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An Enhanced QAOM-Based MAGDM Framework: Integrating Entropy Weighting and Expert Judgment Aggregation

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Abstract. Addressing complex decision-making scenarios, particularly those involving multiple criteria and expert perspectives, often requires robust frameworks capable of managing uncertainty and qualitative assessments. The Qualitative Absolute Order-of-Magnitude (QAOM) model offers a flexible approach for expressing subjective evaluations through linguistic terms with adjustable levels of detail. However, practical challenges remain in applying QAOM, including the absence of an inherent system for deriving attribute weights, limitations in coherently synthesizing the judgments from multiple experts, and the lack of systematic normalization procedures for negatively oriented attributes. To address these issues, this paper proposes an advanced multi-attribute group decision-making (MAGDM) framework fully embedded within the QAOM paradigm. The proposed solution introduces a mathematically consistent metric for comparing linguistic assessments, an entropy-based attribute weighting approach rooted in qualitative information, and an aggregation process that reflects expert diversity. Furthermore, a specialized normalization protocol is developed to handle negative attributes across heterogeneous scales. The feasibility and advantages of the method are validated through comprehensive examples and comparative analyses, highlighting improvements over traditional techniques in terms of objectivity, flexibility, and analytical depth. Overall, these developments markedly enhance the capabilities of QAOM based MAGDM, equipping decision-makers with more nuanced and reliable tools for tackling complex problems characterized by imprecision and divergent expert opinions.

Keywords. Qualitative reasoning, Multi-attribute decision making, Ranking, Qualitative absolute order-of-magnitude.

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1 Introduction

Complex decision-making, characterized by multiple conflicting criteria, is pervasive in science and industry. Addressing this complexity requires structured approaches, leading to the development and adoption of diverse decision-support methodologies across many fields [8, 22, 30, 44, 38]. Structured methodologies like multi-attribute decision-making (MADM) and its extension, multi-attribute group decision-making (MAGDM), are indispensable for addressing complex problems characterized by multiple, often conflicting, performance attributes [1, 17, 26, 34, 45]. However, a significant and recurrent challenge emerges when decision criteria are expressed qualitatively rather than quantitatively. In these contexts, expert assessments are often inherently subjective, relying on interpretive linguistic descriptors rather than strict numerical values [3, 4, 6, 22, 28, 40, 41]. Decision-makers commonly employ verbal appraisals, such as “satisfactory,” “inadequate,” or “high risk”, with the degree of specificity shaped by the complexity of the attribute or the inherent ambiguity involved. While intuitively appealing, the adoption of linguistic variables introduces its own set of complexities, notably vagueness and information gaps that arise from imprecise categories and incomplete knowledge [6]. To mitigate these uncertainties, a variety of advanced techniques have been developed in recent years, including but not limited to: models leveraging linguistic fuzzy sets [25, 33, 36], Neutrosophic set theory [8, 10, 11, 32, 45], Dempster-Shafer evidence frameworks [31, 41], prospect theory [16], and advanced fuzzy models such as complex Fermatean fuzzy approaches [7, 9, 35].

Within Artificial Intelligence (AI), the paradigm of Qualitative Reasoning (QR) offers significant tools for the representation and inference of information that is inherently qualitative, incomplete, or expressed in non-numerical terms [14, 23, 29]. QR techniques are particularly adept at handling linguistic variables, adapting to scenarios where assessments are not strictly quantitative. Among these, the Qualitative Absolute Order-of-Magnitude (QAOM) model, originally conceptualized by Dubois [21], is especially suitable for decision-making applications.

A distinguishing feature of the QAOM framework is its ability to systematically encode subjective evaluations through a hierarchical scheme of linguistic descriptors. This flexibility allows QAOM to accurately capture nuanced expert opinions, utilizing interval-based linguistic expressions (for example, “very bad to medium”) and thereby accommodating diverse levels of detail and certainty [4, 18, 26, 42]. The model’s suitability for situations with varying expert confidence across decision criteria is further supported by its capacity for multi-resolution linguistic representation. Building on its foundational strengths, several studies have explored the integration of QAOM with established MADM techniques such as Technique for Order Preference by Similarity to the Ideal Solution (TOPSIS) [2, 3, 4, 5, 19, 20]. For instance, Agell et al. [5] introduced the use of location-functions to express linguistic variables and relied on Euclidean distance metrics for the comparison of alternatives. Similarly, adaptations of the

TOPSIS method within the QAOM setting have been proposed for applications such as site selection and the appraisal of energy alternatives [2, 4].

Despite these notable developments and demonstrated practical utility, the broader advancement of QAOM, particularly in the context of multi-attribute group decision-making, remains constrained by several unresolved challenges. Addressing these limitations constitutes the main motivation for the present study.

Firstly, current QAOM-based approaches do not provide an objective, internally consistent strategy for determining the importance (weights) of attributes within the QAOM environment itself. Conventional methods have either utilized subjective weights assigned by human experts or depended on objective weighting schemes such as entropy measures, which are computed externally based on numerical approximations of qualitative inputs [4]. This disconnect introduces potential bias and methodological inconsistency, as the computation of attribute importance may not fully reflect the qualitative essence of the original data. The absence of an integrated, objective approach for weight determination is thus a significant limitation, undermining the reliability and credibility of resulting recommendations.

Secondly, conventional QAOM methods exhibit notable weaknesses in the realm of true multi-attribute group decision-making . While the technique of applying Euclidean distances to sets of concatenated location-function vectors provides a solution in scenarios restricted to single attributes or cases with only one expert [2, 5], it fails to address the challenge of synthesizing multiple, potentially conflicting evaluative judgments provided by several experts for each attribute. Achieving an effective and systematic consensus, especially when accounting for the differing credibility or influence of each expert, is essential for robust MAGDM, yet remains unaddressed in prevailing QAOM methodologies.

Thirdly, practical decision-making frequently involves attributes with a negative orientation, commonly referred to as cost-type attributes, where lower attribute values are favored (such as expenditure, risk, or environmental impact). Present QAOM frameworks do not incorporate a comprehensive, built-in normalization process that permits the consistent treatment of such cost-type attributes alongside benefit-type attributes, particularly in situations where the descriptions of attributes are expressed using multi-level or interval-based linguistic terms. The absence of a dedicated normalization approach imposes a significant limitation on the practical deployment of QAOM, especially in decision scenarios characterized by the need to balance beneficial and adverse criteria.

In response to the aforementioned methodological gaps, this study endeavors to substantially refine the QAOM approach for use in advanced group decision-making contexts. First, a principled ranking mechanism for qualitative linguistic labels is introduced, relying on a newly developed distance metric to enable comparative analysis across varying degrees of linguistic granularity. At the same time, an entropy-inspired weighting methodology is proposed that draws directly from the qualitative structure of the input data and is seamlessly integrated within the QAOM framework. The method further innovates by implementing a systematic procedure for synthesizing expert evaluations, allowing for graded influence reflecting the expertise and credibility of individual contributors. Finally, a unified normalization protocol is articulated, ensuring the consistent and simultaneous treatment of both benefit-based and cost-based criteria, even in scenarios that involve interval-valued or multi-resolution QAOM descriptors. Through these advances, the proposed framework offers a cohesive and flexible solution to previously unresolved challenges in complex MAGDM problems, substantially broadening the practical relevance and analytic rigor of QAOM-based decision support.

The remainder of this paper is organized as follows: Section 2 reviews the theoretical background. Section 3 elaborates the proposed framework and its core modules. Section 4 offers illustrative applications and comparative analyses. Section 5 presents conclusions, and Section 6 discusses limitations and potential directions for future research.

2 Materials and Methods

2.1 Qualitative Absolute Order-of-Magnitude Framework

In this subsection, we present the essential principles underlying the QAOM approach, rearticulated for clarity and independence from conventional text. The QAOM model builds upon a lexicon of n ordered linguistic terms, each serving as a qualitative indicator for a specific variable. This set is formally defined as [43]:

$$S_n^* = \{B_1, B_2, B_3, \ldots, B_n\}, \qquad B_1 < B_2 < B_3 < \cdots < B_n,$$

where the relation '$<$' establishes an intrinsic ranking among the linguistic descriptors.

Each element B_i in this set can be conceptually associated with a segment of an underlying continuous scale, delineated by $n+1$ real-valued thresholds $\{a_1, a_2, a_3, \ldots, a_{n+1}\}$, such that B_i corresponds to the interval $[a_i, a_{i+1}]$ for $i = 1, \ldots, n$. As a concrete illustration, when $n = 5$, one may define the linguistic levels as follows: B_1 (strongly opposed), B_2 (opposed), B_3 (neutral), B_4 (in favor), and B_5 (strongly in favor).

To effectively capture varying degrees of subjectivity and ambiguity, the QAOM framework extends the set of basic terms to include interval-valued labels, each comprising a contiguous group of basic categories. Accordingly, the full set for qualitative assessment is described by [5, 18, 37]:

$$S_n = S_n^* \cup \{[B_i, B_j] \, : \, B_i, B_j \in S_n^*, \, i < j\}, \qquad n = 1, 2, 3, \ldots. \tag{1}$$

In this expression, each interval $[B_i, B_j]$ (with $i < j$) refers to the collection $\{B_i, B_{i+1}, \ldots, B_j\}$. Any singleton label B_k may be rewritten as the degenerate interval $[B_k, B_k]$. The widest expression of linguistic uncertainty—meaning maximal ambiguity or lack of information—is assigned to the interval $[B_1, B_n]$, which can also be indicated with a '?' symbol. Figure 1 offers a graphical depiction of this label space. Notably, by defining S_n as above, the QAOM framework can accommodate multiple granularities, permitting flexible adjustment of linguistic resolution and accuracy in decision analysis [37].

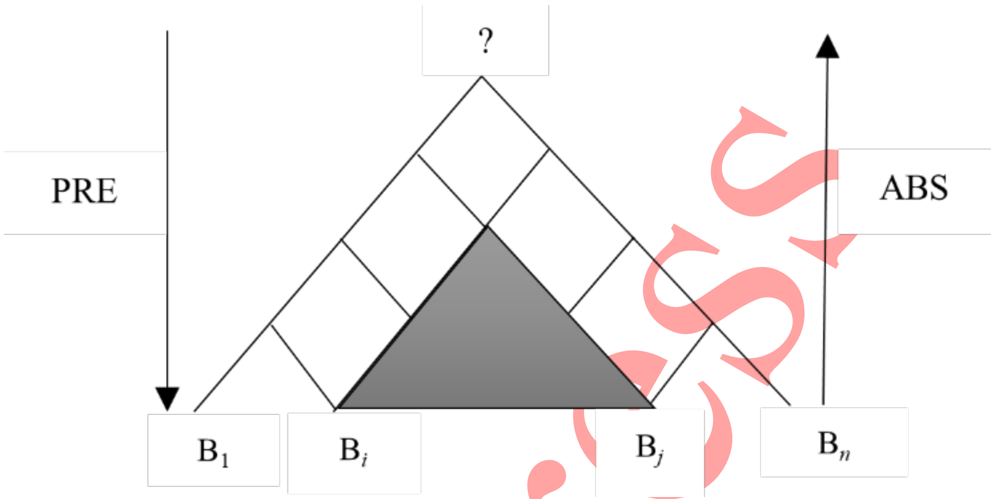

Figure 1: The space of S_n [37].

Definition 1. If $[B_i, B_j]$ and $[B_r, B_s]$ are two elements of set S_n, then $[B_i, B_j]$ is said to be *superior* to $[B_r, B_s]$ if and only if $B_i \geq B_r$ and $B_j \geq B_s$ [37].

Definition 2. Normalizer: The normalizer of the set S_n is defined as a mapping μ:

$$\mu : S_n \to [0, 1]$$

such that

$$\mu([B_i, B_j]) = \sum_{k=i}^{j} \mu(B_k), \qquad \mu(B_k) = \frac{1}{n}, \qquad \sum_{B_k \in S_n^*} \mu(B_k) = 1. \tag{2}$$

This mapping assumes that the linguistic labels $B_1, B_2, \ldots, B_n$ are uniformly and symmetrically distributed [5].

Definition 3. Union and Intersection: If $[B_{i_1}, B_{j_1}]$ and $[B_{i_2}, B_{j_2}]$ are two elements of S_n, their union and intersection are defined as follows [37]:

$$[B_{i_1}, B_{j_1}] \cup [B_{i_2}, B_{j_2}] = [B_{\min(i_1,i_2)}, B_{\max(j_1,j_2)}], \tag{3}$$

$$[B_{i_1}, B_{j_1}] \cap [B_{i_2}, B_{j_2}] = [B_{\max(i_1,i_2)}, B_{\min(j_1,j_2)}]. \tag{4}$$

Definition 4. Location-function: For a linguistic label $Q = [B_i, B_j] \in S_n$, the location-function is defined as [2]:

$$l : S_n \rightarrow \mathbb{Z}^2$$

where

$$l([B_i, B_j]) = \left(-\sum_{s=1}^{i-1} \mu(B_s), \quad \sum_{s=j+1}^{n} \mu(B_s) \right) = (l_1(Q), l_2(Q)). \tag{5}$$

Example 1. In Definition 4, suppose $n = 7$. We can represent any $l([B_i, B_j])$ on a horizontal axis; for example, the value of $l([B_4, B_6])$ is shown in Figure 2.

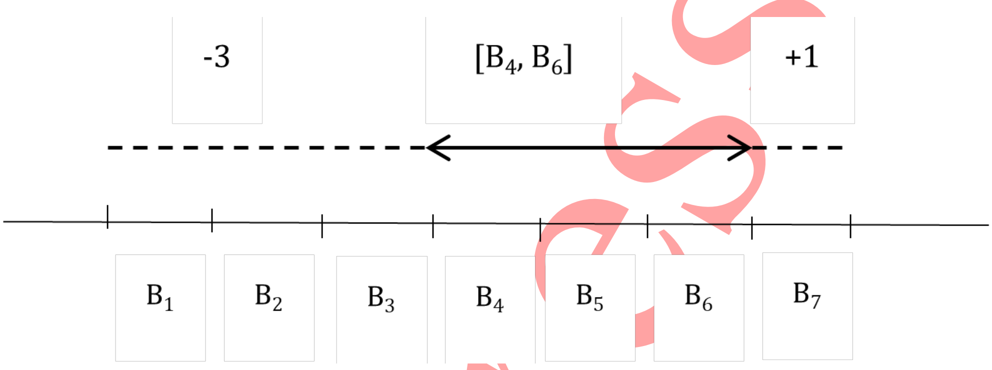

Figure 2: Location-function of linguistic labels.

Definition 5. Vector Location-function: If $A = (Q_1, Q_2, Q_3, \ldots, Q_m)$ represents an m-dimensional vector of linguistic labels in S_n, then its location-function is defined as a $2m$-dimensional vector [5]:

$$L(A) = (l_1(Q_1), l_2(Q_1), l_1(Q_2), l_2(Q_2), \ldots, l_1(Q_m), l_2(Q_m)). \tag{6}$$

Remark 1. According to the above definition, the Euclidean distance between two m- dimensional vectors $A_1 = (Q_{11}, Q_{12}, \ldots, Q_{1m})$ and $A_2 = (Q_{21}, Q_{22}, \ldots, Q_{2m})$ of linguistic labels in S_n can be calculated as follows:

$$d(A_1, A_2) = \sqrt{\sum_{j=1}^{m} [(l_1(Q_{1j}) - l_1(Q_{2j}))^2 + (l_2(Q_{1j}) - l_2(Q_{2j}))^2]}. \tag{7}$$

Numerous adaptations of the qualitative TOPSIS (Q-TOPSIS) methodology have leveraged the concept of Euclidean distance to discriminate among alternatives within a qualitative attribute space [2, 4]. Within this context, the relative proximity of each alternative to the positive ideal solution ($d^+(A_i)$), as well as to the negative ideal solution ($d^-(A_i)$), is systematically quantified by means of a weighted Euclidean metric. This approach facilitates the ranking of alternatives by translating interval-based linguistic evaluations into a normalized geometric framework, as expressed by the following equations:

$$d^+(A_i) = d(A_i, A^+) = \sqrt{\sum_{j=1}^{m} w_j \left[(l_1(Q_{ij}) - l_1(B_n))^2 + (l_2(Q_{ij}) - l_2(B_n))^2\right]}, \quad (8)$$

$$d^-(A_i) = d(A_i, A^-) = \sqrt{\sum_{j=1}^{m} w_j \left[(l_1(Q_{ij}) - l_1(B_1))^2 + (l_2(Q_{ij}) - l_2(B_1))^2\right]}, \quad (9)$$

where w_j is the weight of the j-th attribute, and vectors

$$A^+ = (B_n, B_n, \dots, B_n) \text{ and } A^- = (B_1, B_1, \dots, B_1),$$

are the m-dimensional positive ideal solution (PIS) and negative ideal solution (NIS) vectors, respectively. The relative closeness coefficient is then calculated as:

$$QCC_i = \frac{d^-(A_i)}{d^+(A_i) + d^-(A_i)}. \quad (10)$$

2.2 Extensions to the Basic Definitions of QAOM

To further enrich the QAOM framework, we introduce a set of supplementary definitions that build upon the foundational principles detailed previously. These refinements, inspired by and expanding upon the original work of Agell et al. [5], facilitate broader applications and greater analytical depth within qualitative decision environments.

Definition 6. Let Q be any linguistic label within the qualitative domain S_n. We define the cumulative normalizer $\Psi(Q)$ as a measure that aggregates the membership assignments $\mu(Q)$ across relevant linguistic terms. Specifically, if $\mu(Q)$ denotes the membership function associated with Q, then the cumulative normalizer $\Psi(Q)$ is formulated as:

$$\Psi(Q) = \text{Cumulative } \mu(Q), \qquad \frac{1}{n} \leq \Psi(Q) \leq 1, \quad (11)$$

For a basic label, $\Psi(B_i)$ is given by:

$$\Psi(B_i) = \mu(B_1) + \mu(B_2) + \cdots + \mu(B_i) = \frac{i}{n}. \quad (12)$$

For a non-basic label, $\Psi([B_i, B_j])$ is given by:

$$\Psi([B_i, B_j]) = \Psi(B_i) + \frac{\Psi(B_{j-i})}{2} = \frac{i}{n} + \frac{j-i}{2n} = \frac{i+j}{2n}. \quad (13)$$

It is also evident that $\Psi(B_1) = \frac{1}{n}$ and $\Psi(B_n) = 1$.

Example 2. Let $Q_1 = [B_3]$ and $Q_2 = [B_2, B_5]$ be two linguistic labels in S_7. We have:

$$\Psi(Q_1) = \Psi([B_3]) = \frac{3}{7},$$

and therefore,

$$\Psi(Q_2) = \Psi([B_2, B_5]) = \frac{5+2}{14} = \frac{1}{2}.$$

Definition 7. Let $Q = [B_i, B_j]$ be a non-basic linguistic label in S_n. The distance of Q from B_n (the ideal linguistic label) is defined as:

$$\begin{aligned} S(Q, B_n) &= S([B_i, B_j], [B_n]) \\ &= 1 - \Psi([B_j]) + \frac{1}{4}\left[\Psi([B_j]) - \Psi([B_i])\right] \\ &= 1 - \frac{j}{n} + \frac{j-i}{4n} = \frac{4n-3j-i}{4n}, \quad i, j = 1, 2, \ldots, n. \end{aligned} \tag{14}$$

In addition, if $Q = [B_i]$ is a basic linguistic label:

$$S(Q, B_n) = S([B_i], [B_n]) = S([B_i, B_i], B_n) = \frac{4n-3i-i}{4n} = \frac{n-i}{n}, \tag{15}$$

for $i = 1, 2, \ldots, n$.

It should be noted that $S(Q, B_n)$ represents the distance of the linguistic label Q from the ideal linguistic label B_n.

Example 3. For $n = 7$:

$$S([B_1, B_3], [B_7]) = 1 - \Psi([B_3]) + \frac{1}{4}(\Psi([B_3]) - \Psi([B_1])) = \frac{9}{14}.$$

The results of this distance measurement is illustrated in Figure 3.

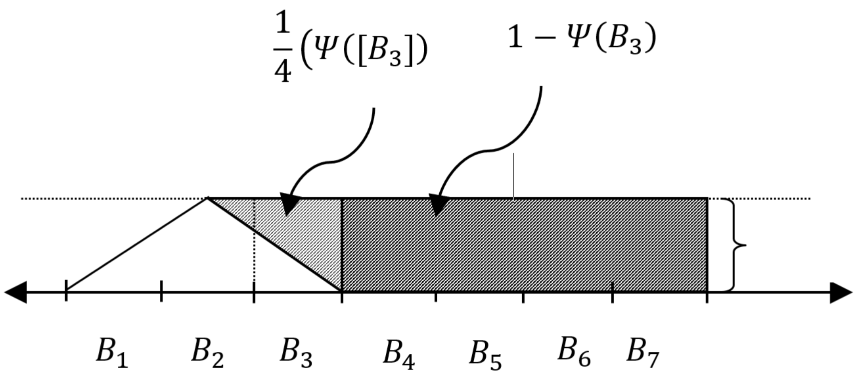

Figure 3: Distance of a linguistic label $[B_1, B_3]$ from the ideal linguistic label.

Example 4. If $Q_1 = [B_2, B_4]$ and $Q_2 = [B_1]$ are two linguistic labels in S_7, then:

$$S(Q_1, [B_7]) = S([B_2, B_4], [B_7]) = \frac{4 \times 7 - 3 \times 4 - 2}{4 \times 7} = \frac{1}{2},$$

and

$$S(Q_2, [B_7]) = \frac{7-1}{7} = \frac{6}{7}.$$

Definition 8. Let $Q = [B_i, B_j]$ be a linguistic label in S_n. The *rank* of Q is defined as:

$$\text{Rank}(Q) = 1 - S(Q, B_n) = 1 - \frac{4n - 3j - i}{4n}, \qquad i, j = 1, 2, \ldots, n. \tag{16}$$

This definition provides a consistent way to order QAOM labels.

Example 5. Assume $n = 7$. Therefore:

$$\begin{aligned}
\text{Rank}([B_1, B_3]) &= 1 - \frac{4 \times 7 - 3 \times 3 - 1}{4 \times 7} = \frac{10}{28}, \\
\text{Rank}([B_7]) &= 1 - \frac{4 \times 7 - 3 \times 7 - 7}{4 \times 7} = 1, \\
\text{Rank}([B_1]) &= 1 - \frac{4 \times 7 - 3 \times 1 - 1}{4 \times 7} = \frac{4}{28}, \\
\text{Rank}([B_1, B_7]) &= 1 - \frac{4 \times 7 - 3 \times 7 - 1}{4 \times 7} = \frac{22}{28}.
\end{aligned}$$

Definition 9. Let Q_1 and Q_2 be two linguistic labels in S_n. We define:

1. $Q_1 \succ Q_2$ if $\text{Rank}(Q_1) > \text{Rank}(Q_2)$,
2. $Q_1 \approx Q_2$ if $\text{Rank}(Q_1) = \text{Rank}(Q_2)$,
3. $Q_1 \succeq Q_2$ if $Q_1 \succ Q_2$ or $Q_1 \approx Q_2$, i.e., $\text{Rank}(Q_1) \geq \text{Rank}(Q_2)$.

It can be readily seen that the proposed ranking possesses the following properties.

Theorem 1. Let Q_1, Q_2, and Q_3 be arbitrary linguistic labels in S_n. Then, the following properties hold:

P1. $Q_1 \succeq Q_1$ (Reflexivity),

P2. If $Q_1 \succeq Q_2$ and $Q_2 \succeq Q_1$, then $Q_1 \approx Q_2$ (Antisymmetry),

P3. If $Q_1 \succeq Q_2$ and $Q_2 \succeq Q_3$, then $Q_1 \succeq Q_3$ (Transitivity).

Proof. P1. By Definition 9, for any $Q_1 \in S_n$, we have $\text{Rank}(Q_1) \geq \text{Rank}(Q_1)$, hence $Q_1 \succeq Q_1$.

P2. If $Q_1 \succeq Q_2$ and $Q_2 \succeq Q_1$, it means $\text{Rank}(Q_1) \geq \text{Rank}(Q_2)$ and $\text{Rank}(Q_2) \geq \text{Rank}(Q_1)$. Thus, $\text{Rank}(Q_1) = \text{Rank}(Q_2)$, i.e., $Q_1 \approx Q_2$.

P3. Since $Q_1 \succeq Q_2$ and $Q_2 \succeq Q_3$ implies $\text{Rank}(Q_1) \geq \text{Rank}(Q_2) \geq \text{Rank}(Q_3)$, therefore $\text{Rank}(Q_1) \geq \text{Rank}(Q_3)$, i.e., $Q_1 \succeq Q_3$. □

Remark 2. In practical multi-attribute decision-making (MADM) contexts, it is common to encounter attributes where lower (i.e., smaller) values correspond to preferable outcomes, typified by cost-like criteria such as fuel consumption in vehicle evaluation. To enable a coherent and equitable evaluation framework, these negatively oriented attributes must be appropriately normalized to allow direct comparison with benefit-type (positively-oriented) criteria. Nevertheless, the literature on QAOM-based MADM approaches has frequently underscored the lack of a unified normalization strategy for such cost-type attributes [2, 4]. Addressing this methodological gap, we introduce a systematic procedure that transforms all negatively oriented attributes into their positive counterparts. This transformation is intended to ensure analytic consistency and interpretative clarity within the QAOM-based MADM paradigm, thereby facilitating robust aggregation and ranking of alternatives irrespective of attribute orientation.

Definition 10. Let Q be a linguistic label in S_n. The inverse of Q is defined as follows:

- If $Q = B_i$ is a basic linguistic label:

$$Q^{-1} = [B_i]^{-1} = [B_{(n+1)-i}], \qquad i = 1, 2, \ldots, n. \tag{17}$$

- If $Q = [B_i, B_j]$ is a non-basic linguistic label:

$$Q^{-1} = [B_i, B_j]^{-1} = [B_{(n+1)-j}, B_{(n+1)-i}], \qquad i, j = 1, 2, \ldots, n. \tag{18}$$

where Q^{-1} denotes the inverse of Q.

Example 6. For $n = 7$, we have:

$$\begin{aligned}
[B_1]^{-1} &= [B_{8-1}] = [B_7], \\
[B_5]^{-1} &= [B_{8-5}] = [B_3], \\
[B_4]^{-1} &= [B_{8-4}] = [B_4], \\
[B_5, B_7]^{-1} &= [B_{8-7}, B_{8-5}] = [B_1, B_3], \\
[B_1, B_7]^{-1} &= [B_{8-7}, B_{8-1}] = [B_1, B_7].
\end{aligned}$$

2.3 Determining Weights of Attributes Based on the Entropy Concept

Assigning appropriate weights to attributes constitutes a pivotal step in MADM, as it inherently shapes the relative significance assigned to each alternative. In practice, decision-makers may occasionally lack definitive information or consensus regarding the underlying importance of individual attributes. In such circumstances, a robust weighting mechanism must either be inferred from available objective data or supplemented with expert-driven, subjective assessments. Even when direct expert estimations are accessible, it is prudent to corroborate or refine these subjective judgments using objective methodologies, in order to strengthen the reliability and internal coherence of the resulting weight structure.

To address the potential limitations associated with subjective weighting and to promote transparency and reproducibility in the analysis, the present study incorporates an entropy-based objective weighting procedure. Rooted in the foundational work of Shannon [39], the entropy method quantitatively evaluates the informational value and discriminative capacity of each attribute. Within the context of the proposed QAOM framework, this technique exploits the inherent ranking patterns revealed by the linguistic evaluations, providing an impartial estimate of attribute relevance. Consequently, the entropy-based scheme enhances the methodological rigor and robustness of the decision process, establishing a defensible weighting paradigm that complements subjective expert reasoning.

Definition 11. Let $D = [Q_{ij}]_{m\times n}$ be a decision matrix, where Q_{ij} is the linguistic label assigned to the jth attribute of the ith alternative. The entropy of the jth attribute, C_j, is defined as:

$$H(C_j) = -\frac{1}{\ln m}\sum_{i=1}^{m}[p_{ij}\ln p_{ij}], \tag{19}$$

where

$$p_{ij} = \frac{\text{Rank}(Q_{ij})}{\sum_{i=1}^{m}\text{Rank}(Q_{ij})}. \tag{20}$$

In addition, unreliability (deviation) is defined as:

$$d(C_j) = 1 - H(C_j). \tag{21}$$

Accordingly, the weight of the jth attribute, w_j, is computed as:

$$w_j = \frac{d_j}{\sum_{j=1}^{n} d_j}. \tag{22}$$

3 Proposed Q-TOPSIS Method

As discussed in the Introduction, this research aims to overcome persistent limitations in prior qualitative MAGDM methodologies, particularly those based on the QAOM framework. We introduce a unified, robust, and extensible MAGDM approach within the QAOM paradigm that systematically addresses these unresolved issues.

Among alternatives in multi-attribute analysis, the TOPSIS, originally developed by Hwang and Yoon [27], has emerged as an influential method for the ranking of alternatives according to their geometric proximity to both ideal and anti-ideal solutions. The conceptual clarity and demonstrated success of TOPSIS across various domains have contributed to its adoption as a reference standard in decision-analysis literature [11, 13, 15, 17, 24]. Building on this foundation, our study synthesizes the formal strengths of TOPSIS with the flexibility of QAOM reasoning in a novel MAGDM framework. Distinctively, the proposed methodology advances the state of the art by (i) introducing a consistent ranking procedure for interval-based qualitative assessments; (ii) natively incorporating an entropy-based, objective approach to attribute weighting that moves beyond ad hoc or externally determined weight assignments; (iii) establishing a systematic method for aggregating linguistic judgments from multiple experts, including provisions for divergent influence or expertise; and (iv) implementing an integrated normalization protocol for both positive (benefit-type) and negative (cost-type) attributes expressed in QAOM labels. Collectively, these innovations overcome the methodological barriers associated with prior QAOM-based MAGDM methods—most notably, the lack of natively integrated objective weighting, the absence of a unified aggregation technique for group judgments, and difficulties in managing negatively oriented criteria.

In summary, the enhanced QAOM-TOPSIS framework presented herein not only strengthens the analytical processing of qualitative and group information, but also extends the practical reach and reliability of qualitative decision-making models in complex environments.

Consider a MAGDM problem with m alternatives

$$A_1, A_2, \ldots, A_m,$$

and n attributes

$$C_1, C_2, \ldots, C_n,$$

where the value of each attribute is determined by k decision makers $D_1, D_2, \ldots, D_k$ with corresponding weights $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_K)$ in a QAOM environment. The following steps detail the Q-TOPSIS method:

Step 1: Construction of the Decision Matrix. Each decision-maker provides an evaluation for every attribute using an appropriate linguistic label from S_n. For cost-type (negative) attributes, the transformation described in Definition 10 is applied to obtain their suitable representation. Consequently, the decision matrices are assembled as $D^k = [Q_{ijk}]_{m\times n}$, where $Q_{ijk} \in S_n$ denotes the linguistic assessment assigned by the kth expert to the jth criterion of the ith alternative.

Step 2: Evaluating the Integrated Decision Matrix. After forming the matrices $D^k = [Q_{ijk}]_{m\times n}$, the rank of each linguistic label is computed using Definition 8, yielding matrices $AD^k = [R(Q_{ijk})]_{m\times n}$, where $R(Q_{ijk}) = \text{Rank}(Q_{ijk})$. By integrating the evaluations of all decision makers, the aggregated decision matrix is calculated as follows:

$$R(Q_{ij}) = \sum_{k=1}^{K} \pi_k\, R(Q_{ijk}), \qquad i = 1, 2, \ldots, m, \;\; j = 1, 2, \ldots, n, \tag{23}$$

where,

$$\sum_{k=1}^{K} \pi_k = 1.$$

The integrated decision matrix is then $AD = [R(Q_{ij})]_{m \times n}$.

Step 3: Constructing the Weighted Decision Matrix. Using the method described in Definition 11, the weight vector can be determined. Let $\mathbf{W} = (w_1, w_2, \ldots, w_n)$ denote the vector of attribute weights, where w_j is the weight of the jth attribute. The weighted decision matrix, denoted by $\mathbf{AD}_V = [v_{ij}]_{m \times n}$, is calculated as follows:

$$v_{ij} = w_j \cdot R(Q_{ij}), \qquad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n. \tag{24}$$

Step 4: Determining PIS and NIS. Let $\mathbf{v}^+ = (v_1^+, v_2^+, \ldots, v_n^+)$ be the Positive Ideal Solution (PIS) and $\mathbf{v}^- = (v_1^-, v_2^-, \ldots, v_n^-)$ the Negative Ideal Solution (NIS), defined as:

$$v_j^+ = \max_i \{v_{ij}\}, \qquad j = 1, 2, \ldots, n, \tag{25}$$

$$v_j^- = \min_i \{v_{ij}\}, \qquad j = 1, 2, \ldots, n. \tag{26}$$

Step 5: Calculating the Distance of Alternatives from PIS and NIS. The distance of the ith alternative from the PIS and NIS is calculated as:

$$d^+(A_i) = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^+)^2}, \qquad i = 1, 2, \ldots, m, \tag{27}$$

$$d^-(A_i) = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^-)^2}, \qquad i = 1, 2, \ldots, m, \tag{28}$$

where $d^+(A_i)$ and $d^-(A_i)$ are the distances of the ith alternative from the PIS and NIS, respectively.

Step 6: Calculating the Qualitative Closeness Coefficient of Alternatives. Finally, the qualitative closeness coefficient for each alternative is calculated as:

$$CC(A_i) = \frac{d^-(A_i)}{d^-(A_i) + d^+(A_i)}, \qquad i = 1, 2, \ldots, m. \tag{29}$$

In this equation, $CC(A_i)$ represents the qualitative closeness coefficient of the ith alternative. The alternatives are ranked in descending order of $CC(A_i)$ for $i = 1, 2, \ldots, m$. Therefore, A^* is the best alternative if and only if $CC(A^*) = \max_i\{CC(A_i)\}$.

4 Numerical Examples and Validation

In this section, practical Examples 7 and 8 are used to investigate the potential and efficiency of the proposed method in real-world decision-making problems. Then, in Example 9, we demonstrate the advantages of the proposed method over previous approaches.

Example 7. In [2], a MADM scenario was analyzed to determine the most suitable wind farm location in Catalonia, northeastern Spain. This problem involved a single decision-maker who assigned equal weights to all attributes. The TOPSIS method was employed within the QAOM framework to evaluate seven alternatives against nine attributes encompassing economic, social, environmental, and technical factors. Only basic linguistic labels from class S_7 were utilized. In the following, we re-evaluate this problem using the proposed method.

Step 1: Constructing the Decision Matrix. In this step, we constructed the decision matrix using the alternatives and attributes defined in Table 1. The summarized decision matrix is presented in Table 3. Definition 10 was used to convert the negative attributes C_5, C_6, and C_8 into positive ones. The normalized decision matrix is shown in Table 4. For example, if we consider S_7 as the reference scale, in Table 3 we have $C_{1,5} = \text{L} = B_2$, so based on Definition 10, $[C_{1,5}]^{-1} = [B_2]^{-1} = [B_{(7+1)-2}] = B_6$ (see Table 4). Similarly, all elements in C_6 and C_8 are converted as seen in Table 4.

Table 1: Attributes and alternatives in Example 7.

Attribute	Code	Alternative description	Code
Land owner's income (+)	C_1	CB-Pre: Coma Bertran preliminary project.	A_1
Economic activity tax (+)	C_2	CB: Coma Bertran project.	A_2
Construction tax (+)	C_3	ST: Serra del Tallat project.	A_3
Number of jobs (+)	C_4	CBST: Combination of CB and ST projects.	A_4
Visual impact (−)	C_5	L: Based on CB and ST projects, considers windmills located at least 1.5 km from population centres and tourist attractions (Santuari del Tallat).	A_5
Deforestation (−)	C_6	R: Attempts to move windmills away from population centres presenting higher resistance (Senan and Montblanc).	A_6
Avoided CO_2 emissions (+)	C_7	NP: Possibility of constructing no project at all.	A_7
Noise (−)	C_8	–	–
Installed capacity (+)	C_9	–	–

Table 2: Linguistic terms and corresponding linguistic labels in Example 7.

Linguistic label	Linguistic term
B_1	Very Low (VL)
B_2	Low (L)
B_3	Medium Low (ML)
B_4	Medium (M)
B_5	Medium High (MH)
B_6	High (H)
B_7	Very High (VH)

Table 3: Decision matrix based on linguistic terms in Example 7.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A1	ML	L	ML	ML	L	MH	L	MH	L
A2	L	L	ML	L	L	MH	L	VH	L
A3	MH	MH	MH	H	MH	M	H	H	MH
A4	VH	VH	VH	VH	VH	VH	VH	VH	VH
A5	M	M	H	MH	M	L	MH	VH	M
A6	M	M	ML	MH	ML	L	M	MH	M
A7	VL	VL	VL	VL	VL	VL	VL	VL	VL

Table 4: Normalized decision matrix based on linguistic labels in Example 7.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A1	B_3	B_2	B_3	B_3	B_6	B_3	B_2	B_3	B_2
A2	B_2	B_2	B_3	B_2	B_6	B_3	B_2	B_1	B_2
A3	B_5	B_5	B_5	B_6	B_3	B_4	B_6	B_2	B_5
A4	B_7	B_7	B_7	B_7	B_1	B_1	B_7	B_1	B_7
A5	B_4	B_4	B_6	B_5	B_4	B_6	B_5	B_1	B_4
A6	B_4	B_4	B_3	B_5	B_5	B_6	B_4	B_3	B_4
A7	B_1	B_1	B_1	B_1	B_7	B_7	B_1	B_7	B_1

Step 2: Evaluating the integrated decision matrix. Using Definition 8, we calculated the rank of each element in the decision matrix. Since only a single expert is involved in this example, integration of multiple expert opinions was not required.

Step 3: Constructing the weighted decision matrix. Following Afsordegan et al. [2], we assigned equal weights to all attributes. The normalized weighted decision matrix, constructed using Equation (24), is presented in Table 5.

Step 4: Determining PIS and NIS. In this step, the Positive Ideal Solution (PIS, v^+) and the Negative Ideal Solution (NIS, v^-) are determined using Equations (25)– (26). Specifically, for this case:

$$v^+ = \left(\max_i \{v_{ij}\}\right) = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1),$$
$$v^- = \left(\min_i \{v_{ij}\}\right) = (0.143,\ 0.143,\ 0.143,\ 0.143,\ 0.143,\ 0.143,\ 0.143,\ 0.143,\ 0.143).$$

These reference values are also given in Table 5.

Table 5: Normalized weighted decision matrix in Example 7.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.429	0.286	0.429	0.429	0.857	0.429	0.286	0.429	0.286
A_2	0.286	0.286	0.429	0.286	0.857	0.429	0.286	0.143	0.286
A_3	0.714	0.714	0.714	0.857	0.429	0.571	0.857	0.286	0.714
A_4	1.000	1.000	1.000	1.000	0.143	0.143	1.000	0.143	1.000
A_5	0.571	0.571	0.857	0.714	0.571	0.857	0.714	0.143	0.571
A_6	0.571	0.571	0.429	0.714	0.714	0.857	0.571	0.429	0.571
A_7	0.143	0.143	0.143	0.143	1.000	1.000	0.143	1.000	0.143
A^+	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
A^-	0.143	0.143	0.143	0.143	0.143	0.143	0.143	0.143	0.143

Step 5: Calculating the distance of each alternative from PIS and NIS. The distances of each alternative from the PIS and NIS are calculated using Equations (27)–(28). The results are summarized in Table 6.

Step 6: Calculating the qualitative closeness coefficient of alternatives. The relative qualitative closeness coefficient for each alternative is calculated using Equation (29), as shown in Table 7.

To further assess the effectiveness of the proposed approach, the resulting rankings were compared with those obtained using the method by Afsordegan et al. [2]. As shown in Table 8, the proposed method produced a different ranking sequence, most notably in the positions of A_5 and A_6, thereby demonstrating the added value and differentiating mechanism of the new approach.

Table 6: Distance of each alternative from PIS (d^+) and NIS (d^-) in Example 7.

Alternative	$d^+(A_i)$	$d^-(A_i)$
A_1	1.784	0.990
A_2	1.990	0.881
A_3	1.178	1.616
A_4	1.485	2.100
A_5	1.450	1.552
A_6	1.784	1.436
A_7	1.990	1.485

Table 7: Relative closeness coefficient and ranking of alternatives in Example 7.

Alternative	Closeness Coefficient $CC(A_i)$	Rank
A_1	0.357	6
A_2	0.307	7
A_3	0.578	2
A_4	0.586	1
A_5	0.517	4
A_6	0.534	3
A_7	0.414	5

Table 8: Comparison of ranking results in Example 7.

Method	Ranking
Afsordegan et al. (2015)	$A_4 \succ A_3 \succ A_5 \succ A_6 \succ A_7 \succ A_1 \succ A_2$
Proposed method	$A_4 \succ A_3 \succ A_6 \succ A_5 \succ A_7 \succ A_1 \succ A_2$

Example 8. In this practical example, we apply the proposed method to the green energy selection problem introduced by Afsordegan et al. [4]. This Multiple Attribute Group Decision-Making case involves nine attributes and seven alternatives, as detailed in Table 9. Attribute values were elicited from a panel of three experts. Attribute weights (determined via AHP by Afsordegan et al. [4]) are also shown in Table 9. In this application, all attributes are considered benefit-type.

Step 1: Constructing the decision matrix. The decision matrix is formed based on group expert judgments, using linguistic terms (see Table 10).

Table 9: Attributes, weights, and alternatives in Example 8.

Attribute	Description	Weight	Alternative	Code
C1	Efficiency	0.0900	Conventional	A1
C2	Exergy (rational efficiency)	0.1000	Nuclear	A2
C3	Investment cost	0.1000	Solar	A3
C4	Operation and maintenance cost	0.1100	Wind	A4
C5	NO_X emission	0.1300	Hydraulic	A5
C6	CO_2 emission	0.1500	Biomass	A6
C7	Land use	0.1100	Combined Heat & Power	A7
C8	Social acceptability	0.0900	–	–
C9	Job creation	0.1200	–	–

Table 10: Decision matrix based on linguistic terms in Example 8.

		C1	C2	C3	C4	C5	C6	C7	C8	C9
E1	A1	H	H	MH	MH	VL	VL	L	ML	MH
	A2	VH	M	VL	VH	ML	ML	ML	L	H
	A3	M	M	M	M	VH	H	VH	H	M
	A4	ML	MH	H	H	H	VH	VH	VH	M
	A5	MH	H	MH	M	ML	L	ML	M	H
	A6	M	MH	M	M	H	H	MH	H	H
	A7	M	MH	M	ML	M	M	MH	H	MH
E2	A1	VH	MH	H	M	VL	ML	VL	L	H
	A2	H	VH	ML	VH	ML	ML	VL	ML	H
	A3	ML	M	MH	M	VH	H	H	H	MH
	A4	M	MH	H	H	H	VH	H	VH	M
	A5	M	H	MH	M	ML	L	ML	M	MH
	A6	M	M	MH	M	H	H	MH	H	H
	A7	MH	M	M	ML	M	M	H	MH	MH
E3	A1	VH	VH	MH	MH	ML	ML	L	ML	MH
	A2	VH	VH	VL	VH	L	ML	ML	ML	H
	A3	M	M	M	M	H	H	H	H	M
	A4	L	MH	H	VH	VH	VH	H	VH	M
	A5	H	H	MH	M	ML	L	ML	M	H
	A6	M	MH	M	MH	H	H	MH	H	MH
	A7	MH	M	MH	M	M	M	MH	H	MH

Step 2: Evaluating the integrated decision matrix. The rank of each linguistic label is determined using Definition 8 (see Table 11); the integrated decision matrix (per Equation (23)) is shown in Table 12.

Step 3: Constructing the integrated weighted decision matrix. Using the weights above, the weighted matrix is computed as shown in Table 13.

Table 11: Ranks of linguistic labels in Example 8 (excerpt).

		C1	C2	C3	C4	C5	C6	C7	C8	C9
E1	A1	0.8571	0.8571	0.7143	0.7143	0.1429	0.1429	0.2857	0.4286	0.7143
	A2	1.0000	0.5714	0.1429	1.0000	0.4286	0.4286	0.4286	0.2857	0.8571
	A3	0.5714	0.5714	0.5714	0.5714	1.0000	0.8571	1.0000	0.8571	0.5714
	A4	0.4286	0.7143	0.8571	0.8571	0.8571	1.0000	1.0000	1.0000	0.5714
	A5	0.7143	0.8571	0.7143	0.5714	0.4286	0.2857	0.4286	0.5714	0.8571
	A6	0.5714	0.7143	0.5714	0.5714	0.8571	0.8571	0.7143	0.8571	0.8571
	A7	0.5714	0.7143	0.5714	0.4286	0.5714	0.5714	0.7143	0.8571	0.7143

Table 12: Integrated decision matrix in Example 8.

	C1	C2	C3	C4	C5	C6	C7	C8	C9
A1	0.9524	0.8571	0.7619	0.6667	0.2381	0.3333	0.2381	0.3810	0.7619
A2	0.9524	0.8571	0.2381	1.0000	0.3810	0.4286	0.3333	0.3810	0.8571
A3	0.5238	0.5714	0.6190	0.5714	0.9524	0.8571	0.9048	0.8571	0.6190
A4	0.4286	0.7143	0.8571	0.9048	0.9048	1.0000	0.9048	1.0000	0.5714
A5	0.7143	0.8571	0.7143	0.5714	0.4286	0.2857	0.4286	0.5714	0.8095
A6	0.5714	0.6667	0.6190	0.6190	0.8571	0.8571	0.7143	0.8571	0.8095
A7	0.6667	0.6190	0.6190	0.4762	0.5714	0.5714	0.7619	0.8095	0.7143
w_j	0.0900	0.1000	0.1000	0.1100	0.1300	0.1500	0.1100	0.0900	0.1200

Table 13: Integrated weighted decision matrix in Example 8.

	C1	C2	C3	C4	C5	C6	C7	C8	C9
A1	0.0857	0.0857	0.0762	0.0733	0.0310	0.0500	0.0262	0.0343	0.0914
A2	0.0857	0.0857	0.0238	0.1100	0.0495	0.0643	0.0367	0.0343	0.1029
A3	0.0471	0.0571	0.0619	0.0629	0.1238	0.1286	0.0995	0.0771	0.0743
A4	0.0386	0.0714	0.0857	0.0995	0.1176	0.1500	0.0995	0.0900	0.0686
A5	0.0643	0.0857	0.0714	0.0629	0.0557	0.0429	0.0471	0.0514	0.0971
A6	0.0514	0.0667	0.0619	0.0681	0.1114	0.1286	0.0786	0.0771	0.0971
A7	0.0600	0.0619	0.0619	0.0524	0.0743	0.0857	0.0838	0.0729	0.0857

Step 4: Determining PIS and NIS. The Positive Ideal Solution (PIS, v^+) and Negative Ideal Solution (NIS, v^-) are determined using Equations (25) and (26):

$$v^+ = (0.0860, 0.0857, 0.0857, 0.1100, 0.1238, 0.1500, 0.0995, 0.0900, 0.1029),$$
$$v^- = (0.0386, 0.0571, 0.0238, 0.0524, 0.0310, 0.0429, 0.0262, 0.0343, 0.0686).$$

Step 5: Calculating the distance of alternatives from PIS and NIS. Using Equations (27) and (28), Table 14 presents the computed distances.

Step 6: Calculating the qualitative closeness coefficient of alternatives. The relative qualitative closeness coefficient is calculated (Equation (29)), producing the results in Table 15. This yields the final ranking: $A_4 \succ A_6 \succ A_3 \succ A_7 \succ A_2 \succ A_5 \succ A_1$.

This ranking exactly matches that obtained using the method by Afsordegan et al. [4], as shown in Table 16.

Table 14: Distances of alternatives from PIS and NIS in Example 8.

Alternative	$d^+(A_i)$	$d^-(A_i)$
A1	0.1690	0.0820
A2	0.1540	0.0920
A3	0.0810	0.1580
A4	0.0610	0.1840
A5	0.1510	0.0780
A6	0.0710	0.1450
A7	0.1120	0.1040

Table 15: Relative qualitative closeness coefficient and ranking of alternatives in Example 8.

Alternative	$CC(A_i)$	Rank
A1	0.3270	7
A2	0.3740	5
A3	0.6610	3
A4	0.7500	1
A5	0.3400	6
A6	0.6710	2
A7	0.4810	4

Table 16: Comparison of ranking results in Example 8.

Method	Ranking
Afsordegan et al. [4]	$A_4 \succ A_6 \succ A_3 \succ A_7 \succ A_2 \succ A_5 \succ A_1$
Proposed method	$A_4 \succ A_6 \succ A_3 \succ A_7 \succ A_2 \succ A_5 \succ A_1$

Example 9. Consider a Multi-Attribute Group Decision-Making problem for selecting the most suitable supplier for a manufacturer. In this scenario, there are four suppliers (S_1, S_2, S_3, and S_4) and four attributes: cost (C_1), services (C_2), quality (C_3), and on-time delivery (C_4), where cost is a negative attribute and the others are positive. The attribute weights are unknown and must be estimated. Attribute values are assessed by three decision makers (D_1, D_2, and D_3), with a weights vector $\boldsymbol{\pi} = (0.2, 0.4, 0.4)$. The steps of the proposed method are as follows:

Step 1: Constructing the decision matrix. According to the proposed method, the decision matrix is initially formed based on the decision makers' judgments, as shown in Table 17. Note that C_1 is a negative attribute. Therefore, using Definition 10, the values of the negative attribute are inverted, as presented in Table 18.

Table 17: Decision matrix based on linguistic terms in Example 9.

DMs	Alternative	C1 (-)	C2	C3	C4
D1	S_1	[B1, B2]	B5	[B6, B7]	[B6, B7]
	S_2	B1	[B6, B7]	B6	[B6, B7]
	S_3	[B1, B2]	[B6, B7]	B7	B4
	S_4	[B2, B4]	[B6, B7]	[B6, B7]	B5
D2	S_1	[B1, B2]	[B6, B7]	B7	[B4, B6]
	S_2	[B2, B4]	B5	[B6, B7]	[B3, B5]
	S_3	[B1, B2]	B7	[B4, B6]	[B3, B5]
	S_4	[B2, B4]	[B6, B7]	B6	B7
D3	S_1	B2	[B4, B6]	[B4, B6]	[B6, B7]
	S_2	[B2, B4]	B6	[B6, B7]	B7
	S_3	[B1, B2]	[B4, B6]	[B6, B7]	[B6, B7]
	S_4	[B1, B2]	B7	[B4, B6]	[B4, B6]

Step 2: Evaluating the normalized decision matrix. In this step, the negative attribute is converted to a positive one according to Definition 10. The resulting matrix is shown in Table 18.

Table 18: Normalized attributes in Example 9.

DMs	Alternative	C1	C2	C3	C4
D1	S_1	[B6, B7]	B5	[B6, B7]	[B6, B7]
	S_2	B7	[B6, B7]	B6	[B6, B7]
	S_3	[B6, B7]	[B6, B7]	B7	B4
	S_4	[B4, B6]	[B6, B7]	[B6, B7]	B5
D2	S_1	[B6, B7]	[B6, B7]	B7	[B4, B6]
	S_2	[B4, B6]	B5	[B6, B7]	[B3, B5]
	S_3	[B6, B7]	B7	[B4, B6]	[B3, B5]
	S_4	[B4, B6]	[B6, B7]	B6	B7
D3	S_1	B6	[B4, B6]	[B4, B6]	[B6, B7]
	S_2	[B4, B6]	B6	[B6, B7]	B7
	S_3	[B6, B7]	[B4, B6]	[B6, B7]	[B6, B7]
	S_4	[B6, B7]	B7	[B4, B6]	[B4, B6]

Step 3: Evaluating the integrated decision matrix. The rank of each linguistic label is determined using Definition 8, as shown in Table 19. Then, the aggregated values for each attribute are computed using Equation (24), and the integrated matrix is displayed in Table 20.

Table 19: Rank of attributes in Example 9.

DMs	Alternative	C1	C2	C3	C4
D1	S_1	0.9643	0.7143	0.9643	0.9643
	S_2	1	0.9643	0.8571	0.9643
	S_3	0.9643	0.9643	1	0.5714
	S_4	0.7857	0.9643	0.9643	0.7143
D2	S_1	0.9643	0.9643	1	0.7857
	S_2	0.7857	0.7143	0.9643	0.6429
	S_3	0.9643	1	0.7857	0.6429
	S_4	0.7857	0.9643	0.8571	1
D3	S_1	0.8571	0.7857	0.7857	0.9643
	S_2	0.7857	0.8571	0.9643	1
	S_3	0.9643	0.7857	0.9643	0.9643
	S_4	0.9643	1	0.7857	0.7857

Table 20: Integrated matrix in Example 9.

Alternative	C1	C2	C3	C4
S_1	0.921	0.843	0.907	0.893
S_2	0.829	0.821	0.943	0.850
S_3	0.964	0.907	0.900	0.757
S_4	0.857	0.979	0.850	0.857
w_j	0.267	0.357	0.102	0.273

Step 4: Constructing the weighted decision matrix. Using Definition 11 and Equations (18)–(20), the weights of the attributes are determined (see the last row of Table 20). The weighted decision matrix is displayed in Table 21.

Step 5: Determining PIS and NIS. Based on Equations (22)–(23), the Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS) are as follows:

$$\mathbf{v}^{+} = \left(\max_{i}\{v_{ij}\}\right) = (0.257, 0.349, 0.096, 0.244),$$

$$\mathbf{v}^{-} = \left(\min_{i}\{v_{ij}\}\right) = (0.221, 0.293, 0.087, 0.207).$$

Table 21: Weighted decision matrix in Example 9.

Alternative	C1	C2	C3	C4
S_1	0.246	0.301	0.093	0.244
S_2	0.221	0.293	0.096	0.232
S_3	0.257	0.324	0.092	0.207
S_4	0.229	0.349	0.087	0.234

The corresponding distances to PIS and NIS are shown in Table 22.

Table 22: Distance of alternatives from PIS and NIS in Example 9.

Distance from PIS	Value	Distance from NIS	Value
$d^+(S_1)$	0.0499	$d^-(S_1)$	0.0456
$d^+(S_2)$	0.0678	$d^-(S_2)$	0.0271
$d^+(S_3)$	0.0452	$d^-(S_3)$	0.0477
$d^+(S_4)$	0.0317	$d^-(S_4)$	0.0629

Step 6: Calculating the qualitative closeness coefficient of alternatives. Finally, Equation (29) is used to compute the relative qualitative closeness coefficient for each alternative, as presented in Table 23. The alternatives are thus ranked as $S_4 \succ S_3 \succ S_1 \succ S_2$.

Table 23: Relative qualitative closeness coefficient of alternatives in Example 9.

Relative closeness coefficient	Value	**Rank**
$CC(S_1)$	0.477	3
$CC(S_2)$	0.286	4
$CC(S_3)$	0.514	2
$CC(S_4)$	0.665	1

5 Discussion

In this study, we reassessed several Multi-Attribute Decision-Making scenarios, originally presented by Afsordegan et al., by applying our proposed method to enhance decision quality across three concrete examples.

The first example involved the selection of an optimal location for a wind farm in Catalonia, previously analyzed via the TOPSIS method integrated with QAOM [2]. Our re-evaluation yielded valuable insights, particularly in the ranking of alternatives. The alternative A4, which combines aspects from projects CB and ST, consistently emerged as the most favorable option (see Table 8). This result is in agreement with the original findings, underscoring the soundness and reliability of our method's ranking system when applied to established MADM problems within the QAOM context. The correspondence with Afsordegan's [2] results validates the foundational effectiveness of our approach and highlights its applicability to both environmental and urban planning domains.

In the second example, we applied the proposed method to select among sustainable energy alternatives, a case previously described by Afsordegan et al. [4]. By constructing a decision matrix based on expert opinions and applying our evaluation framework, we identified the optimal energy alternative among seven options characterized by nine attributes. Both our method and that of Afsordegan et al. selected alternative A4 as the superior choice (Table 16), further reinforcing the reliability of our approach for MAGDM instances where attribute weights are predefined or subjectively assigned. The integration of qualitative assessments in the decision matrix proved essential, accommodating the complex, multifaceted nature of sustainability evaluation, and demonstrating the framework's applicability to real-world scenarios where expert judgment is indispensable.

The third example addressed a more complex MAGDM problem: supplier selection among four candidates based on four attributes, including a negative attribute (cost), and requiring the objective determination of attribute weights from qualitative data, with multiple decision-makers of varying influence. Our systematic approach included: (1) constructing decision matrices from expert assessments (Table 17), (2) normalization of attributes, including cost (Table 18), (3) ranking of linguistic labels (Table 19), (4) determining objective attribute weights with the integrated entropy method (Table 20), (5) building the weighted decision matrix (Table 21), and (6) calculating distances to PIS and NIS (Table 22). This comprehensive workflow enabled us to effectively address challenges that previous QAOM-based methodologies could not resolve. The final ranking found S4 to be the most preferred supplier (Table 23), showcasing the proposed method's capacity to handle intricate, realistic decision situations.

The successful handling of Example 3—featuring objective weight calculation, variable decision-maker influence, and negative attributes—highlights the significant methodological advances our approach delivers over prior QAOM-based methods. To explicitly delineate these improvements, Table 24 provides a detailed, feature-by-feature comparison.

As Table 24 illustrates, although the foundational work of [2] established QAOM for MADM and [4] extended its application to basic MAGDM, these earlier solutions remain limited by their inability to: (i) normalize and process negative (cost-type) attributes fully within the QAOM schema; (ii) incorporate varying influence degrees among group members; and (iii) determine attribute weights objectively from qualitative data using a criterion internal to QAOM.

Table 24: Feature Comparison of QAOM-based Decision-Making Methods.

Feature	[2]	[4]	Proposed Method
Handles MADM	Yes	Yes	Yes
Handles MAGDM	No	Yes	Yes
Handles Positive & Negative Attributes in QAOM	No	No	Yes
Incorporates Decision Maker Weights	No	No	Yes
Determines Objective Attribute Weights in QAOM	No	No	Yes

In contrast, our proposed method overcomes all these limitations. The comprehensive solution to Example 9 explicitly demonstrates the practical value of these methodological enhancements. The objective calculation of attribute weights through entropy directly from qualitative input (Table 20), normalization of the negative cost attribute (C_1, Table 18), and the aggregation of weighted expert assessments, reveal a robustness, flexibility, and generality in complex MAGDM contexts not previously available in QAOM-based decision frameworks. This integrated approach enables a more robust, objective, and adaptable analysis of MAGDM problems characterized by qualitative uncertainty, divergent expert opinion, and mixed attribute types, effectively addressing the substantial limitations identified in the Introduction.

6 Conclusion

Qualitative Absolute Order-of-Magnitude (QAOM) is a widely used qualitative reasoning method in artificial intelligence for analyzing multi-attribute decision-making problems characterized by varying degrees of data precision. QAOM allows decision-makers to express subjective judgments through linguistic labels across multiple scales. However, its practical application—particularly in complex, real-world scenarios—has been hindered by several crucial limitations: The absence of an objective, integrated procedure for attribute weighting, the lack of a robust mechanism for aggregating weighted expert opinions in group decision contexts, and challenges in properly normalizing negative (cost-type) attributes within the QAOM framework. This study addresses these limitations by introducing a novel, integrated MAGDM framework fully embedded within the QAOM environment. The proposed approach incorporates several innovations: (i) a mathematically rigorous ranking system for linguistic labels that supports consistent comparison and appropriately handles negative attributes; (ii) an entropy-based method for objectively determining attribute weights directly from qualitative data; and (iii) a structured procedure for aggregating expert judgment, incorporating recognition of differing expertise through decision-maker weights. A systematic normalization procedure for negative attributes is also developed to enhance the framework's completeness. The effectiveness of the proposed framework was validated through re-evaluating practical examples from the literature ([2] and [4]). Results on these benchmark cases (see Tables 8 and 16) aligned with prior findings, confirming the method's correctness in standard scenarios. More importantly, in complex MAGDM settings, the proposed method revealed distinct advantages: as discussed in Section 4 (Table 24), it offers intrinsic mechanisms for objective attribute weighting, systematic treatment of negative attributes, and integration of heterogeneous expert influence capabilities absent from prior QAOM-based methods. The third illustrative example further highlighted how these innovations enable QAOM to resolve particularly challenging cases, significantly improving its applicability and robustness. In summary, the proposed MAGDM framework considerably enhances the flexibility and analytical capability of QAOM-based decision analysis. By offering systematic solutions for objective attribute weighting, expert opinion aggregation, and normalization of both positive and negative attributes within the QAOM environment, this research advances the methodological foundation for tackling complex group decision-making problems under linguistic uncertainty.

7 Limitations and Future Research Directions

Despite the demonstrated merits and advances of the proposed QAOM-based MAGDM approach, several aspects remain open for further exploration. First, the scope of quantitative comparison with other methods is inherently restricted, as the literature on QAOM-based decision frameworks remains relatively limited. Future studies can address this by establishing more comprehensive benchmarks or applying the method to a wider range of established decision problems. Second, while the present framework has shown its feasibility for moderately sized group decision scenarios, additional work is needed to ensure computational scalability and effectiveness in larger-scale or more intricate contexts. Integrating the proposed approach with other established MCDM techniques could further test and possibly enhance its adaptability and robustness. Third, a systematic evaluation of the method's sensitivity to changes in expert assessments and model parameters—such as through sensitivity analysis—was beyond the scope of this article, but represents an important area for building confidence in its practical deployment. Finally, the versatility of the framework could be more convincingly demonstrated through application to a broader variety of real-world case studies spanning diverse domains. Addressing these limitations can help reinforce the method's strengths and broaden its impact in supporting complex group decision-making under uncertainty.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

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Competing Interests

The authors declare that they have no competing interests relevant to the content of this paper.

Authors' Contributions

Conceptualization: Ali Dehghani Filabadi and Hossein Nahid Titkanlue. Writing – Original Draft Preparation: Ali Dehghani Filabadi. Validation: Hossein Nahid Titkanlue and Ali Dehghani Filabadi. Writing – Review & Editing: Hossein Nahid Titkanlue and Ali Dehghani Filabadi. All authors have read and agreed to the published version of the manuscript.

References

[1] Abouhawwash, M., Jameel, M., Askar, S.S. (2023). "Multi-criteria decision making model for analysis hydrogen production for economic feasibility and sustainable energy", *Multicriteria Algorithms With Applications*, 1, 31-41. https://doi.org/10.61356/j.mawa.2023.16161

[2] Afsordegan, A. (2015). "A contribution to multi-criteria decision making in sustainable energy management based on fuzzy and qualitative reasoning", *Doctoral dissertation, Universitat Politècnica de Catalunya*. https://doi.org/10.5821/dissertation-2117-96001

[3] Afsordegan, A., Sánchez, M., Agell, N., Aguado, J.C., Gamboa, G. (2016a). "Absolute order-of-magnitude reasoning applied to a social multi-criteria evaluation framework", *Journal of Experimental & Theoretical Artificial Intelligence*, 28(1-2), 261-274. https://doi.org/10.1080/0952813X.2015.1024489

[4] Afsordegan, A., Sánchez, M., Agell, N., Zahedi, S., Cremades, L.V. (2016b). "Decision making under uncertainty using a qualitative TOPSIS method for selecting sustainable energy alternatives", *International Journal of Environmental Science and Technology*, 13(6), 1419-1432. https://doi.org/10.1007/s13762-016-0982-7

[5] Agell, N., Sánchez, M., Prats, F., Roselló, L. (2012). "Ranking multi-attribute alternatives on the basis of linguistic labels in group decisions", *Information Sciences*, 209, 49-60. https://doi.org/10.1016/j.ins.2012.05.005

[6] Aggarwal, M. (2020). "Entropy-based subjective choice models", *IEEE Transactions on Knowledge and Data Engineering*, 32(5), 2164-2175. https://doi.org/10.1109/TKDE.2020.3012061

[7] Akram, M., Ali, G., Alcantud, J. C. R., Sitara, W. (2022). "Group decision-making with Fermatean fuzzy soft expert knowledge". *Artificial Intelligence Review*, 55(7), 5349–5389. https://doi.org/10.1007/s10462-021-10119-8

[8] Ali, A., Muthuswamy, M. (2023). "Neutrosophic multi-criteria decision-making framework for sustainable evaluation of power production systems in renewable energy sources", *Sustainable Machine Intelligence Journal*, 4(3), 1-10. https://doi.org/10.61185/SMIJ.2023.44103

[9] Ali, R., Rahman, K., Muhammad, J. (2024). "Complex Fermatean fuzzy models and their algebraic aggregation operators in decision-making: A case study on COVID-19 vaccine selection", *Journal of Operational and Strategic Analytics*, 2(2), 119-135. https://doi.org/10.56578/josa020205

[10] Al-Saeed, S., AbdelAziz, N.M. (2023). "Integrated neutrosophic best-worst method for comprehensive analysis and ranking of flood risks: A case study approach from Aswan, Egypt." *Neutrosophic Systems with Applications*, 5(1), 10-26. https://doi.org/10.61356/j.nswa.2023.24

[11] Ayala Ayala, L.R., Hernández Ramos, E.L., Contento Correa, S.A. (2024). "Neutrosophic TOPSIS-OWA framework for evaluating data protection strategies in judicial systems", *Journal of Fuzzy Extension and Applications*, 5(Special Issue), 1-11. https://doi.org/10.22105/jfea.2024.468198.1545

[12] Baykasouglu, A., Gölcük, I. (2015). "Development of a novel multiple-attribute decision making model via fuzzy cognitive maps and hierarchical fuzzy TOPSIS", *Information Sciences*, 301, 75-98. https://doi.org/10.1016/j.ins.2014.12.048

[13] Baykasouglu, A., Gölcük, I. (2017). "Development of an interval type-2 fuzzy sets based hierarchical MADM model by combining DEMATEL and TOPSIS", *Expert Systems with Applications*, 70(15), 37-51. https://doi.org/10.1016/j.eswa.2016.11.001

[14] Cai, L., Janowicz, K., Zhu, R., Yan, B., He, S. (2023). "HyperQuaternionE: A hyperbolic embedding model for qualitative spatial and temporal reasoning", *GeoInformatica*, 27(1), 159-197. https://doi.org/10.1007/s10707-022-00469-y

[15] Chen, C.-T. (2000). "Extensions of the TOPSIS for group decision-making under fuzzy environment", *Fuzzy Sets and Systems*, 114(1), 1-9. https://doi.org/10.1016/S0165-0114(97)00377-1

[16] Chen, Y., Dong, L. (2024). "A framework for assessment of logistics enterprises' safety standardization performance based on prospect theory", *Journal of Operations Intelligence*, 2(1), 153-166. https://doi.org/10.31181/jopi21202418

[17] Chen, S.-M., Lee, L.-W. (2010). "Fuzzy multiple attributes group decision-making based on the interval type-2 TOPSIS method", *Expert Systems with Applications*, 37(4), 2790-2798. https://doi.org/10.1016/j.eswa.2009.09.012

[18] Dehghani Fil Abadi, A. (2023). "Presenting a model for selecting the best supplier in the space of qualitative reasoning, a new approach for multi-attribute decision-making", *Journal of Industrial Engineering Research in Production Systems*, 11(22), 69-81. https://doi.org/10.22084/ier.2023.27636.2120

[19] De Soto, A. R. (2011). "A hierarchical model of a linguistic variable". *Information Sciences*, 181(20), 4394-4408. https://doi.org/10.1016/j.ins.2011.04.006

[20] Dong, Q., Cooper, O. (2016). "An orders-of-magnitude AHP supply chain risk assessment framework". *International Journal of Production Economics*, 182, 144–156. https://doi.org/10.1016/j.ijpe.2016.08.021

[21] Dubois, D. (1980). "Fuzzy sets and systems: Theory and applications", *Academic Press*, 144, 1-393.

[22] Espinilla, M., Liu, J., Martínez, L. (2011). "An extended hierarchical linguistic model for decision-making problems", *Computational Intelligence*, 27(3), 489-512. https://doi.org/10.1111/j.1467-8640.2011.00385.x

[23] Forbus, K. D. (2004). "Qualitative reasoning". In A. B. Tucker (Ed.), *Computer Science Handbook (2nd ed., pp. 20-1–20-25). Chapman and Hall/CRC*. https://doi.org/10.1201/9780203494455

[24] Jin, C., Mi, J., Li, F., et al. (2023). "An improved TOPSIS method for multi-criteria decision making based on hesitant fuzzy β neighborhood". *Artificial Intelligence Review*, 56(Suppl 1), 793–831. https://doi.org/10.1007/s10462-023-10510-7

[25] Gu, Y., Xu, K., Hu, X., Chen, M. (2024). "Research on improving the financial capacity of farmers based on fuzzy analytic hierarchy process", *Management Analytics and Social Insights*, 1(1), 88-102. http://noo.rs/Zj8YP

[26] Hesamian, G., Dehghani, A. (2021). "A preference degree for ranking k-dimensional vectors of linguistic labels and its application in multi-attribute group decision-making", *Journal of Experimental & Theoretical Artificial Intelligence*, 33(6), 933-944. https://doi.org/10.1080/0952813X.2020.1794233

[27] Hwang, C.-L., Yoon, K. (1981). "Multiple attribute decision making: Methods and applications a state-of-the-art survey", *Springer-Verlag*. https://doi.org/10.1007/978-3-642-48318-9

[28] Jia, F., Wang, Y., Su, Y. (2023). "GRA-based dynamic hybrid multi-attribute three-way decision-making for the performance evaluation of elderly-care services", *Mathematics*, 11(14), Article 3176. https://doi.org/10.3390/math11143176

[29] Kuipers, B. J. (1993). "Reasoning with qualitative models". *Artificial Intelligence*, 59(1-2), 125–132. https://doi.org/10.1016/0004-3702(93)90178-E

[30] Kuppusamy, M., Singh, J., Raman, P., Abdolrahimi, N. (2024). "Advanced decision analytics with fuzzy logic integrating AI and computational thinking for personnel selection", *Journal of Fuzzy Extension and Applications*, 5(4), 679-699. https://doi.org/10.22105/jfea.2024.477833.1621

[31] Liu, Z. (2024). “A belief similarity measure for Dempster-Shafer evidence theory and application in decision making”, *Journal of Soft Computing and Decision Analytics*, 2(1), 213-224. https://doi.org/10.31181/jscda21202443

[32] Mohamed, Z., Ismail, M., Abd El-Gawad, A. (2023). “Sustainable supplier selection using neutrosophic multi-criteria decision making methodology”, *Sustainable Machine Intelligence Journal*, 3(2), 1-9. https://doi.org/10.61185/SMIJ.2023.33102

[33] Mohamed, M., Salam, A., Ye, J., Yong, R. (2024). “A hybrid triangular fuzzy SWARA-MAROCS approach for selecting optimal and smart logistic enterprise based on IoT, blockchain, and UAVs”, *Multicriteria Algorithms with Applications*, 4, 1-15. https://doi.org/10.61356/j.mawa.2024.4241

[34] Morente-Molinera, J.A., Wu, X., Morfeq, A., Al-Hmouz, R., Herrera-Viedma, E. (2020). “A novel multi-criteria group decision-making method for heterogeneous and dynamic contexts using multi-granular fuzzy linguistic modelling and consensus measures”, *Information Fusion*, 53, 240-250. https://doi.org/10.1016/j.inffus.2019.06.028

[35] Qin, H., Peng, Q., Ma, X., Zhan, J. (2023). “A new multi-attribute decision making approach based on new score function and hybrid weighted score measure in interval-valued Fermatean fuzzy environment”, *Complex & Intelligent Systems*, 9(5), 5359-5376. https://doi.org/10.1007/s40747-023-01021-7

[36] Rodriguez, R.M., Martinez, L., Herrera, F. (2012). “Hesitant fuzzy linguistic term sets for decision making”, *IEEE Transactions on Fuzzy Systems*, 20(1), 109-119. https://doi.org/10.1109/TFUZZ.2011.2170076

[37] Roselló, L., Prats, F., Agell, N., Sánchez, M. (2010). “Measuring consensus in group decisions by means of qualitative reasoning”, *International Journal of Approximate Reasoning*, 51(4), 441-452. https://doi.org/10.1016/j.ijar.2010.01.002

[38] Salehi, M., Jabarpour, E. (2020). “Modeling and solving a multi-objective location-routing problem considering the evacuation of casualties and homeless people and fuzzy paths in relief logistics”, *Control and Optimization in Applied Mathematics*, 5(1), 41-65. https://doi.org/10.30473/coam.2021.53682.1142

[39] Shannon, C.E. (2001). “A mathematical theory of communication”, *ACM SIGMOBILE Mobile Computing and Communications Review*, 5(1), 3-55. https://doi.org/10.1145/584091.584093

[40] Sivadas, A., John, S.J. (2024). “Entropy and similarity measures of q-rung orthopair fuzzy soft sets and their applications in decision making problems”, *Journal of Fuzzy Extension and Applications*, 5(4), 660-678. https://doi.org/10.22105/jfea.2024.456028.1466

[41] Titkanloo, H.N., Keramati, A., Fekri, R. (2018). “Data aggregation in multi-source assessment model based on evidence theory”, *Applied Soft Computing*, 69, 443-452. https://doi.org/10.1016/j.asoc.2018.05.001

[42] Travé-Massuyès, L., Piera, N. (1989). "The orders of magnitude models as qualitative algebras", *extit Proceedings of the Eleventh International Joint Conference on Artificial Intelligence (IJCAI'89)* (1261–1266). Morgan Kaufmann. https://dx.doi.org/10.5555/1623891.1623957(https://hal.science/hal-03325964v1)

[43] Travé-Massuyès, L., Prats, F., Sánchez, M., Agell, N. (2005). "Relative and absolute order-of-magnitude models unified", *Annals of Mathematics and Artificial Intelligence*, 45(3-4), 323-341. https://doi.org/10.1007/s10472-005-9002-1

[44] Yaghoubi, M., Dadmand, F. (2024). "A multi-objective model for humanitarian logistics model during an earthquake crisis: A case study of Iran", *Control and Optimization in Applied Mathematics*, 9(2), 21-41. https://doi.org/10.30473/coam.2024.68031.1238

[45] Ye, J., Yong, R., Du, W. (2024). "MAGDM model using single-valued neutrosophic credibility matrix energy and its decision-making application", *Neutrosophic Systems with Applications*, 17, 1-20. https://doi.org/10.61356/j.nswa.2024.17243