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A Hybrid Numerical Approach for Solving Nonlinear Optimal Control Problems

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Abstract. This paper presents an iterative computational method for addressing constrained nonlinear optimal control problems, specifically those involving terminal state, state saturation, and control saturation constraints. The proposed approach reformulates the original problem into a sequence of constrained linear time-varying quadratic optimal control problems. This is achieved by iteratively approximating the nonlinear dynamic system using constrained linear time-varying models. Each reformulated problem is then converted into a standard quadratic programming problem by applying Chelyshkov polynomials in conjunction with a collocation method. Finally, the resulting problems are solved to obtain optimal control solutions.

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1 Introduction

Optimal control of dynamic systems subject to realistic constraints on input signals and state variables represents a crucial area in control theory. Numerous practical control challenges can be framed as optimization problems, thereby creating a significant demand for efficient numerical algorithms capable of delivering solutions. Optimal control theory focuses on finding control functions that optimize a specific performance criterion, typically expressed as a *cost function*, in various domains, including engineering, economics, and biology. Mathematically, these problems are formulated as the minimization of the cost function J, defined as:

$$J = \int_{t_0}^{T_F} L(x(t), u(t), t)\, dt + \phi(x(T_F)),$$

where L denotes the running cost, ϕ indicates the terminal cost, $x(t)$ represents the state variable at time t, and $u(t)$ signifies the control variable at time t. The dynamic evolution of the system is governed by:

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0.$$

Due to the inherent complexity of nonlinear systems, obtaining exact solutions is often infeasible, necessitating the deployment of numerical methods [4].

This paper introduces a novel polynomial basis aimed at addressing longstanding challenges in optimal control challenges, specifically issues related to numerical instability, computational cost, and the enforcement of terminal condition. This development suggests a wider applicability for alternative polynomial representations in the context of dynamical systems.

Numerical methods typically discretize optimal control problems (OCPs) to facilitate computational solutions. The selection of a particular method is influenced by the accuracy requirements and computational feasibility [11].

In recent decades, iterative techniques have been extensively employed to solve OCPs. These methods approximate optimal solutions through successive refinements, with prominent approaches including:

- Gradient-Based Methods: These techniques optimize the control function by iteratively minimizing J in the direction of the cost function's gradient [4].
- Sequential Quadratic Programming (SQP): This method addresses nonlinear control problems by solving a sequence of quadratic approximations, demonstrating robust performance in constrained problems [8].
- Dynamic Programming (DP): DP decomposes complex problems into simpler stages, and is widely applied in nonlinear and high-dimensional contexts [10].

Alipour et al. developed an iterative method that integrates homotopy analysis and parametrization approaches to address these problems [2]. In [1], an iterative method was employed to solve a quadratic optimal control problem (QOCP) using the state parameterization technique alongside scaling Boubaker polynomials. Jaddu proposed a method utilizing the second quasi-linearization technique and state parameterization with Chebyshev polynomials to approach nonlinear OCPs, including state and control saturation constraints [14, 16].

Polynomial basis functions serve to approximate control and state variables, offering a structured representation within bounded intervals. Specifically, polynomial basis approximations express control $u(t)$ and state $x(t)$ as:

$$u(t) \approx \sum_{j=0}^{N} c_j \psi_j(t),$$
$$x(t) \approx \sum_{j=0}^{N} d_j \psi_j(t),$$

where $\psi_j(t)$ represents polynomial basis functions, and c_j and d_j are the corresponding coefficients [10].

The integration of iterative techniques with polynomial basis functions offers an effective numerical methods for addressing complex optimal control challenges [7]. This hybrid approach enhances solution accuracy and flexibility, handling the effective management of nonlinearity and complex constraints. Applications of the iterative-polynomial approach span various fields, including robotics, aerospace, and finance, and continues to evolve alongside advancements in machine learning and adaptive algorithms. Future research aims to refine these hybrid techniques to confront emerging challenges in large-scale and real-time control systems [4, 22].

In this study, we seek to enhance the method presented in [15] while building upon the research conducted by Banks et al. [3, 20]. Our objective is to advance the solutions for nonlinear quadratic OCPs subject to terminal state constraints, and saturation constraints on both state and control variables. By expanding these methodologies, we aim to establish a more robust framework for addressing these complex control problems through the use of Chelyshkov polynomials [19].

The remainder of this paper is structured as follows. Section 2 formally defines the optimal control problem under consideration and presents the necessary preliminaries for implementing the proposed method. Section 3 outlines the development and execution of the proposed method. Section 4 presents numerical examples that illustrate the accuracy and effectiveness of the method. Finally, Section 5 concludes the paper and discusses potential avenues for future research.

2 Preliminaries

2.1 The Problem

We consider the following nonlinear optimal control problem, which we aim to solve numerically:

Find the optimal control $u(t) = u^*(t)$ that minimizes the performance function defined as follows:

$$J = x(T_F)^t S\, x(T_F) + \int_0^{T_F} (x^t Q\, x + u^t Ru)\, dt, \tag{1}$$

The state of the dynamic system is governed by:

$$\dot{x} = f(x(t), u(t), t), \tag{2}$$

subject to the following conditions:

- Initial and terminal conditions:

$$x(0) = x_0, \tag{3}$$
$$\lambda(x(T_F), T_F) = 0, \tag{4}$$

- State and control constraints:

$$\begin{aligned} X_m &\le x(t) \le X_M, \\ U_m &\le u(t) \le U_M. \end{aligned} \tag{5}$$

In this problem, Q and S are in $\mathbb{R}^{n\times n}$, assumed to be positive semidefinite matrices, while $R \in \mathbb{R}^{m\times m}$ is a positive definite matrix. The vector $x \in \mathbb{R}^n$ represents the state vector, $u \in \mathbb{R}^m$ denotes the control vector, $x_0 \in \mathbb{R}^n$ is the initial condition vector. The function f is a continuously differentiable nonlinear function with respect to all its variables. Additionally, we assume that $m \le n$ and that the bounds $X_M, X_m \in \mathbb{R}^n$ and $U_M, U_m \in \mathbb{R}^m$ are specified fixed values, with T_F being a known constant.

In this paper, we address problem (1)–(5) by transforming it using an iteration method into a series of time-varying linear quadratic optimal problems with constraints. In the first iteration, the nonlinear state equation is approximated by linear state equations, which in the subsequent iteration are reformulated into a second-degree programming problem utilizing ChPs.

Given that Chelyshkov polynomials are defined over the interval $[0, 1]$ and knowing any closed interval $[T_S, T_F]$ can be mapped linearly to $[0, 1]$ using the transformation $\tau = \frac{t-T_S}{T_F-T_S}$, $t \in [T_S, T_F]$. We will assume, for the sake of simplicity, that the problem is defined on the interval $[0, 1]$.

2.2 Chelyshkov Polynomials

Chelyshkov polynomials (ChPs) are inherently orthogonal on the interval $[0,1]$ under a uniform weight function. Given that optimal control problems are typically formulated over finite time horizons (e.g., $t \in [0,T]$), normalizing to the interval $[0,1]$ eliminates the need for coordinate transformations required for polynomials such as Legendre (orthogonal on $[-1,1]$) or Chebyshev polynomials. This normalization not only simplifies implementation but also reduces computational overhead, and preserves numerical stability.

The uniform weight function associated with ChPs aligns with the standard L^2 inner product space commonly employed in optimal control formulations. This congruence facilitates the derivation of operational matrices (e.g., integration and differentiation matrices) for spectral methods such as Galerkin or collocation, potentially leading to increased efficiency. In our prior research involving partial differential equations (PDEs) [19], Chelyshkov wavelets exhibited advantages in addressing sharp gradients and discontinuities due to their localization properties. Hence, optimal control problems governed by PDEs or those displaying analogous solution structures may derive substantial benefits from these characteristics, suggesting a promising extension to control theory.

While classical polynomials bases are well-established, exploring newer bases such as ChPs serves to expand the methodological toolkit available to the research community. Their relative novelty, having been introduced in 2006 [5], indicates potential that remains largely unexplored in control applications. This paper aims to rigorously evaluate their efficacy in this domain. Importantly, our motivation extends beyond novelty; the structural properties of ChPs (e.g., boundary adaptability and sparsity in operational matrices) suggest benefits worthy of investigation. We recognize the necessity of benchmarking our findings against classical methods. While this study primarily focuses on establishing feasibility and methodology, we plan to conduct explicit comparative studies in future work to quantitatively assess convergence rates, stability, and computational cost relative to Legendre/Chebyshev bases.

Chelyshkov polynomials $P_m(t)$ are defined as follows [19]:

$$P_m(t) := P_{M,m}(t) = \sum_{j=0}^{M-m} a_{m,j}\, t^{m+j}\,, \quad m = 0,1,\dots,M, \tag{6}$$

where the coefficients are given by:

$$a_{m,j} = (-1)^j \binom{M-m}{j}\binom{M+m+j+1}{M-m},$$

and M is a fixed predetermined integer. Under the uniform weight function $w(t) = 1$, ChPs are orthogonal on the interval $[0,1]$:

$$\int_0^1 P_n(t)\, P_m(t)\, dt = \frac{\delta_{mn}}{m+n+1}, \tag{7}$$

where δ_{mn} is the Kronecker delta. Additionally, we have:

$$\int_0^1 P_n(t), dt = \frac{1}{M+1}, \quad n = 0, 1, \dots, M.$$

Fixing the integer M, it is evident from Eq. (6) that each polynomial $P_m(t)$ where $m = 0, 1, \dots, M$ is of degree M. The set of ChPs $\{P_m(t) \,|\, m = 0, 1, \dots, M\}$ forms an orthogonal basis for $\prod_M(t)$, (the space of polynomials of degree at most M). Consequently, any function $f(t) \in L^2[0,1]$ can be approximated in terms of ChPs follows:

$$f(t) \simeq \sum_{m=0}^{M} c_m \, P_m(t) = \mathbf{C}^{\mathsf{T}} \mathbf{P}_M(t), \tag{8}$$

where $\mathbf{C}^{\mathsf{T}} = [c_0, c_1, \dots, c_M]$ and the coefficients c_m can be approximated as

$$c_m \simeq \frac{\langle f, P_m \rangle}{\|P_m\|^2} = (2m+1) \int_0^1 f(t)\, P_m(t)\, dt,$$

and $\mathbf{P}_M(t) = [P_0(t), P_1(t), \dots, P_M(t)]^{\mathsf{T}}$.

Lemma 1. Let $N = [\mu_{ij}]_{(M+1)\times(M+1)}$ be a matrix. Then:

$$\begin{aligned}
&\mathbf{P}_M^{\mathsf{T}}(t)\, N\, \mathbf{P}_M(t) = \\
&\quad \mu_{11} P_0(t) P_0(t) + 2\mu_{12} P_0(t) P_1(t) + 2\mu_{13} P_0(t) P_2(t) + \cdots + 2\mu_{1,M+1} P_0(t) P_M(t) \\
&\qquad + \mu_{22} P_1(t) P_1(t) + 2\mu_{23} P_1(t) P_2(t) + \cdots + 2\mu_{1,M+1} P_0(t) P_M(t) \\
&\qquad\quad + \mu_{33} P_2(t) P_2(t) + \cdots + 2\mu_{1,M+1} P_0(t) P_M(t) \\
&\qquad\qquad \vdots \\
&\qquad\qquad + \mu_{(M+1),(M+1)} P_M(t) P_M(t).
\end{aligned}$$

Thus, using (7), we obtain:

$$\int_0^1 (\mathbf{P}_M^{\mathsf{T}}(t)\, N\, \mathbf{P}_M(t)) dt = \sum_{i=1}^{M+1} \frac{\mu_{ii}}{2i-1}.$$

Proof. Based on the properties of matrix (vector) multiplication properties and certain calculations, the proof is straightforward. □

Theorem 1. Suppose $u(x) \in C^n[0,1]$ and $u_n(x)$ is its expansion in terms of ChPs, as described in (8). Then we have:

$$\|u(x) - u_n(x)\|_2 \leq \frac{M_n}{(n+1)! 2^{2n+1}},$$

where M_n is a constant such that

$$|u^{(n+1)}(x)| \leq M_n, \quad x \in [0,1].$$

Proof. Let $p_n(t)$ be the interpolating polynomial for u at the nodes t_i, where $t_i,\ i = 0, 1, \ldots, n$ are the roots of the $(n + 1)$-degree shifted Chebyshev polynomial on $[0, 1]$. For any $t \in [0, 1]$, we can express:

$$u(t) - p_n(t) = \frac{u^{(n+1)}(\xi_t)}{(n+1)!} \prod_{i=0}^{n} (t - t_i).$$

□

Since the interpolating nodes are Chebyshev nodes, we find:

$$u(t) - p_n(t) = \frac{M_n}{(n+1)! 2^{2n+1}}, \quad x \in [0, 1].$$

Since $u_n(t)$ represents the least squares best approximation of $u(t)$, [17, Example 3.4.8] , it follows that:

$$\|u(t) - u_n(t)\|_2 \leq \|u(t) - p_n(t)\|_2 = \frac{M_n}{(n+1)! 2^{2n+1}}, \quad x \in [0, 1].$$

Thus, we obtain the desired result.

2.3 Iterative Technique

Consider a nonlinear system described by the following equation [21]:

$$\dot{x} = f(x) = A(x)\, x, \qquad x(0) = x_0 \in \mathbb{R}^n, \tag{9}$$

where $A(x)$ is a locally Lipschitz matrix function. This system can be approximated by a sequence of linear time-varying equations represented as follows:

$$\begin{cases} \dot{x}^{(1)} = A(x^{(0)})\, x^{(1)}, & x^{(1)}(0) = x(0), \\ \vdots & \\ \dot{x}^{(i)} = A(x^{(i-1)})\, x^{(i)}, & x^{(i)}(0) = x(0), \end{cases} \tag{10}$$

for $i = 1, 2, \ldots$.

This formulation allows us to avoid directly addressing the complexities inherent in the nonlinear system by expressing it as a series of linear equations that evolve over time. Consequently, this approach simplifies the analysis and control of the system, thereby facilitating the prediction and optimization of its behavior. Such a technique is particularly advantageous in control theory and engineering, where linear models are typically more tractable and can yield significant insights into the dynamics of more complex nonlinear systems.

Theorem 2. [20]. Suppose that $A(x) : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and that the nonlinear equation (9) possesses a unique solution $x(t)$ on the interval $[0, T]$. Then the sequence defined by (10) converges uniformly to $x(t)$.

3 Main Results

We employ the iterative method to address the OCP defined in equations (1)–(5), leading to the following reformulated problem:

$$J^{[k]} = x^{[k]}(1)^t\, S\, x^{[k]}(1) + \int_0^1 (x^{[k]}(t)^t Q\, x^{[k]}(t) + u^{[k]}(t)^t, R\, u^{[k]}(t))dt, \tag{11}$$

subject to

$$\dot{x}^{[k]}(t) = A(x^{[k-1]}(t))x^{[k]}(t) + B(x^{[k-1]}(t))u^{[k]}(t), \quad x^{[k]}(0) = 0, \tag{12}$$

with the terminal condition given by

$$\lambda(x^{[k]}(1), 1) = 0, \tag{13}$$

and bounds constraints for $k = 1, 2, \dots$ represented as:

$$X_m \le x^{[k]}(t) \le X_M, \qquad U_m \le u^{[k]}(t) \le U_M. \tag{14}$$

The constrained linear time-varying quadratic OCPs, as delineated in Equations (11)–(14), can be effectively resolved by transforming them into quadratic programming problems. This transformation is facilitated through the application of state parameterization which simplifies the original control problems and makes them more tractable to computational solutions. For a comprehensive discussion and detailed explanation of the state parameterization technique, refer to e.g., [15].

Next, we approximate both the state and control variables using ChPs as follows [19]:

$$x^{[k]}(t) = M_x^{[k]}\mathbf{P}_N(t), \tag{15}$$

$$u^{[k]}(t) = M_u^{[k]}\mathbf{P}_N(t), \tag{16}$$

where, $\mathbf{P}_N(t) = [P_0(t), P_1(t), \dots, P_N(t)]^\mathsf{T}$ is the vector of ChPs, $M_x \in \mathbb{R}^{n\times(N+1)}$ and $N_u \in \mathbb{R}^{m\times(N+1)}$ are matrices of unknown parameters that will be determined later, and N denotes the degree of the ChPs.

Substituting (15) and (16) in Equation (11) yields:

$$J^{[k]} = \mathbf{P}_N^\mathsf{T}(1)M_x^{[k]^\mathsf{T}}\, S\, M_x^{[k]}\mathbf{P}_N(1) + \int_0^1 (\mathbf{P}_N^\mathsf{T}(t)M_x^{[k]^\mathsf{T}} Q M_x^{[k]}\mathbf{P}_N(t) + \mathbf{P}_N^\mathsf{T}(t)M_u^{[k]^\mathsf{T}} R M_u^{[k]}\mathbf{P}_N(t))dt. \tag{17}$$

Noting that $\mathbf{P}_N(1) = [(-1)^N, (-1)^{N+1}, \dots, (-1)^{2N}]^\mathsf{T}$, we can express as

$$J^{[k]} = \sum_{i=0}^{N}\sum_{j=0}^{N}(-1)^{i+j}\gamma_{ij} + \sum_{i=1}^{N+1}\frac{\eta_{ii} + \zeta_{ii}}{2i-1}, \tag{18}$$

where $M_x^{[k]^\mathsf{T}} S M_x^{[k]} = [\gamma_{ij}]_{(N+1)\times(N+1)}$, η_{ii} and ζ_{ii} are diagonal entries of $M_x^{[k]^\mathsf{T}} Q M_x^{[k]} = [\eta_{ij}]_{(N+1)\times(N+1)}$ and $M_u^{[k]^\mathsf{T}} R M_u^{[k]} = [\zeta_{ij}]_{(N+1)\times(N+1)}$ respectively. The last term in Equation (18) can be rewritten as:

$$\sum_{i=1}^{N+1} \frac{\eta_{ii}+\zeta_{ii}}{2i-1} = \mathbf{P}_N^\mathsf{T}(1)\bar{D}\mathbf{P}_N(1),$$

where $\bar{D}$ is a diagonal matrix whose elements are defined as $\bar{d}_{ii} = \frac{\eta_{ii}+\zeta_{ii}}{2i-1}$.

Consequently, based on the preceding discussion and referencing results from [9], [12] and [13], Equation (17) or equivalently Equation (18) can be represented in the following quadratic form:

$$J^{[k]} = \frac{1}{2} y^\mathsf{T} H y, \tag{19}$$

where y is a vector containing the unknown elements of matrices M_x and M_u arranged in suitable order, and H is a positive definite Hessian matrix. Since the matrices $A(x^{[k-1]}(t))$ and $B(x^{[k-1]}(t))$ are time-dependent, each element can be expressed as $A_{ij}(t) = f_A(x^{[k-1]}(t), t)$ and $B_{ij}(t) = g_B(x^{[k-1]}(t), t)$. These can then be approximated using ChPs as follows:

$$A_{ij}(t) = W^\mathsf{T}\mathbf{P}_N(t), \tag{20}$$

$$B_{ij}(t) = V^\mathsf{T}\mathbf{P}_N(t). \tag{21}$$

The initial and terminal conditions $x^{[k]}(0) = 0$ and $\lambda(x^{[k]}(1), 1) = 0$ can also be expressed in terms of ChPs. Now, considering the imposed constraints, we have:

$$\begin{aligned} x^{[k]}(t) &\le X_M, \\ -x^{[k]}(t) &\le -X_m, \\ u^{[k]}(t) &\le U_M, \\ -u^{[k]}(t) &\le -U_m. \end{aligned}$$

Currently, we utilize collocation method by discretizing the interval $[0,1]$ into $r+1$ points as follows :

$$0 = t_0 < t_1 < \cdots < t_r = 1.$$

Subsequently, applying the ChP approximations leads to:

$$M_x^{[k]}\mathbf{P}_N(t) \le X_M, \tag{22}$$

$$-M_x^{[k]}\mathbf{P}_N(t) \le -X_m, \tag{23}$$

$$M_u^{[k]}\mathbf{P}_N(t) \le U_M, \tag{24}$$

$$-M_u^{[k]}\mathbf{P}_N(t) \le -U_m. \tag{25}$$

Equations (22)–(25) are subsequently replaced by the following inequality constraints:

$$M_x^{[k]}\mathbf{P}_N(t_s) \leq X_M, \tag{26}$$

$$-M_x^{[k]}\mathbf{P}_N(t_s) \leq -X_m, \tag{27}$$

$$M_u^{[k]}\mathbf{P}_N(t_s) \leq U_M, \tag{28}$$

$$-M_u^{[k]}\mathbf{P}_N(t_s) \leq -U_m, \tag{29}$$

for $s = 0, 1, \ldots, r$.

Consequently, we can formulate the following standard quadratic programming problem:

$$\min_y \frac{1}{2} y^\mathsf{T} H y \tag{30}$$

subject to the constraints:

$$A_1 y = b_1, \tag{31}$$

$$A_2 y \leq b_2, \tag{32}$$

where the equality constraints are arise from the initial and terminal conditions, alongside Equations (20) and (21), while the inequality constraints stem from the saturation constraints as expressed in Equations (26)–(29).

At this point, the problem defined by Equations (30)–(32) can be solved using relevant software or implemented in any suitable programming language. In this study, we utilize the `NMinimize` command in WOLFRAM MATHEMATICA 14.0 to address the optimization problem.

4 Illustrative Examples

Example 1. Consider the well-known Van der Pol oscillator problem, which can be articulated as follows:

$$
\begin{aligned}
Min \ \ J &= \frac{1}{2}\int_0^5 (x_1^2 + x_2^2 + u^2)dt \\
s.t &: \\
\dot{x}_1 &= x_2, \quad x_1(0) = 1, \\
\dot{x}_2 &= -x_1 + (1 + x_1^2)x_2 + u, \\
x_2(0) &= 0, \quad x_1(5) = -1, \quad x_2(5) = 0, \\
|u(t)| &\leq \frac{3}{4}.
\end{aligned}
$$

In this context, we define the matrix $A(x)$ as:

$$A(x) = A(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} 0 & 1 \\ -1 & 1 + x_1^2 \end{pmatrix}.$$

Assuming $x = \binom{x_1}{x_2}$, $y = \binom{y_1}{y_2}$ and $\|x - y\| < r$, we have:

$$\|A(x) - A(y)\| = |x_1^2 - y_1^2| = |x_1 + y_1|\,|x_1 - y_1| < 2r\,|x_1 - y_1| \leq 2r\,\|x - y\|.$$

Thus, $A(x)$ is locally Lipschitz and meets the conditions of Theorem 2, confirming that the method converges. The problem was solved using the proposed iterative method with $N = 7$ and $r = 10$, and the results are displayed in Table 1 and illustrated in Figure 1.

Table 1: Values of J with $N = 7$ and $r = 10$ for the Van der Pol oscillator problem (Example 1)

k	$J^{[k]}$
0	1.450427989
1	2.411326825
2	2.264578128
3	2.780074587

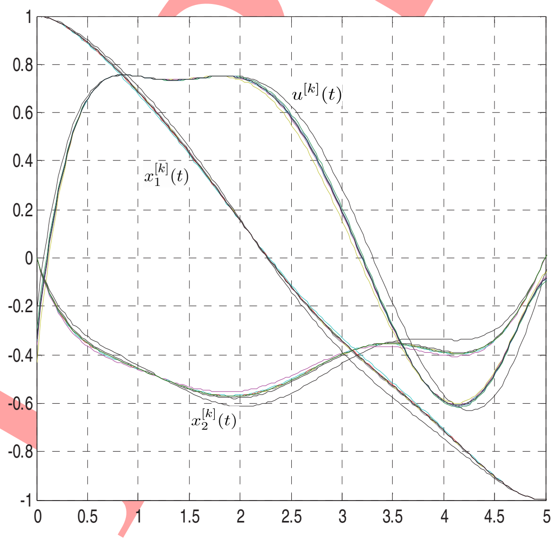

Figure 1: Results for $x_1^{[k]}(t)$, $x_2^{[k]}(t)$ and $u^{[k]}(t)$ from the Van der Pol oscillator problem (Example 1 with $N = 7$ and $r = 10$ and for $k = 1, 2, 3, 4, 5, 6$.

Example 2. We now consider the following nonlinear optimal control problem as discussed in [6, 18]:

$$Min\ \ J = \frac{1}{2}\int_0^1 (x^2 + u^2)\,dt$$

$$s.t:$$
$$u(t) = \dot{x},$$
$$x(0) = 0, \quad x(1) = \frac{1}{2},$$

The exact solution to this problem is given by:

$$x(t) = \frac{e(e^t - e^{-t})}{2(e^2 - 1)},$$
$$u(t) = \frac{e(e^t + e^{-t})}{2(e^2 - 1)},$$
$$J = 0.328258821374830.$$

In the problem, we have $A(x) = 0$, which is evident and satisfies the conditions of locally Lipschitz. The results obtained from solving this problem using the proposed method and comparing them with the results obtained in the [6, 18] are summarized in the Table 2. Additionally, Figure 2 illustrates the behavior of the L_∞ error as a function of N. As supported by theoretical expectations, the error decreases with an increase in N.

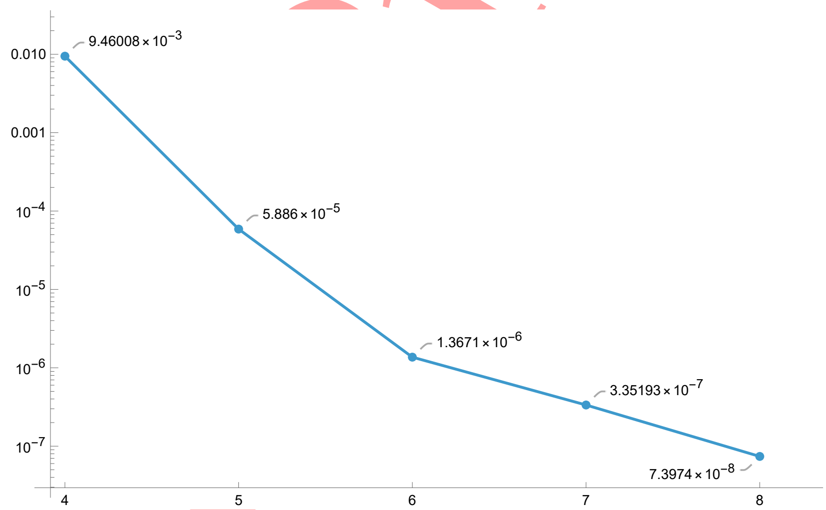

Figure 2: L_∞ error for the nonlinear optimal control problem (Example 2) with varying $N = 4, 5, 6, 7, 8$ and $r = 8$ for $k = 5$.

Table 2: L_∞ Error of J with $N = 5$ and $r = 8$ for the nonlinear optimal control problem (Example 2).

	Proposed method	Method in [6]	Method in [18]
L_∞ Error	5.886×10^{-5}	2.03089×10^{-4}	2.1×10^{-4}

5 Conclusions

In this study, we have introduced a hybrid approach for effectively solving optimal control problems (OCPs). Our approach integrates three approximation techniques: an iterative method, Chebyshev polynomial approximation, and collocation. The numerical examples serve to illustrate the computational efficiency and accuracy inherent in the proposed method, highlighting its effectiveness in addressing complex OCPs. Looking ahead, future research will focus on several key areas. These include the exploration of Chelyshkov wavelet approximation to enhance the flexibility and accuracy of our solutions; testing the method across a broader spectrum of OCPs to validate its robustness and adaptability; extending the framework to accommodate multi-objective optimization problems, which are vital in various application domains; and devising adaptive parameter selection strategies to further refine the method's performance. These advancements hold the promise of significantly enhancing the method's efficacy and expanding its applicability across diverse fields, ultimately contributing to the resolution of increasingly intricate optimization challenges.

Declarations

Availability of Supporting Data
All data generated or analyzed during this study are included in this published paper.

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Competing Interests
The authors declare that they have no competing interests relevant to the content of this paper.

Authors' Contributions
The main text of manuscript is collectively written by the authors.

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