2. If $s<2$ and $t>2$ (or $s>2$ and $t<2$), then we observe

$$|s-2|+|t-2|>|s+t-4|,$$

implying that $S(G)>S(G/e)$.

Therefore, the theorem has been proven. □

Theorem 11. Let G be a unicyclic graph with n vertices and let $e=uv \in E(G)$ such that G/e is also a unicyclic graph. The following statements hold:

(i) If $\deg(u) \geq 2$ and $\deg(v) \geq 2$ or $\deg(u) \leq 2$ and $\deg(v) \leq 2$, then $Var(G/e) \geq Var(G)$.

(ii) If $\deg(u) > 2$ and $\deg(v) = 1$ or $\deg(v) > 2$ and $\deg(u) = 1$, then

$$(n-1)Var(G/e) < nVar(G).$$

Proof. Assume $\deg(u) = s$ and $\deg(v) = t$.

1. If $s, t \geq 2$ or $s, t \leq 2$, then we have:

$$(s-2)^2+(t-2)^2 \leq (s+t-4)^2.$$

It follows that

$$Var(G)=\frac{1}{n}[\sum_{i=1}^{n}(d_i-2)^2] \leq \frac{1}{n-1}[\sum_{i=1\&d_i\neq s,t}^{n}(d_i-2)^2+(s+t-4)^2]=Var(G/e).$$

2. If $s>2$ and $t=1$, then we find that

$$(s-2)^2+1>(s-3)^2.$$

Therefore,

$$nVar(G)=\sum_{i=1}^{n}(d_i-2)^2 > \sum_{i=1\&d_i\neq s,t}^{n}(d_i-2)^2+(s-3)^2=(n-1)Var(G/e).$$

□

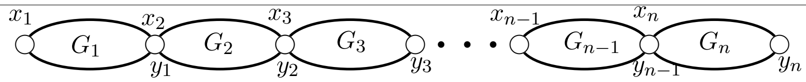

Figure 3: Chain of graphs

3 Results for Certain Cactus Graphs

In this section, we calculate the degree deviation and the degree variance of several cactus graphs. Consider a series of disjoint connected graphs, denoted as $G_1, G_2, \ldots, G_n$. We construct a chain of these graphs, referred to as $C(G_1, \ldots, G_n)$, by linking them sequentially. Specifically, for each adjacent pair of graphs, G_i and G_{i+1}, we merge a selected vertex y_i from G_i with a selected vertex x_{i+1} from G_{i+1} (see Figure 3).

Proposition 2. Let $X_{n,m}$ be a chain cactus graph in which each block consists of a cycle of length m, where $n \geq 2$ represents the number of cycles. Then the following holds:

(i) $S(X_{n,m}) = \frac{4(m-2)n^2-4(m-4)n-8}{(m-1)n+1}$.

(ii) $Var(X_{n,m}) = \frac{4(m-2)n^2-4(m-4)n-8}{[(m-1)n+1]^2}$.

Proof. (i) Since all of these chain cactus graphs exhibit bidegree properties, we apply Lemma 1 to find that

$$S(X_{n,m}) = IRD(X_{n,m}) = \frac{2[n-1][n(m-2)+2]}{[n-1]+[n(m-2)+2]}(4-2)$$
$$= \frac{4(m-2)n^2-4(m-4)n-8}{(m-1)n+1}.$$

(ii) The variance is given by $Var(X_{n,m}) = \frac{(4-2)}{2[n(m-1)+1]} S(X_{n,m}) = \frac{4(m-2)n^2-4(m-4)n-8}{[(m-1)n+1]^2}$. □

Example 1. (i) Consider the chain triangular cactus $T_n = X_{n,3}$ for $n \geq 2$, as illustrated in the Figure 4. By applying Proposition 2, we obtain the following results:

$$S(T_n) = \frac{4(n-1)(n+2)}{2n+1},$$

and

$$Var(T_n) = \frac{4(n-1)(n+2)}{(2n+1)^2}.$$

(ii) Let $Q_n = X_{n,4}$ for $n \geq 2$ represent a para-chain square cactus, as depicted in Figure 5. According to Proposition 2, we have:

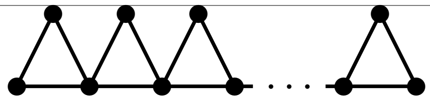

Figure 4: The graph T_n

$$S(Q_n) = \frac{8(n-1)(n+1)}{3n+1},$$

and

$$Var(Q_n) = \frac{8(n-1)(n+1)}{(3n+1)^2}.$$

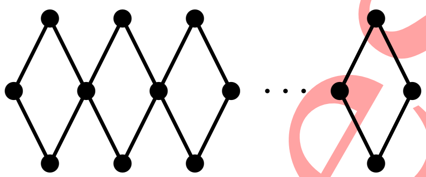

Figure 5: The graph Q_n

Theorem 12. For any integers $s, t \in \mathbb{N}$, such that $3 \leq s < t$, it holds that

$$S(X_{n,s}) < S(X_{n,t})$$

Proof. It suffices to demonstrate the assertion for the specific case where $s = m$ and $t = m+1$. By applying Proposition 2, we aim to show that $S(X_{n,m+1}) - S(X_{n,m}) > 0$:

$$S(X_{n,m+1}) - S(X_{n,m}) = \frac{4(m-1)n^2 - 4(m-3)n - 8}{mn+1}$$
$$- \frac{4(m-2)n^2 - 4(m-4)n - 8}{(m-1)n+1} = \frac{2(n-2) + mn(m-2) + 3}{[mn+1][(m-1)n+1]} > 0.$$

□

Corollary 7. For every $m \geq 3$, it follows that $S(T_n) \leq S(X_{n,m})$.

Corollary 8. For any integers $s, t \in \mathbb{N}$ such that $3 \leq s < t$, it holds that

$$[(s-1)n+1]Var(X_{n,s}) < [(t-1)n+1]Var(X_{n,t}).$$

Proof. The result follows from Lemma 1 and Theorem 12. □

Theorem 13. For any integer $m \geq 3$ and $n \geq 2$, we have

$$S(X_{n,m}) < S(X_{n+1,m}).$$

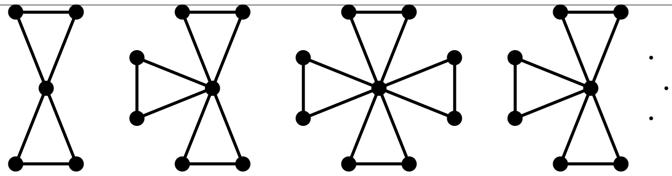

Figure 6: Friendship graph F_2, F_3, F_4 and F_n, respectively

Proof. Utilizing Proposition 2, we compute:

$$S(X_{n+1,m}) - S(X_{n,m}) = \frac{4(m-2)(n+1)^2 - 4(m-4)(n+1) - 8}{(m-1)(n+1)+1}$$
$$- \frac{4(m-2)n^2 - 4(m-4)n - 8}{(m-1)n+1}$$
$$= \frac{4(m-2)^2n^2 + 4m(mn-2) + 8}{[(m-1)(n+1)+1][(m-1)n+1]} > 0.$$

□

Definition 2. The friendship graph F_n is defined as the graph formed by merging n copies of the cycle graph C_3 at a common vertex. It is evident that the friendship graph is a bidegree cactus graph, as illustrated in Figure 6.

Corollary 9. Let F_n denote a friendship graph. Then the degree deviation $S(F_n)$ and the degree variance $Var(F_n)$ can be expressed as follows:

$$S(F_n) = \frac{8n(n-1)}{2n+1} \quad \text{and} \quad Var(F_n) = \frac{8n(n-1)^2}{(2n+1)^2}.$$

4 Conclusion

In this study, we have explored two fundamental irregularity measures in graph theory—Bell's degree variance $Var(G)$ and degree deviation $S(G)$— across various significant classes of graphs. Our investigation has resulted in the derivation of precise formulas and bounds for these measures in acyclic graphs (trees), unicyclic graphs, and different forms of cactus graphs, thus contributing to a deeper understanding of graph irregularity. Key findings from our research include:

1. The establishment of exact formulations for $S(T)$ and $\text{Var}(T)$ in trees with n vertices, highlighting their dependence on the number of pendant vertices and the degree distribution (Theorem 3).

2. The derivation of the relationship $S(G) = 2t$ for unicyclic graphs with t pendant vertices (Theorem 6), demonstrating a straightforward linear correlation. We found optimal trees and unicyclic graphs based on these two irregularity measures.

3. The identification of optimal trees and unicyclic graphs based on the two measures of irregularity.

4. The observation that both $S(X_{n,m})$ and $\mathrm{Var}(X_{n,m})$ exhibit monotonically increasing behavior regarding the parameters n and m (Theorems 12 and 13).

Our findings expand the theoretical foundation of graph irregularity measures and provide valuable tools for quantifying structural differences among various graph classes. The inequalities and explicit formulas presented may have practical applications in network analysis, where assessing the deviation from regularity is crucial.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

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Competing Interests

The authors declare that they have no competing interests relevant to the content of this paper.

Authors' Contributions

All authors contributed substantially and equally to the manuscript.

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