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# Optimal Control of Linear Singularly Perturbed Systems via Eigenvalue Assignment

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**Abstract.** Optimal control of certain singularly perturbed systems, with slow and fast dynamics, presents notable challenges, including ill-conditioning, high dimensionality, and ill-posed algebraic Riccati equations. In this paper, we introduce a novel inverse optimal control method based on the eigenvalue assignment approach to address these issues. The proposed method optimizes the objective function while ensuring system stability through the strategic placement of eigenvalues in the singular perturbed closed-loop system. To facilitate analysis and support the implementation, a new theorem is proved, and a corresponding algorithm is developed. The proposed algorithm is free of ill-conditioned numerical problems, making it more robust in terms of numerical diffusion and perturbation measurement. Finally, two simulation examples are presented to illustrate the advantages of the proposed method, demonstrating improvement in controller robustness, substantial reductions in cost functions, and decreased control amplitudes.

**Keywords.** Singularly perturbed systems, Optimal control, Eigenvalue placement, Certainty matrix.

**MSC.** 93D05; 93D09.

## 1 Introduction

Linear Singularly Perturbed Systems (LSPSs) are extensively utilized in modeling complex physical systems that involve both fast and slow dynamics. These systems are characterized by the presence of a small perturbation parameter that introduces significant coupling between the fast and slow states. While this structure provides a powerful framework for describing real-world phenomena, it also presents substantial challenges in control design, particularly in achieving numerical stability and computational efficiency. Consequently, the optimal control of LSPSs has attracted significant attention in the field of control theory [20]. Traditional methods for controlling LSPSs often rely on solving the Algebraic Riccati Equation (ARE) or using iterative algorithms such as the Kleinman algorithm [12]. These approaches have been shown to perform well in certain cases but can encounter severe difficulties when the system exhibits ill-conditioning or indefinite quadratic terms in the ARE [15, 20]. Additionally, these methods are computationally expensive for high-dimensional systems, which further motivate the need for innovative control strategies that address these limitations.
One promising approach to tackling the complexity of LSPSs is the application of singular perturbation theory. This method decomposes the original system into reduced-order subsystems corresponding to the fast and slow dynamics. Such decomposition simplifies the control design process, as demonstrated in [13, 14, 23]. For instance, [23] proposed a decentralized composite control strategy for two-time-scale networks, effectively addressing the ill-conditioning and dimensionality challenges by combining model-based fast controllers with data-driven slow controllers. Similarly, [13] introduced a novel dual-input control strategy, where separate inputs are used for fast and slow states, enabling more flexible and efficient control of these systems.

Recent advancements in reinforcement learning (RL) have provided a model-free alternative for controlling LSPSs, particularly when system dynamics are partially or entirely unknown. In RL- based methods, neural networks are used to approximate given cost functions and reconstruct unmeasurable states, which enable the design of adaptive controllers without relying on precise system models [14, 16]. Despite their effectiveness, RL-based approaches often suffer from high computational costs and extended learning times, which limit their practical applicability, especially for systems with stringent performance requirements [14, 21].
In [5], a mixed $H_2/H_\infty$ feedback approach using reduced-order models achieved near-optimal performance with lower computational effort. Similarly, [4] employed balanced model reduction and perturbation approximations to simplify feedback control problems, alleviating numerical stiffness and ensuring robust performance for large-scale systems.

One of the methods for controlling linear systems is to use the pole assignment method in linear systems with state feedback. Karbassi and Bell [11] introduced an algorithm to derive an explicit parametric controller matrix through elementary similarity transformations that convert the controllable pair into a primary vector companion form.

This paper introduces a novel approach to the optimal control of LSPSs by leveraging eigenvalue placement through the Certainty Matrix. Unlike traditional methods that rely on reduced-order modeling [22] or RL-based algorithms [20], the proposed method directly incorporates the system's eigenstructure into the control design. By doing so, it provides a robust framework for addressing the ill-posedness of Riccati equations and ensuring the convergence of iterative algorithms such as Kleinman's method. Furthermore, the proposed method facilitates the optimal placement of eigenvalues, enabling simultaneous control of fast and slow dynamics with guaranteed stability and effective computational cost reduction.

In recent years, several notable advancements in the field of optimal control have been proposed that offer novel perspectives on dealing with time delays, fractional dynamics, and nonlinearities.For instance, an efficient finite difference method has been applied to handle optimal control problems with time-varying delays [8], and a new modal series representation was proposed to solve nonlinear optimal control problems [9]. Moreover, a novel fractional model and control framework has been introduced for tumor-immune systems using non-singular derivatives [2], while a precise finite difference formula was recently developed for numerically solving fractional optimal control problems with delay dependencies [1]. These advancements reflect the evolving nature of optimal control theory and further motivate the development of robust and generalizable approaches such as the one presented in this paper.

Also significant attention has been devoted to the design of precise and robust control strategies for nonlinear systems and the optimization of their performance. In this regard, Ebrahimipour and Mirhosseini-Alizamini [6] proposed an optimal adaptive sliding mode controller for a class of nonlinear affine systems, ensuring system stability and desired performance. Furthermore, Hashemi Borzabadi et al. [7] introduced a sub-ordinary approach to obtain near-exact solutions for optimal control problems, demonstrating high accuracy and computational efficiency.

In this paper, a new robust optimal algorithm based on eigenvalue assignment in a prescribed region is designed. The proposed method is free on ill-conditioned numerical issue and guarentee the stability of the system for the purpose a theorem is proved for stability analysis and analytical support of the proposed method. Compared to model reduction techniques that may compromise performance by ignoring high-frequency dynamics, our approach fully preserves the system's structure and ensures robustness against singular perturbations. This distinction makes the proposed method more reliable and efficient for real-time and high-dimensional control tasks.

The proposed method assigns the best eigenvalues of the LSPS, which overcomes to numerical ill-conditioning, in an inverse parametric optimal control approach. Additionally, a novel algorithm is developed based on the invers parametric optimal control approach to ensure robustness and stability of the LSPS control. Also, the computational efficiency of the method is increased by avoiding solving complex the Riccati equation. This paper is organized in five sections. Linear singularly perturbed system dynamic and its challenges are introduced in section 2. Section 3 presents an inverse optimal control method, a theorem and a new algorithm to facilitate application of the proposed method. Two example are simulated in section 4. Finally, the conclusion of this paper is presented in section 5.

## 2 Problem Statement

Consider the following linear singularly perturbed optimal control system:

$$\begin{bmatrix} \dot{x_1}(t) \\ \epsilon \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \tag{1}$$

where the objective is to minimize the cost function:

$$J(x,t) = \int_t^{\infty} \left( x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau) \right) d\tau, \tag{2}$$

with weighting matrices $Q \geq 0$ and $R > 0$. The singular perturbation parameter satisfies $0 < \epsilon \ll 1$ . The state vector is defined as $x(t) = \left[ x_1^T(t), x_2^T(t) \right]^T \in \mathbb{R}^n$, where $x_1(t) \in \mathbb{R}^{n_1}$ contains the fast states and $x_2(t) \in \mathbb{R}^{n_2}$ the slow states, with $n = n_1 + n_2$. The control input is denoted by $u(t) \in \mathbb{R}^m$, and the system matrices $A_{ij}$ and $B_i$ (for $i, j = 1, 2$) have suitable dimensions. Let

$$A_\epsilon = \begin{bmatrix} A_{11} & A_{12} \\ \epsilon^{-1} A_{21} & \epsilon^{-1} A_{22} \end{bmatrix}, \quad B_\epsilon = \begin{bmatrix} B_1 \\ \epsilon^{-1} B_2 \end{bmatrix}, \tag{3}$$

where the dimensions of the matrices $A_{ij}$ and $B_i$ are specified as follows:

- $A_{11} \in \mathbb{R}^{n_1 \times n_1}$: interactions among fast states.
- $A_{12} \in \mathbb{R}^{n_1 \times n_2}$: coupling from slow to fast states.
- $A_{21} \in \mathbb{R}^{n_2 \times n_1}$: effect of fast states on slow states.
- $A_{22} \in \mathbb{R}^{n_2 \times n_2}$: interactions among slow states.
- $B_1 \in \mathbb{R}^{n_1 \times m}$: control input influence on fast states.
- $B_2 \in \mathbb{R}^{n_2 \times m}$: control input influence on slow states.

The dimensions of the matrices above play a crucial role in formulating the system's dynamics and guiding the analysis of the perturbation effects in both fast and slow subsystems. This formulation provides a proper modeling of singularly perturbed systems, allowing for the analysis of the system's behavior under different perturbation scales and guiding the design of

effective control strategies that balance the fast and slow dynamics.
The goal of this paper is to design an optimal controller for system prescribed in (1), (2). The proposed controller can be considered as follows:

$$u(t) = -K_\epsilon x(t). \tag{4}$$

**Assumption 1:** Suppose the following conditions hold:

a. The pair $(A_\epsilon, B_\epsilon)$ is controllable,

b. The pair $(A_\epsilon, \sqrt{Q})$ is observable.

The main challenge addressed in this study is the difficulty of achieving optimal control for Singularly Perturbed Systems (SPS), which exhibit both fast and slow dynamics, making them prone to numerical instability and high computational cost. In this paper, a robust eigenvalue assignment-based inverse optimal control technique is proposed to overcome these challenges, which reduces computational cost while maintaining system stability and optimality. This approach provides a robust and efficient alternative to conventional control methods.

## 3 Inverse Optimal Control of Linear Singularly Perturbed System

Consider a controllable time-invariant linear dynamical system as follows:

$$\dot{x}(t) = A_\epsilon x(t) + B_\epsilon u(t), \tag{5}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, and the matrices $A_\epsilon$ and $B_\epsilon$ are real constant matrices of dimensions $n \times n$ and $n \times m$, respectively, with $\text{rank}(B_\epsilon) = m$, as defined in (3). The goal of eigenvalue assignment is to design a state feedback controller

$$u(t) = K_\epsilon x(t), \tag{6}$$

such that the optimal control system prescribed in Equation (1) and (2) is controlled and stability of the system be guaranteed. If $K_\epsilon \in \mathbb{R}^{m \times n}$ is the state feedback matrix. According to (5) and (6), the closed-loop dynamical system is obtained as follows:

$$\dot{x}(t) = (A_\epsilon + B_\epsilon K_\epsilon)x(t). \tag{7}$$

To design a stable inverse controller for (7), the companion vector form of $A_\epsilon$ and $B_\epsilon$ can be computed using the following relations

$$\tilde{A}_\epsilon = T_\epsilon^{-1} A_\epsilon T_\epsilon = \begin{bmatrix} & G_{0,\epsilon} \\ I_{n-m} & 0_{n-m,m} \end{bmatrix}, \tag{8}$$

$$\tilde{B}_\epsilon = T_\epsilon^{-1} B_\epsilon = \begin{bmatrix} B_{0,\epsilon} \\ 0_{n-m,m} \end{bmatrix}, \tag{9}$$

where $T_\epsilon$ is a similarity matrix [11]. Also, $G_{0,\epsilon}$ is an $m \times n$ matrix and $B_{0,\epsilon}$ is an upper triangular $m \times m$ matrix. It is worth noting that if the Kronecker variables [17] of the pair $A_\epsilon$ and $B_\epsilon$ are regular, then $\tilde{A}_\epsilon$ and $\tilde{B}_\epsilon$ always comply with this structure [10]. The state feedback controller, which assigns zero to all eigenvalues for the transformed pair $A_\epsilon$ and $B_\epsilon$, is defined as follows:

$$u = -B_{0,\epsilon}^{-1} G_{0,\epsilon} x = \tilde{K}_\epsilon x. \tag{10}$$

Also, $K_\epsilon$ in (7) can be obtained using the following relation:

$$K_\epsilon = \tilde{K}_\epsilon T_\epsilon^{-1}.$$

Since in the presence of a singular disturbance parameter, eigenvalues located near the origin lead to high system sensitivity and reduced robustness, it is necessary to shift the eigenvalues from the origin to a specified region on the left-hand side in the complex plane. To achieve this objective, we employ the certainty matrix technique. The reader is referred to [19] for more detailed explanations.

Assume that $\tilde{A}_{\Lambda,\epsilon}$ is a matrix in the companion vector form with eigenvalue spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, which is

$$\tilde{A}_{\Lambda,\epsilon} = \begin{bmatrix} \multicolumn{2}{c}{G_{\Lambda,\epsilon}} \\ I_{n-m} & O_{n-m,m} \end{bmatrix}, \tag{11}$$

then

$$\tilde{K}_\epsilon = -B_{0,\epsilon}^{-1}(G_{0,\epsilon} - G_{\Lambda,\epsilon}), \tag{12}$$

is the feedback matrix that assigns the spectrum of eigenvalues $\Lambda$ to the closed-loop matrix $\tilde{\Gamma}_\epsilon = \tilde{A}_\epsilon + \tilde{B}_\epsilon \tilde{K}_\epsilon$. Furthermore,

$$K_\epsilon = \tilde{K}_\epsilon T_\epsilon^{-1} = -B_{0,\epsilon}^{-1}(G_{0,\epsilon} - G_{\Lambda,\epsilon}) T_\epsilon^{-1}, \tag{13}$$

is a feedback matrix that assigns the set of eigenvalues to the pair $(B_\epsilon, A_\epsilon)$.

In this paper, a matrix called the certainty matrix corresponding to system (1) is used to assign eigenvalues in a specific region of the complex plane based on an explicit parametric control rule with nonlinear parameters.

The certainty matrix is considered as follows

$$\tilde{A}_{c,\epsilon} = \begin{bmatrix} \multicolumn{2}{c}{G_{c,\epsilon}^k} \\ I_{n-m} & O_{n-m,m} \end{bmatrix}. \tag{14}$$

It is obvious that for the $\Lambda$ eigenvalue spectrum of the above parametric matrix and the corresponding characteristic equation of the spectrum of $\Lambda$, the following relationship holds

$$P_n(\lambda) \ = \det(\tilde{A}_{c,\epsilon} - \lambda I) = 0. \tag{15}$$

Let $f_i(g_{11}, g_{12}, \dots, g_{1n}, g_{21}, \dots, g_{mn})$ be coefficients of $\lambda^{(i)}$ for $i = 1, \dots, n$ in $\det(\tilde{A}_{c,\epsilon} - \lambda I)$ and $c_i$ be coefficient of $\lambda^{(i)}$ in characteristic polynomial $P_n(\lambda)$ of the spectrum $\Lambda$. Now, according to (15), it can be inferred

$$f_i(g_{11}, g_{12}, \dots, g_{1n}, g_{21}, \dots, g_{mn}) = C_i, \qquad i = 1, 2, \dots, n \tag{16}$$

where the elements $g_{ij}$ of the matrix $\tilde{G}_{c,\epsilon}$ are defined as follows:

$$[\tilde{G}^k_{c,\epsilon}]_{ij} = \begin{cases} g_{kj}, & i = k, \\ 1, & i \neq k. \end{cases}$$

Also, $k$ represents an arbitrary row and the $k$-th row of $\tilde{G}^k_{c,\epsilon}$ can be determined by solving the linear system (16) . The objective is to find the $c_i$ values by solving the system, such that all eigenvalues of the closed-loop matrix corresponding to (5) satisfy the following conditions:

$$\beta \leq \mathrm{Re}(\lambda_i) \leq \alpha, \qquad i = 1, 2, \dots, n. \tag{17}$$

Now, based on the definition of the characteristic equation, we have

$$t_{i1} < c_i < t_{i2},$$

such that

$$\begin{cases} t_{i1} = \min\left\{(-1)^i \binom{n}{i} \alpha^i, (-1)^i \binom{n}{i} \beta^i\right\}, \\ t_{i2} = \max\left\{(-1)^i \binom{n}{i} \alpha^i, (-1)^i \binom{n}{i} \beta^i\right\}, \end{cases} \tag{18}$$

where $t_{i1}$ and $t_{i2}$ represent the lower and upper bounds of the coefficient for the $i$-th term, respectively. Now, assume that $\ell \in (0, 1)$, then

$$C'_i = \ell t_{i1} + (1 - \ell) t_{i2}, \tag{19}$$

the variable $\ell$ is considered for constructing the convex combination of the upper and lower bounds of each inequality. Therefore, the system of (16) will be solved using the new coefficients obtained from (19) as follows:

$$f_i(g_{11}, g_{12}, \dots, g_{1n}, g_{21}, \dots, g_{mn}) = C'_i \qquad i = 1, 2, \dots, n \tag{20}$$

This is a linear system with $n$ equations and $n$ unknown parameters $g_{kj}$, which after solving, by choosing $\ell$ in the interval $(0, 1)$ and calculating the certainty matrix $A_{c,\epsilon}$, the state feedback matrix $K_\epsilon$ is obtained in such a way that the eigenvalues of the closed-loop matrix are located in the desired enclosed region. Finally, for the feedback matrix $K_\epsilon$, we have

$$K_\epsilon = \tilde{K}_\epsilon T_\epsilon^{-1} = -B_{0,\epsilon}^{-1}(G_{0,\epsilon} - G_{\lambda,\epsilon} - G_{c,\epsilon})T_\epsilon^{-1}. \tag{21}$$

Suppose we aim to place the eigenvalues of the closed-loop matrix on the left side of the line $x = \alpha$. The weighted matrix $\hat{Q}$ is designed to achieve this placement while ensuring system stability and optimizing the performance function (2) [18].

For this purpose, let

$$M_{c,\epsilon} = \begin{bmatrix} B_\epsilon & A_\epsilon B_\epsilon & \cdots & A_\epsilon^{n-1} B_\epsilon \end{bmatrix}, \tag{22}$$

and, construct the gain controllability matrix $V_\epsilon$ as follows:

$$V_\epsilon = K_{m\times n,\epsilon} \times M_{c_{n,mn},\epsilon}, \tag{23}$$

where $K_{m\times n,\epsilon}$ represents the control gain matrix (21).

Additionally, compute the weighted matrix $\widehat{Q}$ using the following relations

$$\widehat{Q} = \left(V_{m\times mn} M^{+}_{c_{mn\times n}}\right)^T \left(V_{m\times mn} M^{+}_{c_{mn\times n}}\right), \tag{24}$$

where

$$M^{+}_{c_{mn\times n}} = \left(M_c^T\right)_{nm\times n} \left(M_{c_{n\times mn}} M_c^T\right)^{-1}_{n\times n}. \tag{25}$$

Now, for designing the optimal controller, the following Algebraic Riccati Equation (ARE) can be solved:

$$A_\epsilon^T P + P A_\epsilon - K_\epsilon^T R K_\epsilon + \widehat{Q} = 0, \tag{26}$$

where $P$ is a positive definite matrix. After solving this equation, the optimal control gain $K_\epsilon$ is as follows:

$$K_\epsilon = R^{-1} B_\epsilon^T P. \tag{27}$$

By solving this optimal control problem, the stability of the system is systematically guaranteed by assigning eigenvalues in a specific region of the complex Z- plane while minimizing the cost function and avoiding the undesirable effects of a singular disturbance, which appears in direct control in solving the Riccati equation.

The proposed method enhances the quality of eigenvalue assignment by enabling direct and precise placement of eigenvalues within a desired region of the complex plane, without relying on iterative tuning or solving Riccati equations. Unlike traditional methods, it allows explicit enforcement of spectral constraints, leading to improved robustness, better damping characteristics, and superior control over the closed-loop dynamics. This results in a more reliable and performance-oriented system design, especially in cases where sensitivity and numerical conditioning are critical.

Now, consider the following theorem to facilitate the analytical investigation of the stability property of the proposed method.

**Theorem 1.** Consider the singularly perturbed closed-loop system (1). If $\widehat{Q}$ and $K_\varepsilon$ are determined by equations (24) and (27), respectively, and if $R$ is a positive definite matrix, then the optimal control system with $u = -K_\varepsilon x$ is asymptotically stable.

*Proof.* Suppose that $P$ is the solution to the following Riccati equation

$$A_\epsilon^T P + PA_\epsilon - K_\varepsilon^T RK_\varepsilon + \widehat{Q} = 0. \tag{28}$$

Also, consider $V(x) = x^T Px$ as a Lyapunov function. Therefore,

$$\dot{V}(x) = x^T \left[ (A_\varepsilon - B_\varepsilon K_\varepsilon)^T P + P(A_\varepsilon - B_\varepsilon K_\varepsilon) \right] x. \tag{29}$$

By some simple computations, we have

$$\dot{V}(x) = -x^T \left( \widehat{Q} + K_\varepsilon^T RK_\varepsilon \right) x. \tag{30}$$

If $\widehat{Q}$ and $K_\varepsilon^T RK_\varepsilon$ are positive definite, then for any $x \neq 0$, we have $\dot{V}(x) < 0$.
Now, we show that, based on the designed controller, $\widehat{Q}$ and $K_\varepsilon^T RK_\varepsilon$ are positive definite.

First, we prove that $\widehat{Q}$ is positive definite.

Based on equations (22) - (25), the following relation can be established:

$$VM_c^+ = (K_\varepsilon M_c)(M_c^T)(M_c M_c^T)^{-1} \tag{31}$$

$$= (K_\varepsilon M_c)(M_c^T)(M_c^T)^{-1}(M_c^{-1}) = K_\varepsilon \tag{32}$$

thus:

$$\widehat{Q} = (VM_c^+)^T (VM_c^+) = K_\varepsilon^T K_\varepsilon > 0. \tag{33}$$

On the other hand, since $R > 0$, it is straightforward to verify that $K_\varepsilon^T RK_\varepsilon > 0$. Therefore, $\dot{V}(x) < 0$ always holds. Moreover, $\dot{V}(x) = 0$ holds only at the equilibrium point, which completes the proof.

□

**Note:** Considering that the eigenvalues of the closed-loop matrix $\dot{x} = (A_\epsilon - B_\epsilon K_\epsilon)x$ are calculated by solving a system of linear equations (21) based on the bounds from equation (18) and their convex combination. The very small value of $\epsilon$ does not significantly affect on solving this equation system. Moreover, since the eigenvalues are assigned within a specified region on the left-hand side of the complex plane, by choosing an appropriate $r$ and given that $\beta \leq \text{Re}(\lambda) \leq \alpha$, it follows that:

$$\begin{aligned} &\lambda_{\text{min}} \geq \beta + re^{i\theta}, \qquad 0 \leq \theta \leq 2\pi, \\ &\lambda_{\text{max}} \leq \alpha + re^{i\theta}, \end{aligned} \tag{34}$$

then the condition number of the closed-loop matrix $\Gamma_\epsilon = A_\epsilon - B_\epsilon K_\epsilon$ can be computed as follows [20]:

$$\kappa(\Gamma_\epsilon) = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \leq \frac{\alpha + re^{i\theta}}{\beta + re^{i\theta}} = \mathcal{O}(1). \tag{35}$$

Therefore, the proposed method is free of ill-conditioning.

**Algorithm 1** Inverse optimal control of linear singularly perturbed systems

**Input:** $A_\epsilon$, $B_\epsilon$, Q, R, $\epsilon$ and $\Lambda$.

**Step 1.** Set $\alpha = a$ and $\beta = \alpha - 1$, where $a$ is a non-positive integer.

**Step 2.** Using equation (8), compute the companion form of $A_\epsilon$ and $B_\epsilon$ using (9).

**Step 3.** For the given $\Lambda$ compute the matrix $\tilde{A}_{\Lambda,\varepsilon}$

$$\tilde{A}_{\Lambda,\varepsilon} = \begin{bmatrix} & G_{\Lambda,\varepsilon} & \\ I_{n-m} & & O_{n-m,m} \end{bmatrix}.$$

**Step 4.** To assign eigenvalues in the region $\beta \leq \text{Re}(\lambda_i) \leq \alpha$, compute the $\tilde{A}_{c,\varepsilon}$ matrix using equation (21)

$$\tilde{A}_{c,\varepsilon} = \begin{bmatrix} & G_{c,\varepsilon} & \\ I_{n-m} & & O_{n-m,m} \end{bmatrix}.$$

**Step 5.** Calculate the state feedback matrix $K_\varepsilon$ as

$$K_\varepsilon = B_{0,\varepsilon}^{-1} \left( G_{0,\varepsilon} - G_{\Lambda,\varepsilon} - G_{c,\varepsilon} \right) T_\varepsilon^{-1}.$$

**Step 6.** Compute the matrix $\hat{Q}$ using the state feedback matrix from Step 5 as

$$\widehat{Q} = \left( V_\epsilon M_{c,\epsilon}^{+} \right)^T \left( V_\epsilon M_{c,\epsilon}^{+} \right),$$

where

$$M_{c,\epsilon}^{+} = \left( M_{c,\epsilon}^T \right) \left( M_{c,\epsilon} M_{c,\epsilon}^T \right)^{-1}, \quad V_\epsilon = K_\varepsilon \times M_{c,\epsilon}.$$

**Step 7.** Determine the matrix $P$ such that

$$A_\varepsilon^T P + P A_\varepsilon - K_\varepsilon^T R K_\varepsilon + \widehat{Q} = 0.$$

**Step 8.** Compute the final controller matrix $K_\varepsilon$ using (27).

## 4 Numerical Examples

**Example 1.** Consider the system described by (1) and the obtained results are applied to an RC ladder circuit system, which is widely used for analog-to-digital conversion applications. The system matrices for this circuit are provided in [20] as follows

$$A_\epsilon = \begin{bmatrix} -\frac{3}{2RC} & \frac{1}{RC} & 0 & 0 \\ \frac{1}{RC} & -\frac{2}{RC} & 0 & 0 \\ 0 & 0 & -\frac{2}{\epsilon RC} & \frac{1}{\epsilon RC} \\ 0 & 0 & \frac{1}{\epsilon RC} & -\frac{3}{2\epsilon RC} \end{bmatrix}, \quad B_\epsilon = \begin{bmatrix} \frac{1}{2RC} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{2\epsilon RC} \end{bmatrix}, \tag{36}$$

where $R = 5 \times 10^3\,\Omega$, $C = 100 \times 10^{-6}\,\text{F}$, and $\epsilon = 0.05$. The weighting matrix of the cost function defined in equation (2) is selected as $R = I_2$, and the matrix $Q$ is given by:

$$Q = \begin{bmatrix} 50 & 0 & 0.5 & 1 \\ 0 & 100 & 0.5 & 1.5 \\ 0.5 & 0.5 & 1 & 0 \\ 1 & 1.5 & 0 & 1 \end{bmatrix}. \tag{37}$$

The objective is to determine the state feedback matrix $K_\epsilon$ such that all eigenvalues of the closed-loop matrix are placed in the left-hand side of the complex plane. The parameters are set as $a = 0$, $\alpha = 0$, $\beta = -1$ and $l = 0.75$. In this example, the results of the proposed method are compared with [20]. To ensure a fair comparison, both the proposed method and the method in [20] use the same initial condition: $x_0 = [1, 1, -1, -1]^T$. This comparison demonstrates the effectiveness of the proposed method particularly with respect to state variables, control inputs, cost function and cumulative i.e., $\int_0^t \left(x^T Q x + u^T R u\right) d\tau$.

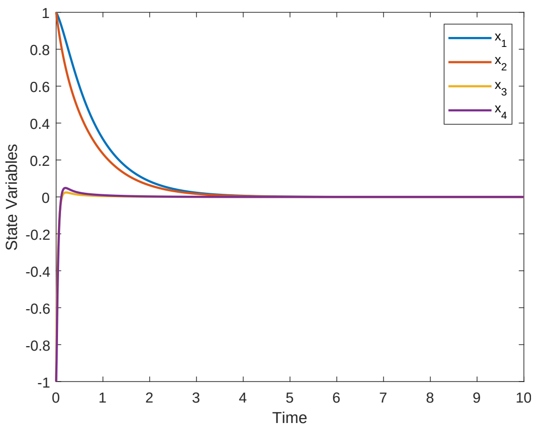


**Figure 1:** The proposed state variables.

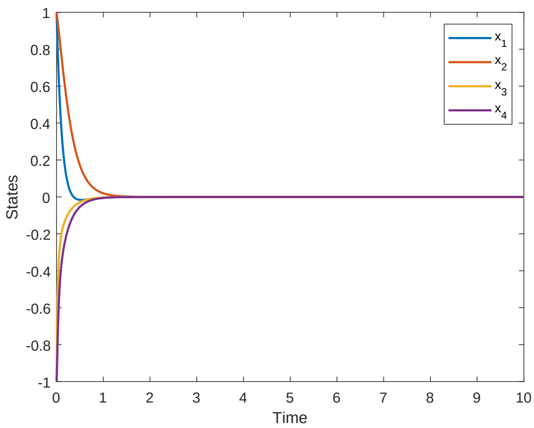


**Figure 2:** The state variables in [20].

Figures 1 and 2 illustrate the behavior of state variables in the proposed method and the method presented in [20], respectively. As observed, in both methods, the state variables gradually converge to the equilibrium state.

Figures 3 and 4 describe the control signals generated by the two different methods. In the proposed method (Figure 3), the amplitude of the control inputs is significantly reduced. The reduction in control amplitude is a crucial feature, as it indicates lower energy consumption in the system and improved efficiency of the controller.

Figures 5 and 6 illustrate the cost function values for both methods. It can be observed that the cost function of the proposed method is significantly very less than the cost function value in [20]. This implies better system performance and lower control costs.

Additionally, Figures 7 and 8 represent cumulative cost function value in both methods. This value in the proposed method is considerably lower than that of the method presented in [20]. This suggests that in the proposed method, the system achieves stability and desirable performance with minimal cost.

The comparisons between results of the proposed method and the method presented in [20] are shown in Table 1. As one can see, cumulative cost, robustness indicators ($\|K_\epsilon\|_2$, $\|c\|_2$ and $\|c\|_\infty$) and number of iteration is significantly reduced. Where $K_\epsilon$ is optimal feedback gain matrix and vector $c$ with elements $c_j$ is defined as follows:

$$c_j = \frac{\|q_j\|\|p_j\|}{|q_j^T p_j|} \geq 1. \tag{38}$$

Additionally, the proposed method demonstrates a significant improvement in computational efficiency. The reduced number of required iterations compared to the method in [20], indicates lower computational complexity and enhanced algorithmic performance. Notably, while preserving the stability and robustness of the control system, the proposed method minimizes both computational and performance costs.

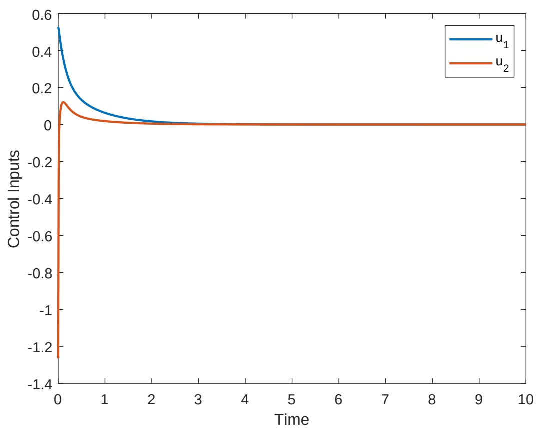


**Figure 3:** The proposed control signal.

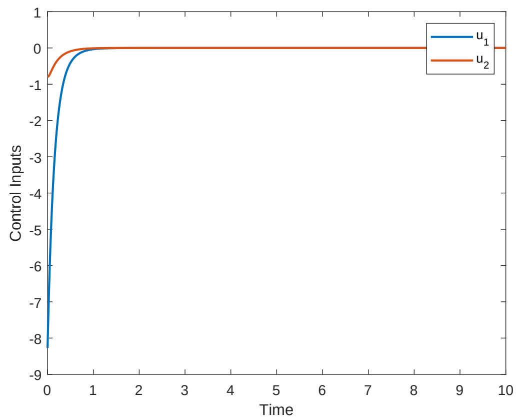


**Figure 4:** Control signal in [20]

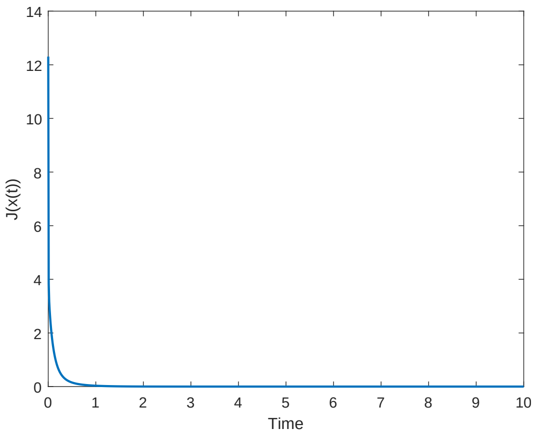


**Figure 5:** The proposed objective function.

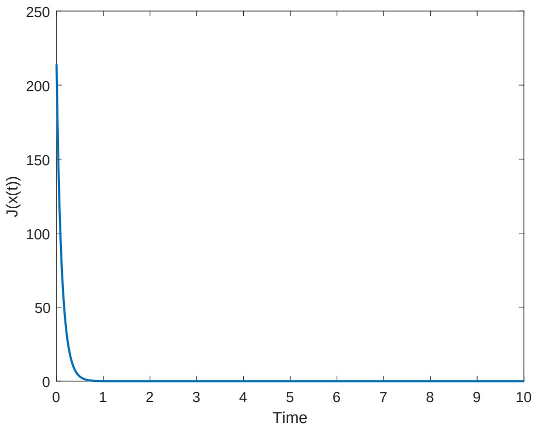


**Figure 6:** The objective function in [20].

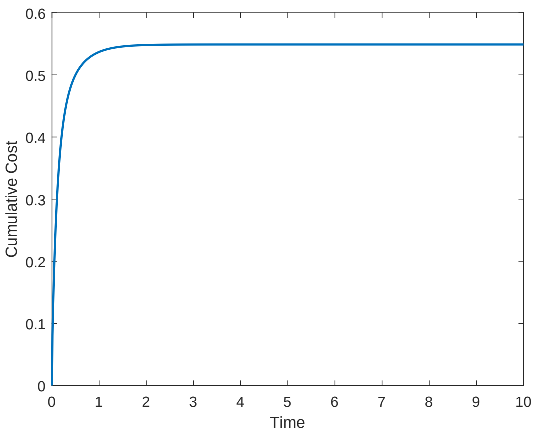


**Figure 7:** Value $\int_0^t \left(x^T Qx + u^T Ru\right) d\tau$ in presented method.

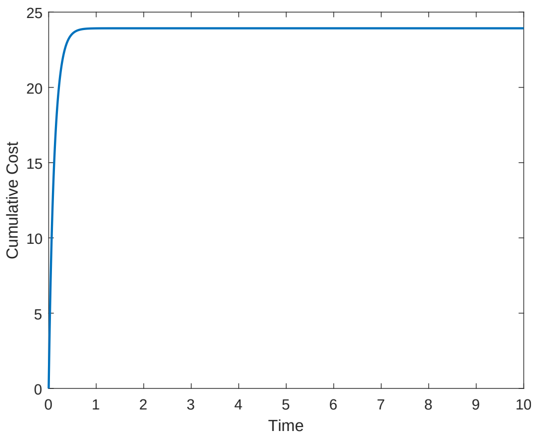


**Figure 8:** Value of $\int_0^t \left(x^T Qx + u^T Ru\right) d\tau$ in [20]

**Table 1:** Comparison between the results of the proposed method and the proposed method in [20].

| Indexes / Methods | $\|c\|_2$ | $\|c\|_\infty$ | $\|K_\epsilon\|_2$ | Cumulative Cost | Number of Iteration |
|---|---|---|---|---|---|
| The proposed method in [20] | 2.4091 | 1.3666 | 6.1755 | 23.93 | 6 |
| The proposed method | 2.3781 | 1.3136 | 6.0046 | 0.5489 | 1 |

**Example 2.** Consider the system described by (1) and a standard SP system [3]. The model matrices are considered as follows

$$A_\epsilon = \begin{bmatrix} 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.345 & 0 \\ 0 & -\frac{0.524}{\epsilon} & -\frac{0.465}{\epsilon} & \frac{0.262}{\epsilon} \\ 0 & 0 & 0 & -\frac{1}{\epsilon} \end{bmatrix}, \quad B_\epsilon = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{\epsilon} \\ \frac{1}{\epsilon} \end{bmatrix}, \tag{39}$$

where $\epsilon = 0.05$. The weighting matrices of the cost function defined in equation (2) are chosen as $Q = 10I$ and $R = I_2$. The goal is to design an optimal control signal of a perturbed singular system with given drift and diffusion Matrixes $A_\epsilon$ and $B_\epsilon$ in (39). The system parameters are set as $a = -2$, $\alpha = -2$, $\beta = -3$, and $l = 0.75$ with initial condition: $x_0 = [1,\ 2,\ 1,\ 0]^T$.

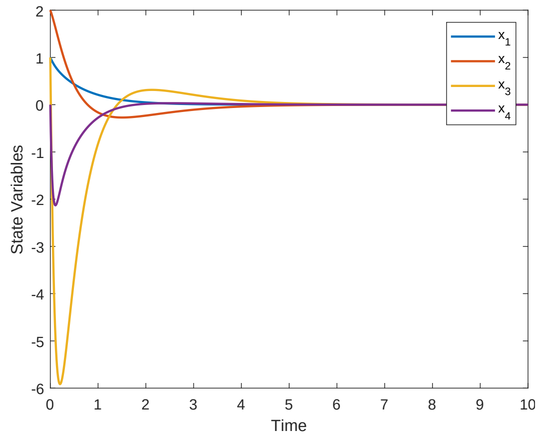


**Figure 9:** The state variables.

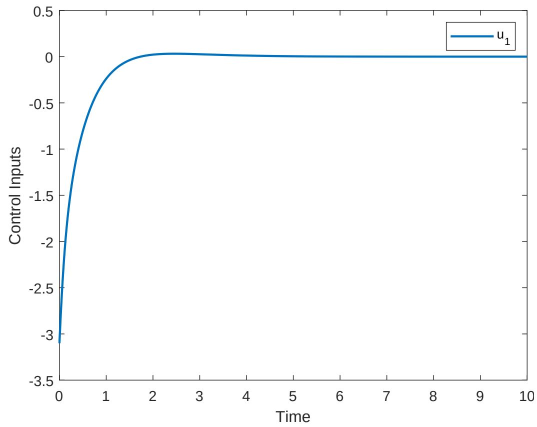


**Figure 10:** Control signal.

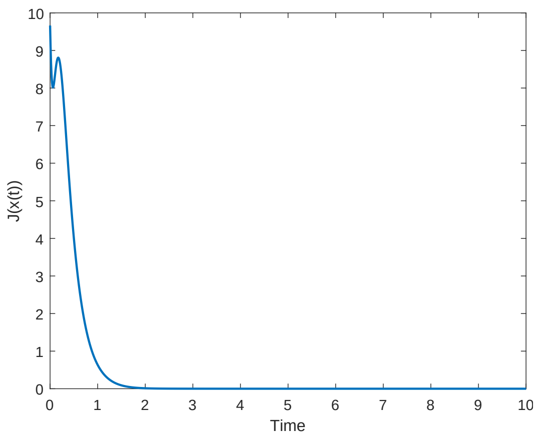


**Figure 11:** The objective function.

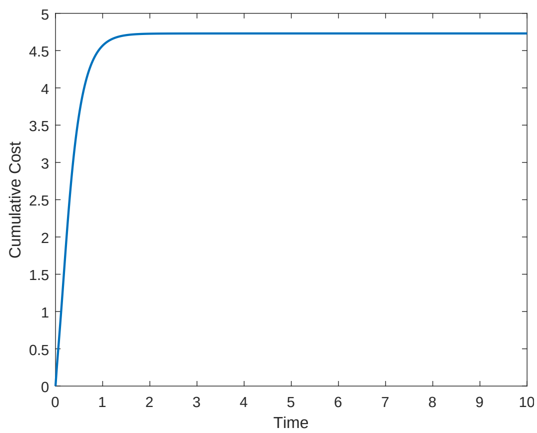


**Figure 12:** Cumulative cost.

Figures 9 and 10 demonstrate the evolution of the state variables and control under the proposed control strategy. As illustrated in Figure 9, the state variables converge smoothly to the equilibrium state, indicating the system's asymptotic stability. Figure 10 shows the corresponding control, which remain bounded with relatively low amplitudes throughout the simulation. This behavior highlights the efficiency of the controller in achieving stability while minimizing control effort. Figures 11 and 12 present the performance of the controller in terms of the cost function. Figure 11 depicts the instantaneous value of the cost function $\int_0^t (x^T Q x + u^T R u)\, d\tau$, while Figure 12 shows its cumulative value.In [3] the control gain matrix $K = [3.16, 1.99]$ only for two slow states. According to this control gain the cumulative cost is 7.29 that is bigger than cumulative value of the proposed method that is $4.7302$. The results indicate that the proposed method maintains a low cost throughout the system operation, which directly reflects reduced energy consumption and improved control efficiency. The significantly lower cumulative cost confirms the superiority of the proposed method in terms of overall performance and resource utilization.

## 5 Conclusion

In this study, a novel inverse optimal control method based on eigenvalue assignment was developed to address challenges such as ill-conditioning, high system dimensionality, and the complexity of solving algebraic Riccati equations. The proposed approach exhibits strong resistance to numerical instabilities, offers robustness against perturbations and numerical diffusion, and ensures system stability through optimal eigenvalue placement in the singularly perturbed closed-loop system. As demonstrated by simulation results, this method significantly enhances controller robustness, while also achieving significant reductions in both the performance cost function and control signal amplitude.

## Declarations

### Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

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### Competing Interests

The authors declare that they have no competing interests relevant to the content of this paper.

### Authors' Contributions

The main text of manuscript is collectively written by the authors.

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