Received: February 25, 2025; **Accepted:** June 24, 2025; **Published:** July 1, 2025.
DOI. 10.30473/coam.2025.73893.1293
Summer-Autumn (2025) Volume 10, Issue 2, (103-133)
Research Article
Open Access
Control and Optimization in Applied Mathematics - COAM

Network Data Envelopment Analysis and Uncertainty in Decision-Making: A Three-Stage Model Based on Liu's Uncertainty Theory

Jafar Pourmahmoud ✉, **Ahad Abbasi**, **Alireza Ghafari-Hadigheh**

Department of Applied Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.

✉ **Correspondence:**
Jafar Pourmahmoud
E-mail:
pourmahmoud@azaruniv.ac.ir

How to Cite
Pourmahmoud, J., Abbasi, A., Ghafari - Hadigheh, A.R. (2025). "Network data envelopment analysis and uncertainty in decision-making: A three-stage model based on Liu's uncertainty theory", Control and Optimization in Applied Mathematics, 10(2): 103-133, doi: 10.30473/coam.2025.73893.1293.

Abstract. Data Envelopment Analysis (DEA) is a well-established methodology for assessing the efficiency of decision-making units. In complex systems comprising multiple interconnected subsections, Network DEA provides a structured framework for efficiency evaluation. However, traditional DEA models rely on the assumption of deterministic data, which inadequately reflects the inherent uncertainty present in real-world scenarios. Traditional uncertainty-handling methods, such as fuzzy logic, stochastic models, and interval-based techniques, often fail when there is limited historical data and when expert opinions significantly influence the dataset. To address these limitations, this study introduces an uncertain network DEA model based on Liu's uncertainty theory, facilitating a more accurate assessment of efficiency under conditions of data imprecision. The proposed model is designed for three interconnected subsections and is further extended into a generalized multi-stage framework, allowing it to adapt to increasingly complex systems. Its effectiveness and practical applicability are demonstrated through two numerical case studies in the banking industry, highlighting its capacity to support decision-making under uncertainty. The findings emphasize the model's potential to enhance efficiency evaluation methods, particularly in environments characterized by limited and uncertain data.

Keywords. Uncertainty theory, Network DEA, Efficiency, Banking system.

MSC. 90C08; 90C90.

1 Introduction

Continuous performance evaluation is essential for organizational growth and development. Data Envelopment Analysis (DEA) has emerged as a powerful methodology for assessing the relative efficiency of homogeneous decision-making units (DMUs) that transform multiple inputs into multiple outputs. While traditional DEA models provide valuable insights into DMUs' efficiency, they fail to address critical questions in complex organizational structures where DMUs consist of interconnected subunits. In such network structures, outputs from one subunit often serve as inputs to others, creating a need for more sophisticated evaluation frameworks.

Traditional DEA approaches face two significant limitations when applied to network structures: (1) inability to identify which specific subunit contributes to inefficiency, and (2) difficulty in comparing performance across subunits. Network DEA (NDEA) was developed to address these limitations by modeling the internal structure of DMUs. However, existing NDEA models typically assume deterministic data, which rarely reflects real-world conditions where data uncertainty is prevalent.

Uncertainty in performance evaluation is applied in various forms, such as:

- Interval data (when only bounds are known)
- Fuzzy data (when values are inaccurate and vague)
- Probabilistic data (when historical distributions exist)
- Uncertain data (when expert opinion is primary)

While approaches exist for each uncertainty type (e.g., stochastic DEA for probabilistic data, fuzzy DEA for fuzzy data), they often require substantial historical data or make restrictive assumptions. The analysis of DEA and network DEA problems typically involves two key constraints: restricted sample sizes and model dependence on data characteristics. Stochastic modeling remains feasible only when sufficient empirical observations exist, which is inappropriate for data-scarce situations requiring expert input. Conversely, while fuzzy set theory offers an alternative for uncertain environments, its application can sometimes generate inconsistent and contradictory solutions that compromise analytical validity.

Liu's uncertainty theory [26] provides a mathematical framework for such scenarios, serving as a practical alternative when probability distributions are unknown or data is limited. While uncertainty theory has been successfully implemented in basic two-stage DEA models [11, 24, 33], its application remains limited for real-world systems that typically feature more complex, multi-stage network structures. Current literature reveals a significant research gap in extending these uncertainty-based approaches to generalized network configurations. This study makes three key contributions:

i. We develop a novel three-stage NDEA model based on Liu's uncertainty theory, specifically designed for cases where expert opinion supplements limited data.

ii. We extend the model to generalized p-stage network structures, significantly expanding its applicability.

iii. We validate the framework through two banking sector case studies, demonstrating its practical utility.

The remainder of this paper is organized as follows: Section 2 reviews relevant literature on non-deterministic NDEA. Section 3 presents our three-stage uncertain NDEA model. Section 4 generalizes the approach to p-stage structures. Section 5 applies the model to banking sector problems. Section 6 concludes with key findings and implications.

2 Literature Review

DEA is a widely used non-parametric approach for evaluating the efficiency of DMUs initially developed by Charnes, Cooper, and Rhodes called CCR model (1978). The CCR model, built upon the constant returns to scale (CRS) assumption, is used as the core structure for deriving other DEA models. Due to its unique characteristics and strength, DEA has seen widespread adoption across various disciplines, including management science, applied mathematics, industrial engineering, and economics. A comprehensive review by Mergoni et al. [27] examines DEA's development over the past five decades. Traditional DEA models treat DMUs as "black boxes", ignoring their internal structures. In contrast, Network DEA combines optimization methods with network modeling to incorporate internal relationships, thereby improving the accuracy of efficiency analysis by better reflecting real-world operations. The foundational work of Färe et al. [9] introduced network-based concepts to DEA. This concept has been studied by many researchers, the most comprehensive one compiled by Kao [21].

The advanced formulations offer deeper structural insights and greater analytical precision, allowing for more accurate efficiency assessments [4, 51]. Network DEA offers several distinct advantages when assessing the efficiency of DMUs with internal network structures. It accurately replicates the internal structure of complex systems, enabling simultaneous evaluation of both overall system efficiency and individual sub-process performance. This approach provides decomposable efficiency measures, reducing the number of unreal efficient DMUs and generating more accurate efficiency estimates. Network DEA facilitates optimal resource allocation across network components and examination of interdependencies between sub-processes [2, 17].

Both traditional DEA and NDEA approaches demonstrate significant sensitivity to data variations. While these methodologies traditionally assume deterministic inputs and outputs, real-world applications frequently encounter imprecise or uncertain data. To address this limitation, researchers have developed sophisticated extensions incorporating non-deterministic frameworks to strengthen efficiency analysis under uncertainty. The literature identifies different approaches for managing data uncertainty.

Stochastic DEA applies probability theory to account for random variations. There are three distinct methodological interpretations in this concept:

- Modeling deviations from efficiency frontiers as random variables
- Accounting for random noise in measurements
- Treating the production possibility set (PPS) as random PPS [29].

Stochastic modeling frameworks can only be properly employed when the dataset contains a statistically significant number of observations. Interval DEA was originally proposed by Cooper et al. [5]. It computes efficiency bounds (optimistic and pessimistic) using interval data and enables DMU classification based on efficiency ranges [6, 18].

Fuzzy DEA effectively handles linguistic ambiguity and vagueness. This method incorporates Zadeh's fuzzy set theory [49]. A Comprehensive review of Fuzzy DEA is available in the study of Hatami-Marbini et al. [14] and Emrouznejad et al. [8]. However, fuzzy DEA models present notable limitations such as potential for unbounded optimal values and computational complexity with high solution costs [13, 43]. Some fuzzy DEA models can only be formulated as linear optimization problems when restricted to trapezoidal fuzzy number representations [14]. These challenges highlight the need for careful model selection based on the specific nature of uncertainty in each application context.

Uncertainty theory provides a mathematical framework for quantifying expert opinions when statistical data is unavailable or unreliable. This axiomatic approach proves particularly valuable in (1) forecasting during emergencies such as war or pandemic, (2) analyzing scenarios with scarce historical data, (3) modeling qualitative concepts with linguistic ambiguity, and (4) analyzing dynamic systems subject to continuous-time noise .In such cases, uncertainty theory systematically considers domain experts' opinions [26]. Recently, many studies have applied uncertainty theory to present an applicable efficiency analysis. Lio and Liu [24] developed an innovative CCR model that treats inputs and outputs as uncertain variables, deriving an equivalent deterministic formulation through expected value calculations. In another study, a new model was suggested to achieve the highest degree of belief that the evaluated DMU is efficient [11]. The field has observed further expansion as additional scholars have successfully adapted traditional DEA methodologies to incorporate uncertainty principles [11, 34, 46].

We divide this section to focus on various aspects of DEA, with a particular emphasis on the methodological approaches employed to address the inherent uncertainty associated with efficiency assessment.

2.1 Network DEA

Network DEA combines optimization techniques and network modeling to consider internal structural connections during efficiency assessments. This method increases the realism of DEA models by more precisely simulating the operational mechanisms present in real-world systems [21]. The incorporation of network-based ideas into DEA by Färe et al. [9] broadened traditional methodologies, extending established frameworks like two-stage and hybrid DEA models.

These expanded formulations offer greater structural detail and analytical oversight, permitting modelers to enhance efficiency evaluations [4, 42, 47]. Such network-based modeling methods enable the systematic evaluation of individual components by monitoring efficiency at the sub-component level. They aid in pinpointing underperforming units that obstruct overall system effectiveness, while concurrently emphasizing optimized entities that boost operational performance [2, 17].

Network DEA models fall into two primary categories: static and dynamic. Static DEA depends on predefined elements and relationships that stay unchanged over time, providing a simplified structure for efficiency analysis. Conversely, dynamic DEA adopts a more realistic method by indexing elements across different time periods, permitting the assessment of efficiency changes over time.

Apart from these classifications, some network DEA models include uncertainty in relationships and values. For example, stochastic network DEA incorporates probability theory to improve the reliability of efficiency assessments in complex real-world situations [39, 50]. However, when probability theory is insufficient due to unreliable or inadequate data, alternative nondeterministic approaches—such as fuzzy theory and uncertainty theory—are used to create refined versions of efficiency measurement frameworks [11, 16]. These techniques allow a more adaptable and robust assessment of efficiency under uncertain conditions. The growing availability of big data, alongside progress in computational technologies, has created new opportunities for improving network DEA models by capturing more complex relationships within intricate systems. These technological enhancements have considerably widened the applicability of network DEA, enabling more accurate and dynamic efficiency evaluations. Increased computational power now supports real-time efficiency tracking, further extending the relevance of network DEA to sectors like healthcare and finance, where continuous performance monitoring is crucial. Moving forward, future advancements are anticipated to integrate machine learning techniques to refine efficiency assessments and yield deeper insights into system dynamics, ultimately enhancing decision-making and operational performance [37].

2.2 Non-Determinstic Methods

Stochastic DEA (SDEA) was developed to handle uncertainties in financial activities by integrating random noise into the DEA structure. This technique delivers a more practical evaluation of efficiency in unstable settings, like those in banking [29]. Although SDEA was initially presented by Charnes et al.[3], the inclusion of stochastic components to DEA is credited to Banker's research [30]. The dependability and usefulness of SDEA models were shown in multiple investigations by Korostelev et al. [22], and Simar [40], which prepared the way for incorporating distribution functions, stochastic processes, and bootstrapping methods. Nonetheless, SDEA is vulnerable to sample size and demands strict assumptions about the distributions of noise and inefficiency, which might not always apply in actual situations [41]. Limited datasets can cause skewed efficiency estimates, emphasizing the necessity for models that more effectively manage uncertainty in banking information [10].

Fuzzy DEA (FDEA) is a further expansion of conventional DEA, created to manage vagueness in input-output information through fuzzy set theory [48]. It has been especially beneficial in cases where financial information is inexact, such as risk appraisals or qualitative analyses [1]. However, FDEA brings subjectivity into the examination via the choice of membership functions, which can introduce human prejudice [12]. Furthermore, resolving fuzzy DEA models can be complicated, especially when used on extensive datasets typically present in financial industries [23]. To assess revenue efficiency in fuzzy network data envelopment analysis by converting a fuzzy efficiency model into an exact linear programming issue employing linear ranking functions and triangular fuzzy numbers [38].

To address the constraints of SDEA and FDEA, investigators have suggested expert-based models such as uncertainty theory, established by Liu [26]. Uncertainty theory does not depend on probability distributions, rendering it more flexible to settings with extremely uncertain information, like emerging markets in the banking industry [19, 26]. By integrating expert opinions into the DEA structure, this approach supplies a sturdy option for efficiency measurement in ambiguous settings [28]. Uncertainty theory has proven its utility and significance in representing real-world issues where information is scarce, untrustworthy, or mainly qualitative and grounded in human assessment [25]. However, the precision of these models is highly reliant on the trustworthiness of expert contributions, which can differ in application. Consequently, it is vital to seize the most critical features of the problem and utilize the interdependencies among DMUs to attain outcomes that more accurately represent real-world situations.

Table 1 displays instances of research performed in diverse areas connected to imprecise information. The included studies tackle the difficulties linked to three-stage and generalized network structures in ambiguous data settings. In this paper, we intend to suggest a model within an uncertainty-based structure that efficiently tackles these difficulties, delivering an organized and thorough solution to enhance efficiency assessment under data inaccuracy.

Table 1: Some studies of FDEA, SDEA,GDEA and UDEA.

Author(s)	**Main contributions(FDEA)**	**Studied problem**
Puri J, Yadav SP [36]	fuzzy DEA model with undesirable fuzzy outputs	banking sector in India
Pourbabagol et al. [31]	Fuzzy DEA network based on possibility and necessity measures	supply chain performance
Pourmahmoud, Jafar, and Naser Bafekr [32]	cost efficiency with fuzzy DEA models	-
Author(s)	**Main contributions(SDEA)**	**Studied problem**
Olesen and Petersen [29]	Stochastic data envelopment analysis	A review
Author(s)	**Main contributions(GDEA)**	**Studied problem**
Wang et al. [45]	Efficiency with DEA and Grey theory	estate companies
Pourmahmoud et al. [35]	DEA with three-parameter interval grey number	Health System
Author(s)	**Main contributions(UDEA)**	**Studied problem**
Ghaffari-Hadigheh and Lio [11]	NDEA in uncertain environment	-
Pourmahmoud and Bagheri [33]	uncertain model for a basic two-stage system	-

2.3 DEA in Banking System

Financial systems have utilized efficiency assessment models like DEA more broadly than many other industries. Banks depend significantly on optimizing overall and subdivided processes within their systems, employing various measures to evaluate current efficiency. By recognizing the most efficient units, banks can emulate their behavior and practices, strategize for other units, merge them, or introduce new inputs or outputs to their task networks. DEA models, encompassing two-stage and three-stage frameworks, have been extensively used in the banking sector to evaluate and boost operational efficiency. These models break down banking activities into interlinked sub-processes, enabling a more detailed performance analysis. For example, a two-stage DEA model can separate banking operations into deposit-taking and loan disbursement stages, allowing banks to pinpoint inefficiencies within each phase and execute necessary improvements [15]. Likewise, three-stage DEA models include an additional intermediate stage, offering a more thorough evaluation of the operational structure and improving strategic decision-making [7, 20, 51]. The use of network DEA models aids banks in optimizing resource allocation, lowering operational costs, and enhancing service delivery by detecting bottlenecks and improving process efficiency [44]. As the banking industry grows more complex, combining network DEA models with advanced data analytics techniques further increases their applicability and effectiveness in performance appraisal.

Progressing DEA methodology demands continuous enhancement in uncertainty-handling techniques, particularly in complex network environments where data imprecision is common. As a network-based system, banks depend intensely on optimizing both overall and subdivided processes using various efficiency measures. Nevertheless, in some countries, such as Iran, where inflation is extremely volatile, customer performance can be impacted by temporary fluctuations. Thus, when addressing this specific input, it is essential to recognize that the data is not exact and carries an inherent level of uncertainty. In these instances, the knowledge and viewpoints of experts can be highly beneficial. In this research, we adopt a three-stage network structure to assess the loan allocation process, specifically in cases where expert opinions are pivotal. Moreover, in the second practical illustration, we propose a fresh perspective on the branch merger issue by analyzing all possible merger outcomes, with experts specifying the input data for each case. Based on the suggested model, we determine the best merger strategy and offer it as a data-driven suggestion for bank managers and policymakers.

3 Uncertainty Theory

The content in this section is based on reference [26]. For a detailed discussion and complete proofs of the theorems, please consult the original source.

Let Γ be a nonempty set and $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. A set function $\mathbb{M} : \mathcal{L} \to [0, 1]$ that satisfies the following axioms is known as an uncertain measure:

(Normality Axiom):

$$\mathbb{M}\{\Gamma\} = 1. \tag{1}$$

(Duality Axiom):

$$\mathbb{M}\{\Lambda\} + \mathbb{M}\{\Lambda^c\} = 1, \qquad \Lambda \in \mathcal{L}. \tag{2}$$

(Subadditivity Axiom):

$$\mathbb{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathbb{M}\{\Lambda_i\}, \tag{3}$$

for any countable sequence of events $\Lambda_i \subseteq \mathcal{L}$.

The triplet $(\Gamma, \mathcal{L}, \mathbb{M})$ is called an *uncertainty space*.

The Product Axiom, which differentiates uncertainty theory from probability theory, is formulated as follows:

(Product Axiom): Let $(\Gamma_k, \mathcal{L}_k, \mathbb{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. Then, the product uncertain measure $\mathbb{M}$ on the product σ-algebra $\mathcal{L}_1 \times \mathcal{L}_2 \times \ldots \times \mathcal{L}_n$ satisfies:

$$\mathbb{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathbb{M}_k\{\Lambda_k\}. \tag{4}$$

where $\bigwedge$ denotes the infimum.

Definition 1. An uncertain variable is introduced to facilitate the quantitative modeling of phenomena within uncertainty theory. It is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathbb{M})$ to the set of real numbers. Specifically, for any Borel set B, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\},$$

is an event.

The uncertainty distribution of an uncertain variable ξ is defined as:

$$\Phi(x) = \mathbb{M}\{\xi \leq x\}, \quad x \in \mathbb{R}. \tag{5}$$

Furthermore, a set of uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if, for any Borel sets $B_1, B_2, \ldots, B_n$, the following holds:

$$\mathbb{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathbb{M}\left\{\xi_i \in B_i\right\}. \tag{6}$$

In the literature on uncertainty theory, various types of uncertain variables are explored. One of the simplest among them is the linear uncertain variable. Building on the previously introduced definitions, we now present several commonly used uncertainty distributions.

The most widely applied is the linear uncertainty distribution, defined as follows:

$$\Phi(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{b-a}, & a < x \leq b, \\ 1, & x > b. \end{cases} \tag{7}$$

where a and b are real numbers with $a < b$.

The second most commonly used distribution is the zigzag uncertainty distribution, defined as follows:

$$\Phi(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{2(b-a)}, & a < x \leq b, \\ \frac{x+c-2b}{2(c-b)}, & b < x \leq c, \\ 1, & x > c. \end{cases} \tag{8}$$

where a, b, and c are real numbers with $a < b < c$.

Another commonly used distribution is the normal uncertainty distribution, expressed as follows:

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\,\sigma}\right)\right)^{-1}, \quad x \geq 0,$$

where e and σ are real parameters with $\sigma > 0$.

Definition 2. Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

The inverse distributions corresponding to the three distributions above are as follows, respectively:

$$\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b,$$

$$\Phi(x)^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & \alpha < 0.5, \\ (2-2\alpha)b + (2\alpha-1)c, & \alpha \geq 0.5, \end{cases}$$

$$\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right),$$

for arbitrary Borel sets $B_1, B_2, \ldots, B_n$. An uncertain distribution $\Phi(x)$ is considered regular if it is a continuous and strictly increasing function with respect to x, satisfying $0 < \Phi(x) < 1$, and

$$\lim_{x\to-\infty} \Phi(x) = 0, \qquad \lim_{x\to+\infty} \Phi(x) = 1. \tag{9}$$

Let $\xi_1, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \ldots, \Phi_n$, respectively. Suppose that the function $f(\xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_m$, and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$. Then, f itself is an uncertain variable whose uncertainty distribution can be derived accordingly.

$$\Psi(x) = \sup_{f(x_1,x_2,\ldots,x_n)=x} \left(\min_{1\leq i\leq m} \Phi_i(x_i) \wedge \min_{m+1\leq i\leq n} (1 - \Phi(x_i)) \right). \tag{10}$$

Theorem 1. If the function $f(\xi_1, \ldots, \xi_n)$ is strictly increasing with respect to $\xi_1, \ldots, \xi_m$, and strictly decreasing with respect to $\xi_{m+1}, \ldots, \xi_n$, then f is an uncertain variable whose inverse uncertainty distribution is given by:

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right). \tag{11}$$

The expected value of an uncertain variable ξ is defined as:

$$E[\xi] = \int_0^{+\infty} \mathbb{M}\{\xi > r\}\, dr - \int_{-\infty}^0 \mathbb{M}\{\xi \leq r\}\, dr,$$

provided that at least one of these integrals is finite.

Theorem 2. Given an uncertain variable ξ with uncertainty distribution $\Phi(x)$, its expected value is calculated as follows:

$$E[\xi] = \int_0^{\infty} (1 - \Phi(x))\, dx - \int_{-\infty}^0 \Phi(x)\, dx,$$

Assuming at least one of these integrals converges, the expected value of an uncertain variable with a regular uncertainty distribution $\Phi(x)$, can also be expressed as:

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\, d\alpha.$$

Corollary 1. According to Theorem 2:

- The expected value of a zigzag uncertain variable $\xi \sim \mathcal{Z}(a, b, c)$ is:

$$E[\xi] = \frac{a + 2b + c}{4}.$$

- The expected value of a linear uncertain variable $\xi \sim \mathcal{L}(e, f)$ is:

$$E[\xi] = \frac{e + f}{2}.$$

Furthermore, if ξ and η are independent uncertain variables with finite expected values, then for any real numbers a and b: the expected value of the linear combination $a\xi + b\eta$ is given by:

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{12}$$

4 The Three-Stage Network DEA Model

We employed a three-stage framework to depict the scenario. Following this choice, the uncertain DEA framework is formulated; in this structure, as the variables incorporate an uncertain character, we utilize their expectation to transform the issue into a solvable form.

4.1 The Problem at Focus

In this section, we outline an actual process employed by Iran's banking system for extending loans to individuals, startups, and enterprises according to their business proposals, job generation capacity, and intended services or products. Our objective is to appraise the complete accreditation procedure using an uncertainty theory-based model to gauge the banking system's efficiency.

The framework contains three phases, mirroring the real-world process, with a total of n DMUs participating in judging each applicant's suitability. The procedure commences when an organization conducts a preliminary interview with the applicant to assess the knowledge-oriented character of their business proposal and its capacity to produce employment prospects. Subsequently, in the second phase, the central bank and its partner organizations appraise the proposal's macroeconomic and microeconomic contributions. This analysis seeks to establish the comprehensive effect of the suggested enterprise on both national and regional economies. Ultimately, in the third phase, one or more branches of private or public banks evaluate the loan applicant's repayment capacity. This stage is critical for safeguarding the banking system's financial soundness. Through applying this model, we acquire valuable perspectives on the accreditation process's efficiency. The model's outcomes validate whether the banking system is successfully executing its function in granting loans to qualified applicants.

4.1.1 Why is there a need to use Liu Uncertainty Theory?

In this context, the banking system possesses minimal, if any, information regarding a business plan before it undergoes evaluation through multiple stages and DMUs. Even when lenders obtain the plan, inadequate time prevents comprehensive scrutiny of details or collection of supplementary relevant data. Consequently, only specific indicators can be connected to historical precedents and comparable cases. Additionally, the actual data employed, processed, and depended upon for decisions are the viewpoints of experts engaged across various DMUs at different stages.

Given these constraints, the banking system encounters substantial difficulty in acquiring thorough knowledge about a business proposal. The informational deficit remains until the plan is formally submitted and appraised by successive stages and DMUs. Regrettably, lenders frequently face time limitations and cannot deeply inspect the plan's complexities or assemble additional information. As a result, lenders must depend on restricted signals tied to prior occurrences and similar circumstances. Furthermore, final decisions rest on the judgments of specialists participating in distinct DMUs throughout multiple phases.

Under these conditions, the accessible data is inadequate to effectively employ probability theory as a verification mechanism. Thus, substitute methodologies like Bayesian or Fuzzy theory require examination. Bayesian theory entails considerable computational expenses and difficulties in establishing appropriate prior probabilities, since deriving suitable values from human validators' beliefs lacks clarity. Conversely, Fuzzy theory, as emphasized by Liu, exhibits constraints related to its theoretical basis and axiom selection. Moreover, selecting an apt membership function is chiefly problem-specific and lacks a defined methodology for initiating the process or achieving the intended membership function.

Considering these factors, the present data insufficiency clearly necessitates exploring alternative validation approaches. While Bayesian and fuzzy theories offer potential remedies, each method introduces unique challenges and intricacies requiring careful assessment.

4.2 The Proposed Model

Consider the following three-stage network structure as shown in Figure 1. The deterministic model of this structure, based on Kao's framework [21], is formulated as follows when the data is deterministic.

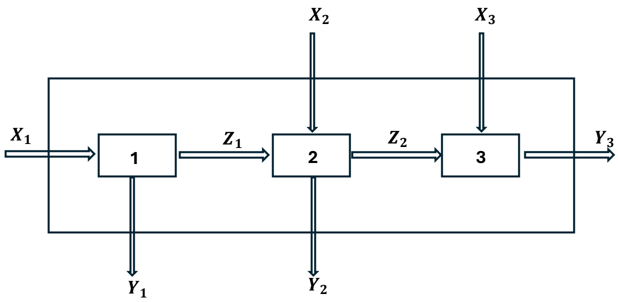

Figure 1: Structure of the three-stage system.

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r Y_{r0} + \sum_{g=1}^{h} w_g Z_{g0} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i X_{i0} + \sum_{g=1}^{h} w_g Z_{g0} = 1, \\
& \textstyle\sum_{r=1}^{s^{(1)}} u_r Y_{rj}^{(1)} + \sum_{g=1}^{h^{(1)}} w_g Z_{gj}^{(1)} - \sum_{i=1}^{m^{(1)}} v_i X_{ij}^{(1)} \leq 0, \\
& \textstyle\sum_{r=s^{(1)}+1}^{s^{(2)}} u_r Y_{rj}^{(2)} + \sum_{g=h^{(1)}+1}^{h^{(2)}} w_g Z_{gj}^{(2)} \\
& \quad -\left(\textstyle\sum_{i=m^{(1)}+1}^{m^{(2)}} v_i X_{ij}^{(2)} + \sum_{g=h^{(0)}+1}^{h^{(1)}} w_g Z_{gj}^{(1)} \right) \leq 0, \\
& \textstyle\sum_{r=s^{(2)}+1}^{s^{(3)}} u_r Y_{rj}^{(3)} - \left(\sum_{i=m^{(2)}+1}^{m^{(3)}} v_i X_{ij}^{(3)} + \sum_{g=h^{(1)}+1}^{h^{(2)}} w_g Z_{gj}^{(2)} \right) \leq 0, \\
& v_i, u_r, w_g \geq \epsilon.
\end{aligned}
\tag{13}
$$

- $X_{ij}^{(k)}$, $k = 1, \dots, 3,\ j = 1, \dots, n,\ i = 1, \dots, m^{(k)}$ are the exogenous inputs. Specially, $X_{i0}^{(1)}$'s are the first stage inputs, $X_{i0}^{(2)}$ are the second stage's exogenous inputs and $X_{i0}^{(3)}$ are the third exogenous inputs of the target DMU. We also assume the all the variables are independent throughout the study.

- $Z_{gj}^{(k)},\ k = 1, 2,\ j = 1, \ldots, n,\ g = h^{(k-1)} + 1, \cdots, h^{(k)}$ are intermediate products at each stage.
- $Y_{rj}^{(k)},\ k = 1, \ldots, 3,\ j = 1, \cdots, n,\ r = 1, \ldots, s^{(k)}$ are final outputs of the model at each stage.
- $v_i, u_r, w_g \geq \epsilon$ are positive weights that are needed to be optimally chosen.

Consider the following sketch of the three-stage model where all the variables are considered to be in an uncertain environment with information provided out of it through the opinions of some experts. In this model, following assumptions are considered. We also note that all the uncertain variable are positive. Therefore, based on our assumptions and the flowchart of the model, we present the uncertain DEA model through the following optimization problem

$$
\begin{aligned}
&\max E\left[\sum_{r=1}^{s} u_r Y_{r0} + \sum_{g=1}^{h} w_g Z_{g0}\right] \\
&\text{s.t.} E\left[\sum_{i=1}^{m} v_i X_{i0} + \sum_{g=1}^{h} w_g Z_{g0}\right] = 1, \\
&E\left[\sum_{r=1}^{s^{(1)}} u_r Y_{rj}^{(1)} + \sum_{g=1}^{h^{(1)}} w_g Z_{gj}^{(1)} - \sum_{i=1}^{m^{(1)}} v_i X_{ij}^{(1)}\right] \leq 0, \\
&E\Bigg[\sum_{r=s^{(1)}+1}^{s^{(2)}} u_r Y_{rj}^{(2)} + \sum_{g=h^{(1)}+1}^{h^{(2)}} w_g Z_{gj}^{(2)} \\
&\quad - \Big(\sum_{i=m^{(1)}+1}^{m^{(2)}} v_i X_{ij}^{(2)} + \sum_{g=h^{(0)}+1}^{h^{(1)}} w_g Z_{gj}^{(1)}\Big)\Bigg] \leq 0, \\
&E\left[\sum_{r=s^{(2)}+1}^{s^{(3)}} u_r Y_{rj}^{(3)} - \Big(\sum_{i=m^{(2)}+1}^{m^{(3)}} v_i X_{ij}^{(3)} + \sum_{g=h^{(1)}+1}^{h^{(2)}} w_g Z_{gj}^{(2)}\Big)\right] \leq 0, \\
&v_i, u_r, w_g \geq \epsilon.
\end{aligned}
\tag{14}
$$

After interviewing various experts across different stages and DMUs, and taking into account each uncertain variable, we represent their beliefs through uncertain distributions. We assume that each uncertain distribution is regular and has a well-defined inverse. Consequently, the uncertain inverse distributions of the model's variables are represented as follows:

$$
\psi_{ij}^{-1(k)} \longrightarrow X_{ij}^{(k)},\ k = 1, \ldots, 3,\ j = 1, \ldots, n,\ i = 1, \ldots, m^{(k)},
$$

$$
\phi_{rj}^{-1(k)} \longrightarrow Y_{rj}^{(k)},\ k = 1, \ldots, 3,\ j = 1, \ldots, n,\ r = 1, \ldots, s^{(k)},
$$

$$
\Gamma_{gj}^{-1(k)} \longrightarrow Z_{gj}^{(k)},\ k = 1, 2,\ \ j = 1, \ldots, n,\ g = h^{(k-1)} + 1, \ldots, h^{(k)}.
$$

Theorem 3. Suppose for any DMU_j, inputs, intermediate and output variables are uncertain then equivalent crisp appearance of model (14) is model (15).

$$
\begin{aligned}
\max \quad & \int_0^1 \left(\sum_{r=1}^{s} u_r \phi_{r0}^{-1}(\alpha) + \sum_{g=1}^{h} w_g \Gamma_{g0}^{-1}(\alpha) \right) d\alpha \\
\text{s.t.} \quad & \int_0^1 \left(\sum_{i=1}^{m} v_i \psi_{i0}^{-1}(\alpha) + \sum_{g=1}^{h} w_g \Gamma_{g0}^{-1}(\alpha) \right) d\alpha = 1, \\
& \int_0^1 \left(\sum_{r=1}^{s^{(1)}} u_r \phi_{rj}^{-1(1)}(\alpha) + \sum_{g=1}^{h^{(1)}} w_g \Gamma_{gj}^{-1(1)}(\alpha) - \sum_{i=1}^{m^{(1)}} v_i \psi_{ij}^{-1(1)}(1-\alpha) \right) d\alpha \leq 0, \\
& \int_0^1 \left(\sum_{r=s^{(1)}+1}^{s^{(2)}} u_r \phi_{rj}^{-1(2)}(\alpha) + \sum_{g=h^{(1)}+1}^{h^{(2)}} w_g \Gamma_{gj}^{-1(2)}(\alpha) \right. \\
& \quad \left. - \left(\sum_{i=m^{(1)}+1}^{m^{(2)}} v_i \psi_{ij}^{-1(2)}(1-\alpha) + \sum_{g=h^{(0)}+1}^{h^{(1)}} w_g \Gamma_{gj}^{-1(1)}(1-\alpha) \right) \right) d\alpha \leq 0, \\
& \int_0^1 \left(\sum_{r=s^{(2)}+1}^{s^{(3)}} u_r \phi_{rj}^{-1(3)}(\alpha) \right. \\
& \quad \left. - \left(\sum_{i=m^{(2)}+1}^{m^{(3)}} v_i \psi_{ij}^{-1(3)}(1-\alpha) + \sum_{g=h^{(1)}+1}^{h^{(2)}} w_g \Gamma_{gj}^{-1(2)}(1-\alpha) \right) \right) d\alpha \leq 0, \\
& v_i, u_r, w_g \geq \epsilon.
\end{aligned}
\tag{15}
$$

Proof. Considering the following relationships:

$$F^0 = \sum_{r=1}^{s} u_r Y_{r0} + \sum_{g=1}^{h} w_g Z_{g0}.$$

F^0 is increasing in relation to Y_{r0}, Z_{g0} then as stated in Theorem 1 the inverse uncertainty distribution of F^0 is:

$$(F^0)^{-1} = \sum_{r=1}^{s} u_r \phi_{r0}^{-1}(\alpha) + \sum_{g=1}^{h} w_g \Gamma_{g0}^{-1}(\alpha),$$

and according to Theorem 2

$$E\left(F^0\right) = \int_0^1 \left(\sum_{r=1}^{s} u_r \phi_{r0}^{-1}(\alpha) + \sum_{g=1}^{h} w_g \Gamma_{g0}^{-1}(\alpha) \right) d(\alpha),$$

for the third constraint of model (14) considering

$$F^3 = \left[\sum_{r=s^{(1)}+1}^{s^{(2)}} u_r Y_{rj}^{(2)} + \sum_{g=h^{(1)}+1}^{h^{(2)}} w_g Z_{gj}^{(2)} - \Big(\sum_{i=m^{(1)}+1}^{m^{(2)}} v_i X_{ij}^{(2)} + \sum_{g=h^{(0)}+1}^{h^{(1)}} w_g Z_{gj}^{(1)} \Big) \right].$$

F^3 is increasing in relation to $Y_{rj}^{(2)}, Z_{gj}^{(2)}$ and decreasing related to $X_{ij}^{(2)}, Z_{gj}^{(1)}$ then as stated in Theorem 1 the inverse uncertainty distribution of F^3 is:

$$
\begin{aligned}
(F^3)^{-1} = & \sum_{r=s^{(1)}+1}^{s^{(2)}} u_r \phi_{rj}^{-1(2)}(\alpha) + \sum_{g=h^{(1)}+1}^{h^{(2)}} w_g \Gamma_{gj}^{-1(2)}(\alpha) \\
& - \left(\sum_{i=m^{(1)}+1}^{m^{(2)}} v_i \psi_{ij}^{-1(2)}(1-\alpha) + \sum_{g=h^{(0)}+1}^{h^{(1)}} w_g \Gamma_{gj}^{-1(1)}(1-\alpha) \right).
\end{aligned}
$$

According to Theorem 2

$$E\Big(F^3\Big) = \int_0^1 \Bigg(\sum_{r=s^{(1)}+1}^{s^{(2)}} u_r \phi_{rj}^{-1(2)}(\alpha) + \sum_{g=h^{(1)}+1}^{h^{(2)}} w_g \Gamma_{gj}^{-1(2)}(\alpha) - \Bigg(\sum_{i=m^{(1)}+1}^{m^{(2)}} v_i \psi_{ij}^{-1(2)}(1-\alpha) + \sum_{g=h^{(0)}+1}^{h^{(1)}} w_g \Gamma_{gj}^{-1(1)}(1-\alpha) \Bigg) \Bigg) d\alpha.$$

Do the same for other conditions. Therefore, using the results of Theorems 1 and 2 the claim is proved. □

Theorem 4. Considering the optimization problem (14), there is at least one feasible solution.

Proof. Since all the uncertain variables are positive, their expected values are also positive. Now we choose the weights of exogenous variables, v_i as following

$$v_i = \frac{1 - \sum_{g=1}^{h} \epsilon(Z_{g0})}{mX_{i0}}. \tag{16}$$

We also assume that

$$w_g = u_r = \epsilon. \tag{17}$$

We place the assumed solution in the model constraints:

$$\begin{aligned}
&E\left[\sum_{i=1}^{m} \frac{1 - \sum_{g=1}^{h} \epsilon(Z_{g0})}{mX_{i0}} X_{i0} + \sum_{g=1}^{h} \epsilon Z_{g0} \right] = 1,\\
&E\left[\sum_{r=1}^{s^{(1)}} \epsilon Y_{rj}^{(1)} + \sum_{g=1}^{h^{(1)}} \epsilon Z_{gj}^{(1)} - \sum_{i=1}^{m^{(1)}} \frac{1 - \sum_{g=1}^{h} \epsilon(Z_{g0})}{mX_{i0}} X_{ij}^{(1)} \right] \leq 0,\\
&E\left[\sum_{r=s^{(1)}+1}^{s^{(2)}} \epsilon Y_{rj}^{(2)} + \sum_{g=h^{(1)}+1}^{h^{(2)}} \epsilon Z_{gj}^{(2)} - \Big(\sum_{i=m^{(1)}+1}^{m^{(2)}} \frac{1 - \sum_{g=1}^{h} \epsilon(Z_{g0})}{mX_{i0}} X_{ij}^{(2)} + \sum_{g=h^{(0)}+1}^{h^{(1)}} \epsilon Z_{gj}^{(1)} \Big) \right]\\
&\leq 0,\\
&E\left[\sum_{r=s^{(2)}+1}^{s^{(3)}} \epsilon Y_{rj}^{(3)} - \Big(\sum_{i=m^{(2)}+1}^{m^{(3)}} \frac{1 - \sum_{g=1}^{h} \epsilon(Z_{g0})}{mX_{i0}} X_{ij}^{(3)} + \sum_{g=h^{(1)}+1}^{h^{(2)}} \epsilon Z_{gj}^{(2)} \Big) \right] \leq 0,\\
&v_i, u_r, w_g \geq \epsilon.
\end{aligned}$$

Since all data values are strictly positive, the proof is finalized by simplifying the given expressions, ensuring logical consistency and correctness. With these choices for v_i, w_g, u_r all the constrained are satisfied and we prove that there is at least one feasible solution. □

Theorem 5. The optimization problem (14) has an optimal solution.

Proof. Based on the first constraint, both $E\left[\sum_{i=1}^{m} v_i X_{i0} \right]$ and $E\left[\sum_{g=1}^{h} w_g Z_{g0} \right]$ are finite. Therefore, using the sum constraints two, three and four we have

$$E\left[\sum_{r=1}^{s} u_r Y_{rj} + \sum_{g=1}^{h} w_g Z_{gj} - \sum_{i=1}^{m} v_i X_{ij}\right] \leq 0,$$
$$E\left[\sum_{r=1}^{s} u_r Y_{rj} + \sum_{g=1}^{h} w_g Z_{gj}\right] \leq \sum_{i=1}^{m} v_i X_{ij}.$$

It is resulted that the objective function of the model (14) is finite, therefore, using the previous theorem, it has an optimal solution. □

4.3 The Model with Linear and Zigzag Uncertain Variables

For the sake of illustration, assume that inputs are all have uncertain linear distribution $\mathcal{L}(e, f)$, intermediates and outputs have uncertain zigzag distribution $\mathcal{Z}(a, b, c)$.

Since all the uncertain variables in the model are independent, the expectation operator can be moved inside the summation symbols. According to Corollary 1 model (14) is given as

$$\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r \tfrac{(a_{r0}+2b_{r0}+c_{r0})}{4} + \sum_{g=1}^{h} w_g \tfrac{(a_{g0}+2b_{g0}+c_{g0})}{4} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i \tfrac{(e_{i0}+f_{i0})}{2} + \sum_{g=1}^{h} w_g \tfrac{(a_{g0}+2b_{g0}+c_{g0})}{4} = 1, \\
& \textstyle\sum_{r=1}^{s^{(1)}} u_r \tfrac{(a_{rj}+2b_{rj}+c_{rj})}{4} + \sum_{g=1}^{h^{(1)}} w_g \tfrac{(a_{gj}+2b_{gj}+c_{gj})}{4} - \sum_{i=1}^{m^{(1)}} v_i \tfrac{(e_{ij}+f_{ij})}{2} \leq 0, \\
& \textstyle\sum_{r=s^{(1)}+1}^{s^{(2)}} u_r \tfrac{(a_{rj}+2b_{rj}+c_{rj})}{4} + \sum_{g=h^{(1)}+1}^{h^{(2)}} w_g \tfrac{(a_{gj}+2b_{gj}+c_{gj})}{4} - \\
& \textstyle\left(\sum_{i=m^{(1)}+1}^{m^{(2)}} v_i \tfrac{(e_{ij}+f_{ij})}{2} + \sum_{g=h^{(0)}+1}^{h^{(1)}} w_g \tfrac{(a_{gj}+2b_{gj}+c_{gj})}{4}\right) \leq 0, \\
& \textstyle\sum_{r=s^{(2)}+1}^{s^{(3)}} u_r \tfrac{(a_{rj}+2b_{rj}+c_{rj})}{4} \\
& \quad \textstyle-\left(\sum_{i=m^{(2)}+1}^{m^{(3)}} v_i \tfrac{(e_{ij}+f_{ij})}{2} + \sum_{g=h^{(1)}+1}^{h^{(2)}} w_g \tfrac{(a_{gj}+2b_{gj}+c_{gj})}{4}\right) \leq 0, \\
& v_i, u_r, w_g \geq \epsilon.
\end{aligned} \tag{18}$$

Therefore, in the original uncertain optimization model, each uncertain variable can be replaced by its expected value, resulting in the deterministic equivalent shown above. This transformation preserves the structure of the optimization while removing the uncertainty, making it solvable by standard mathematical programming methods.

5 Generalizing to a P-Stage Model

In real-world applications, problems often exhibit a higher degree of internal structural complexity. Consequently, accurately modeling such multifaceted issues typically requires the implementation of a generalized P-stage network. In this section, we extend the proposed model to the general network case. For a system with p stages, as illustrated in Figure 2, we have the following Model 19.

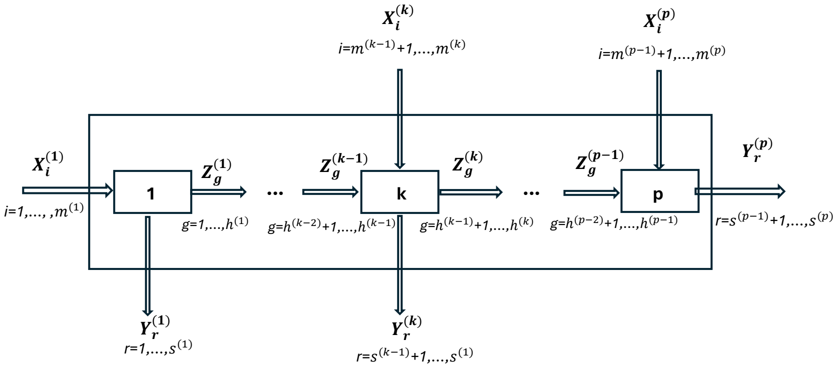

Figure 2: Structure of the p-stage system.

$$
\begin{aligned}
\max\quad & E\left[\sum_{r=1}^{s} u_r Y_{r0} + \sum_{g=1}^{h} w_g Z_{g0}\right] \\
\text{s.t.}\quad & E\left[\sum_{i=1}^{m} v_i X_{i0} + \sum_{g=1}^{h} w_g Z_{g0}\right] = 1, \\
& E\left[\textstyle\sum_{r=1}^{s^{(1)}} u_r Y_{rj}^{(1)} + \sum_{g=1}^{h^{(1)}} w_g Z_{gj}^{(1)} - \sum_{i=1}^{m^{(1)}} v_i X_{ij}^{(1)}\right] \le 0, \\
& E\Bigg[\textstyle\sum_{r=s^{(1)}+1}^{s^{(2)}} u_r Y_{rj}^{(2)} + \sum_{g=h^{(1)}+1}^{h^{(2)}} w_g Z_{gj}^{(2)} \\
& \quad -\left(\textstyle\sum_{i=m^{(1)}+1}^{m^{(2)}} v_i X_{ij}^{(2)} + \sum_{g=h^{(0)}+1}^{h^{(1)}} w_g Z_{gj}^{(1)}\right)\Bigg] \le 0, \\
& E\Bigg[\textstyle\sum_{r=s^{(k-1)}+1}^{s^{(k)}} u_r Y_{rj}^{(k)} + \sum_{g=h^{(k-1)}+1}^{h^{(k)}} w_g Z_{gj}^{(k)} \\
& \quad -\left(\textstyle\sum_{i=m^{(k-1)}+1}^{m^{(k)}} v_i X_{ij}^{(k)} + \sum_{g=h^{(k-2)}+1}^{h^{(k-1)}} w_g Z_{gj}^{(k-1)}\right)\Bigg] \le 0, \quad k = 2,\cdots,p-1, \\
& E\left[\textstyle\sum_{r=s^{(p-1)}+1}^{s^{p}} u_r Y_{rj}^{p} - \left(\sum_{i=m^{(p-1)}+1}^{m^{p}} v_i X_{ij}^{(3)} + \sum_{g=h^{(p-2)}+1}^{h^{(p-1)}} w_g Z_{gj}^{(p-1)}\right)\right] \le 0, \\
& v_i, u_r, w_g \ge \epsilon.
\end{aligned}
\tag{19}
$$

Similarly, we define the distributions of the uncertain variables as follow

$$
\begin{aligned}
&\psi_{ij}^{-1(k)} \longrightarrow X_{ij}^{(k)}, \qquad k = 1,\ldots,p,\ j = 1,\ldots,n,\ i = 1,\ldots,m^{(k)}, \\
&\phi_{rj}^{-1(k)} \longrightarrow Y_{rj}^{(k)}, \qquad k = 1,\ldots,p,\ j = 1,\ldots,n,\ r = 1,\ldots,s^{(k)}, \\
&\Gamma_{gj}^{-1(k)} \longrightarrow Z_{gj}^{(k)}, \qquad k = 1,\ldots,p-1,\ j = 1,\ldots,n,\ g = h^{(k-1)}+1,\ldots,h^{(k)}.
\end{aligned}
$$

Based on the above discussion, we present the following theorems:

Theorem 6. Consider Model (19). Then the following representation holds

$$
\begin{aligned}
\max \quad & \int_0^1 \sum_{r=1}^{s} u_r \phi_{r0}^{-1}(\alpha) + \sum_{g=1}^{h} w_g \Gamma_{g0}^{-1}(\alpha) d\alpha \\
\text{s.t.} \quad & \int_0^1 \sum_{i=1}^{m} v_i \psi_{i0}^{-1}(\alpha) + \sum_{g=1}^{h} w_g \Gamma_{g0}^{-1}(\alpha)\, d\alpha = 1, \\
& \int_0^1 \sum_{r=1}^{s^{(1)}} u_r \phi_{rj}^{-1(1)}(\alpha) + \sum_{g=1}^{h^{(1)}} w_g \Gamma_{gj}^{-1(1)}(\alpha) - \sum_{i=1}^{m^{(1)}} v_i \psi_{ij}^{-1(1)}(1-\alpha)\, d\alpha \le 0, \\
& \int_0^1 \sum_{r=s^{(k-1)}+1}^{s^{(k)}} u_r \varphi_{rj}^{-1(k)}(\alpha) + \sum_{g=h^{(k-1)}+1}^{h^{(k)}} w_g \Gamma_{gj}^{-1(k)} \\
& \quad -\Big(\sum_{i=m^{(k-1)}+1}^{m^{(k)}} v_i \psi_{ij}^{-1(k)}(1-\alpha) + \sum_{g=h^{(k-2)}+1}^{h^{(k-1)}} w_g \Gamma_{gj}^{-1(k-1)}(1-\alpha) \Big)\, d\alpha \le 0, \\
& \quad k = 2, \cdots, p-1, \\
& \int_0^1 \sum_{r=s^{(p-1)}+1}^{s^{(p)}} u_r \phi_{rj}^{-1(p)}(\alpha) \\
& \quad -\left(\sum_{i=m^{(p-1)}+1}^{m^{(p)}} v_i \psi_{ij}^{-1(p)}(1-\alpha) + \sum_{g=h^{(p-2)}+1}^{h^{(p-1)}} w_g \Gamma_{gj}^{-1(p-1)}(1-\alpha) \right) d\alpha \le 0, \\
& v_i, u_r, w_g \ge \epsilon.
\end{aligned}
\tag{20}
$$

Proof. The proof follows similarly to that of Theorem 3, with the argument being repeated for each $k = 4, \ldots, p$. See Appendix A for more details. □

Theorem 7. Consider optimization problem (19). Then there exists at least one feasible solution.

Proof. Given the positivity of all uncertain variables, their expected values are strictly positive. Adopting weighting coefficients similar to Theorem 4 and following the same proof methodology guarantees constraint satisfaction. See Appendix A. □

Theorem 8. The optimization problem (19) admits an optimal solution.

Proof. Observe that the objective function of model (19) is identical to that of model (14). Since model (14) possesses an optimal solution, this property consequently extends to model (19). For complete technical details of this proof, we refer the reader to Appendix A. □

6 Numerical Implementation

Example 1. In this example, we analyze 10 loan applications submitted by companies and business owners seeking financial support from a private bank in Iran. Our analysis focused on four branches of this bank. Among these applicants, some possess strong financial backgrounds, while others were perceived as untrustworthy due to financial instability or previous loan repayment issues. Several applicants had previously applied for loans but were rejected. The primary challenge is to evaluate the capability of each branch to accurately identify suitable candidates for loan approval.

However, the input data provided to these branches fluctuates significantly and is highly influenced by expert opinions. Each branch received diverse input sets characterized by different statistical distributions, contributed by five financial experts offering distinct perspectives on the key variables representing customer financial conditions. Consequently, each branch is assessed using five separate data groups, enabling us to evaluate their ability to analyze customer financial credentials across various scenarios. For instance, some data sets prioritize customer income and estimated business revenue, while others emphasize loan history and geographical location. This approach ensures a broad spectrum of input variables and distributions, providing comprehensive information for accurate financial assessment.

Each branch, analyzed through these five data groups, represents a set of decision-making units (DMUs). Based on these assumptions, the relevant variables are defined as follows:

Initial inputs X_{ij}:

- Applicant's average monthly turnover (income history),
- Declared initial capital or shares by the applicant,
- Business plan score (based on preliminary expert review),
- Number of jobs proposed to be created,
- Applicant's personal financial reliability (e.g., credit history or reputation),
- Industry experience and professional background relevant to the proposed business
- Knowledge-based project score,
- Innovation index,
- Local economic contribution score,
- Alignment with the national economic goals,
- Projected employment impact,
- Loan repayment risk index,
- Market readiness and commercialization potential.

Final output Y_{ij}:

- Overall loan approval score,
- Estimated probability of repayment,
- Anticipated loan return (profitability for the bank),
- Risk-adjusted strength of loan recommendation,
- Applicant's financial leverage and debt-to-income ratio.

Distribution Modeling:

i. For inputs (linear distribution): $\psi_{ij}^{-1(k)}(\alpha) = 0.05a_{ij} + 0.95b_{ij}$,

ii. For intermediate outputs (linear distribution): $\phi_{rj}^{-1(k)}(\alpha) = 0.05c_{rj} + 0.95d_{rj}$

iii. For final outputs (normal distribution):

$$\Gamma_{gj}^{-1(k)}(\alpha) = e_{gj} + \left(\frac{\sigma\sqrt{3}}{\pi} \ln\left(\frac{0.95}{0.05}\right)\right),$$

where (a, b), (c, d) and (e, σ) are parameters provided in the accompanying tables.

Example Variable Explanation:

For instance, setting $\alpha = 0.95$, the scaled value for the "Number of jobs to be created" variable represents the estimated maximum number of jobs that a customer might generate if granted the loan.

Table 2: Exogenous inputs.

DMU	ψ^{-1}(Inputs)
1	[(5.25, 9.5), (5.25, 7.5), (7.25, 9.5), (7, 9), (8.5, 9.5)]
2	[(2.5, 7.0), (5.0, 6.5), (8.5, 10.5), (7.0, 9.0), (6.5, 8.0)]
3	[(4.5, 8.5), (6.5, 9.0), (7.5, 9.5), (7.5, 9.5), (6.0, 8.0)]
4	[(4.0, 7.5), (5.5, 7.0), (8.0, 9.0), (7.0, 8.5), (6.5, 8.0)]
5	[(6.0, 8.5), (7.0, 9.0), (7.0, 9.0), (7.5, 8.5), (6.0, 8.0)]
6	[(4.0, 8.0), (5.0, 6.0), (8.0, 9.5), (7.5, 9.0), (7.0, 9.0)]
7	[(5.5, 7.5), (6.0, 8.0), (8.5, 10.0), (7.5, 9.0), (7.5, 9.5)]
8	[(3.5, 8.0), (5.5, 6.5), (8.5, 9.5), (7.5, 9.0), (6.5, 8.5)]
9	[(2.5, 8.5), (6.5, 8.0), (7.5, 9.0), (6.0, 8.5), (7.0, 9.5)]
10	[(5.0, 8.0), (5.5, 7.5), (8.0, 9.5), (7.5, 9.0), (7.5, 9.0)]
11	[(6.0, 8.5), (7.5, 8.5), (7.5, 9.0), (7.5, 8.5), (6.5, 8.0)]
12	[(5.0, 7.5), (6.5, 7.5), (8.0, 9.5), (7.5, 9.0), (7.5, 9.0)]
13	[(3.0, 7.5), (4.5, 6.0), (7.5, 8.5), (6.5, 8.0), (7.0, 8.5)]
14	[(6.0, 8.5), (7.0, 9.0), (8.0, 9.5), (7.5, 9.0), (6.5, 8.0)]
15	[(4.5, 8.0), (5.5, 6.5), (8.0, 9.0), (7.5, 8.5), (6.5, 8.5)]
16	[(5.0, 8.0), (5.5, 7.5), (8.5, 9.0), (7.5, 8.5), (7.0, 9.0)]
17	[(4.0, 7.0), (5.0, 6.0), (8.5, 9.5), (6.5, 8.0), (7.5, 9.0)]
18	[(5.0, 8.5), (6.0, 8.5), (7.5, 9.0), (7.0, 8.5), (6.5, 8.5)]
19	[(3.5, 7.5), (5.5, 6.5), (8.0, 9.0), (6.5, 8.0), (7.0, 8.5)]
20	[(4.0, 8.5), (6.5, 8.0), (8.5, 9.5), (7.0, 9.0), (6.5, 8.5)]

Table 3: Intermediate products.

DMU	ϕ^{-1}(Intermediate Outputs)
1	[(15.0, 26.5), (18.0, 21.5), (11.5, 16.0), (14, 17), (11, 13.5), (12.5, 14.5)]
2	[(13.0, 22.5), (19.0, 24.5), (14.5, 17.0), (10.0, 12.5), (16.0, 18.5)]
3	[(14.0, 25.5), (18.5, 21.0), (12.5, 15.0), (10.0, 13.5), (16.5, 19.5)]
4	[(13.5, 22.5), (20.0, 23.0), (11.5, 13.5), (11.5, 14.5), (16.0, 17.5)]
5	[(14.5, 25.0), (19.5, 21.0), (12.5, 14.0), (12.5, 15.0), (16.5, 18.0)]
6	[(15.5, 24.5), (17.5, 19.5), (12.0, 13.0), (10.0, 12.5), (16.5, 18.5)]
7	[(14.5, 23.5), (18.0, 21.5), (12.5, 14.5), (12.5, 14.0), (16.0, 17.0)]
8	[(12.5, 23.0), (19.0, 21.0), (11.5, 13.0), (10.0, 12.5), (17.0, 18.5)]
9	[(13.5, 22.5), (18.0, 21.0), (11.5, 13.0), (11.0, 12.5), (16.5, 18.5)]
10	[(15.0, 24.0), (20.0, 22.5), (12.0, 14.5), (12.0, 14.0), (17.0, 19.0)]
11	[(14.5, 24.5), (18.5, 20.5), (12.5, 13.5), (10.5, 12.5), (16.5, 17.5)]
12	[(14.0, 24.0), (19.0, 21.5), (11.5, 12.5), (11.0, 13.0), (16.5, 18.0)]
13	[(13.5, 23.5), (18.5, 20.5), (12.0, 14.5), (10.5, 12.5), (16.0, 18.5)]
14	[(14.5, 24.5), (18.5, 20.5), (12.5, 14.0), (11.5, 13.0), (16.5, 18.5)]
15	[(14.0, 23.5), (19.5, 21.0), (11.5, 13.5), (10.0, 11.5), (16.5, 17.5)]
16	[(14.5, 23.0), (19.5, 22.0), (11.5, 13.5), (10.0, 12.0), (16.0, 18.5)]
17	[(13.0, 22.5), (20.0, 22.0), (12.0, 14.0), (11.0, 13.0), (17.0, 19.0)]
18	[(14.5, 24.5), (19.5, 22.0), (12.0, 13.0), (10.5, 12.5), (17.0, 18.5)]
19	[(12.5, 22.5), (18.5, 21.5), (12.5, 14.0), (10.0, 12.5), (17.0, 19.0)]
20	[(13.0, 23.5), (19.0, 21.5), (12.0, 13.5), (10.5, 12.5), (17.0, 18.0)]

Results and Insights

The analysis indicates that DMUs 11, 13, and 14 are efficient within this network. Their high efficiency scores suggest that these branches have successfully evaluated applicants despite data uncertainties and have demonstrated effective loan allocation or rejection decisions. Conversely, units 9, 8, and 16 exhibit the lowest efficiency scores, indicating subpar performance in customer evaluation and loan decision-making. This inefficiency has led to potential misallocations, such as granting loans to ineligible applicants or denying credit to worthy candidates. These branches should revisit their evaluation criteria, weightings, and data quality to enhance decision accuracy.

Importantly, the integrated uncertainty theory, combined with our proposed model, effectively captures the optimal performance of these DMUs given their inputs and network structure. This approach underscores the potential of uncertainty theory as a viable alternative to probability-based methods in DEA, especially when data is scarce, unreliable, or primarily based on expert judgment.

Table 4: Final outputs.

DMU	Γ^{-1}(Final Outputs)
1	[(3.30, 0.75), (3.5, 0.5), (4, 0.85), (3.5, 0.9)]
2	[(3.50, 0.90), (3.50, 0.85), (6.40, 0.95), (6.80, 0.88)]
3	[(3.40, 0.80), (5.90, 0.90), (6.80, 0.95), (7.90, 0.85)]
4	[(3.60, 0.70), (6.20, 0.85), (7.00, 0.95), (7.10, 0.80)]
5	[(3.30, 0.85), (5.70, 0.90), (6.90, 0.95), (7.30, 0.90)]
6	[(3.40, 0.80), (5.80, 0.85), (7.00, 0.90), (7.20, 0.75)]
7	[(3.30, 0.85), (5.90, 0.95), (7.10, 0.85), (7.40, 0.80)]
8	[(3.60, 0.80), (6.20, 0.90), (7.10, 0.85), (7.30, 0.90)]
9	[(3.50, 0.90), (5.90, 0.85), (6.80, 0.95), (7.20, 0.90)]
10	[(3.50, 0.85), (6.30, 0.90), (7.00, 0.95), (7.60, 0.80)]
11	[(3.40, 0.85), (5.80, 0.90), (7.00, 0.90), (7.30, 0.90)]
12	[(3.60, 0.80), (5.90, 0.90), (7.10, 0.95), (7.40, 0.85)]
13	[(3.40, 0.85), (5.80, 0.85), (7.10, 0.90), (7.30, 0.80)]
14	[(3.40, 0.80), (5.90, 0.85), (7.10, 0.90), (7.50, 0.80)]
15	[(3.50, 0.90), (5.80, 0.90), (7.00, 0.95), (7.30, 0.90)]
16	[(3.50, 0.85), (6.10, 0.85), (7.20, 0.95), (7.60, 0.90)]
17	[(3.60, 0.90), (5.90, 0.90), (7.20, 0.95), (7.30, 0.85)]
18	[(3.40, 0.85), (5.80, 0.90), (7.00, 0.90), (7.20, 0.85)]
19	[(3.40, 0.90), (5.70, 0.85), (7.00, 0.95), (7.30, 0.90)]
20	[(3.50, 0.85), (5.90, 0.90), (7.00, 0.90), (7.40, 0.80)]

Table 5: Efficiency results for DMUs.

DMU	**Efficiency**	**DMU**	**Efficiency**
1	0.6591	11	1.0000
2	0.4088	12	0.7769
3	0.7242	13	1.0000
4	0.7223	14	1.0000
5	0.7362	15	0.9447
6	0.5421	16	0.4900
7	0.6991	17	0.7473
8	0.4894	18	0.8080
9	0.4279	19	0.8529
10	0.8480	20	0.7016

Example 2. Consider a bank operating in a city with six branches. The management realizes that the system is underperforming, as evidenced by a decline in customer numbers and banking activities. To address this, they propose merging the six branches into three. To determine the most efficient combination, we propose a new approach to this task, examining all possible merger scenarios (choosing three out of six). Experts identify relevant data and important indicators for each scenario. Applying the proposed model, we analyze the results to determine the most efficient option for bank managers.

To solve this problem, we employ a three-stage uncertain DEA model with the following considerations:

- x_1: Operational costs
- x_2: Non-operational costs
- y_1: Income from banking facilities
- y_2: Income from service fees
- y_3: Non-performing facilities
- z_1: Personnel exchanged between branches
- z_2: Money transferred between branches

We assume that all these variables are uncertain in nature because their values are derived through multiple layers of filtering and assessment processes. As a result, the reported figures may not fully represent their true origins or underlying causes. This inherent uncertainty stems from the complexity and variability in data collection, processing, and interpretation. To address this, we rely on expert opinions to construct appropriate distributions for each variable, ensuring a more realistic and reliable depiction of the underlying uncertainty.

The variables in our three-stage uncertain DEA model are defined with the following distribution functions:

- For inputs (x_1 and x_2):
$$\psi_{ij}^{-1(k)}(\alpha) = 0.05a_{ij} + 0.95b_{ij},$$
- For intermediate outputs (z_1 and z_2):
$$\phi_{rj}^{-1(k)}(\alpha) = 0.05c_{rj} + 0.95d_{rj},$$
- For final outputs (y_1, y_2, y_3):
$$\Gamma_{gj}^{-1(k)}(\alpha) = e_{gj} + \left(\frac{\sigma\sqrt{3}}{\pi}\ln\left(\frac{0.95}{0.05}\right)\right).$$

The most efficient combination is C15, with an efficiency score of 0.642. This combination corresponds to the following branches:

- Branch 1: Represents strong performance in operational costs and intermediate outputs.
- Branch 3: Provides high final output, including income from banking facilities and service fees.

Table 6: Inputs (x_1, x_2).

Combination	a_{1j}	b_{1j}	a_{2j}	b_{2j}
C1	10	15	5	8
C2	12	18	6	9
C3	14	20	7	10
C4	11	16	5	9
C5	13	17	6	8
C6	9	14	4	7
C7	10	16	5	8
C8	12	17	6	10
C9	11	18	5	9
C10	13	19	7	11
C11	10	15	4	7
C12	12	17	6	9
C13	11	16	5	8
C14	10	14	5	8
C15	6	10	3	5
C16	14	20	7	10
C17	13	19	6	8
C18	12	18	5	9
C19	11	15	4	7
C20	10	16	5	9

Table 7: Intermediate outputs (z_1, z_2).

Combination	c_{1j}	d_{1j}	c_{2j}	d_{2j}
C1	3	5	2	4
C2	4	6	3	5
C3	5	7	4	6
C4	3	6	2	5
C5	4	7	3	6
C6	3	5	2	4
C7	4	6	3	5
C8	5	8	4	6
C9	3	7	2	5
C10	4	9	3	6
C11	3	5	2	4
C12	4	6	3	5
C13	3	7	2	5
C14	4	8	3	6
C15	2	4	1	3
C16	5	7	4	6
C17	4	8	3	6
C18	3	6	2	5
C19	5	7	4	6
C20	4	6	3	5

Table 8: Final outputs (y_1, y_2, y_3).

Combination	e_{1j}	σ_{1j}	e_{2j}	σ_{2j}	e_{3j}	σ_{3j}
C1	20	0.5	30	0.8	15	0.6
C2	18	0.6	28	0.7	16	0.5
C3	22	0.4	32	0.6	17	0.5
C4	19	0.5	29	0.8	15	0.7
C5	21	0.6	31	0.7	18	0.5
C6	20	0.5	30	0.7	16	0.6
C7	19	0.6	29	0.8	15	0.5
C8	23	0.5	33	0.6	18	0.5
C9	21	0.6	31	0.7	17	0.6
C10	22	0.4	34	0.6	19	0.5
C11	20	0.5	30	0.7	15	0.6
C12	19	0.6	29	0.8	16	0.5
C13	21	0.5	31	0.6	17	0.6
C14	20	0.6	30	0.8	15	0.5
C15	35	0.4	40	0.6	30	0.5
C16	22	0.5	32	0.7	18	0.6
C17	23	0.4	33	0.6	19	0.5
C18	21	0.5	31	0.7	16	0.6
C19	20	0.6	30	0.8	15	0.5
C20	22	0.4	32	0.7	18	0.5

Table 9: Rankings and efficiency.

Rank	Combination	Efficiency
1	C15	0.642
2	C10	0.631
3	C8	0.615
4	C16	0.600
5	C5	0.592
6	C20	0.580
7	C9	0.575
8	C18	0.568
9	C12	0.560
10	C7	0.554
11	C4	0.543
12	C2	0.537
13	C13	0.530
14	C3	0.520
15	C6	0.515
16	C1	0.502
17	C17	0.495
18	C19	0.482
19	C11	0.470
20	C14	0.455

- Branch 5: Balances low operational costs with significant intermediate and final output contributions.

These branches together maximize efficiency while minimizing redundancy, making C15 the optimal choice.

7 Conclusion

This research proposes a systematic approach to evaluating efficiency in complex decision-making environments, especially where data uncertainty poses significant challenge. Traditional DEA models often encounter limitations when dealing with uncertain data due to their dependence on deterministic inputs, reducing their effectiveness in practical scenarios characterized by limited or expert-derived information. To address this gap, this study integrates Liu's uncertainty theory into a three-stage network DEA framework, enhancing the accuracy of efficiency evaluations when historical data is sparse. The validation through two case studies in the banking sector highlights the practical utility of the proposed model. Results demonstrate that uncertainty-aware DEA provides more reliable efficiency assessments for activities such as loan distribution and branch consolidation. By effectively capturing uncertainties driven by expert judgement, the model proves particularly beneficial for applications like financial risk analysis, credit evaluation, and strategic resource distribution. Looking forward, several potential avenues could further expand and refine this framework:

- Integration with machine learning: Merging this uncertainty-based DEA with predictive analytics can bolster efficiency forecasts .
- Dynamic uncertainty modeling: Future research could examine how uncertainties evolve over time, enabling real-time decision-making in dynamic markets.
- Extending this approach to sectors such as healthcare, energy, and supply chain management, areas where expert-driven uncertainty significantly influences efficiency assessments.

By advancing the measurement of efficiency under uncertain conditions, this study contributes to a more adaptable and resilient decision-making process, ensuring organizations to better navigate uncertainty with increased confidence.

Appendix. Detailed Proof of Theorems 6 and 7

Proof of Theorem 6

We consider an uncertain optimization model in which both the objective function and constraints involve uncertain variables with known uncertainty distributions. To derive the deterministic equivalent formulation, we employ inverse uncertainty distributions and expected value operators established in uncertainty theory. Our analysis specifically relies on the following fundamental results:

- Theorem 1 (Inverse Distribution Transformation),
- Theorem 2 (Expected Value of Regular Uncertain Variables)

from Section 3.

Step 1: Objective Function

The objective function in the uncertain model is expressed as:

$$\max E\left[\sum_{r=1}^{s} u_r \varphi_{r0} + \sum_{g=1}^{h} w_g \Gamma_{g0}\right].$$

Applying linearity of expectation and Theorem 2, we obtain:

$$E[\varphi_{r0}] = \int_0^1 \varphi_{r0}^{-1}(\alpha)\, d\alpha,$$

$$E[\Gamma_{g0}] = \int_0^1 \Gamma_{g0}^{-1}(\alpha)\, d\alpha.$$

Consequently, the objective simplifies to:

$$\max \int_0^1 \left(\sum_{r=1}^{s} u_r \varphi_{r0}^{-1}(\alpha) + \sum_{g=1}^{h} w_g \Gamma_{g0}^{-1}(\alpha)\right) d\alpha.$$

Step 2: Constraints

Similarly, the first constraint becomes:

$$E\left[\sum_{i=1}^{m} v_i \psi_{i0} + \sum_{g=1}^{h} w_g \Gamma_{g0}\right] = 1,$$

which translates into:

$$\int_0^1 \left(\sum_{i=1}^{m} v_i \psi_{i0}^{-1}(\alpha) + \sum_{g=1}^{h} w_g \Gamma_{g0}^{-1}(\alpha)\right) d\alpha = 1.$$

For the k-th piecewise constraint, where $k = 1, \ldots, p$, the structure involves differences between weighted uncertain outputs and inputs. Utilizing Theorem 1, and noting the monotonicity properties, the inverse distributions involved are:

- $\varphi_{rj}^{-1(k)}(\alpha)$ and $\Gamma_{gj}^{-1(k)}(\alpha)$ for increasing functions,
- $\psi_{ij}^{-1(k)}(1-\alpha)$ for decreasing functions.

Accordingly, the constraints are formulated as follows:

- For $k = 1$:

$$\int_0^1 \left(\sum_{r=1}^{s(1)} u_r \varphi_{rj}^{-1(1)}(\alpha) + \sum_{g=1}^{h(1)} w_g \Gamma_{gj}^{-1(1)}(\alpha) - \sum_{i=1}^{m(1)} v_i \psi_{ij}^{-1(1)}(1-\alpha)\right) d\alpha \le 0.$$

- For $k = 2, \ldots, p-1$:

$$\int_0^1 \left(\sum_{r=s^{(k-1)}+1}^{s^{(k)}} u_r \varphi_{rj}^{-1(k)}(\alpha) + \sum_{g=h^{(k-1)}+1}^{h^{(k)}} w_g \Gamma_{gj}^{-1(k)}(\alpha)\right) -$$

$$\left(\textstyle\sum_{i=m^{(k-1)}+1}^{m^{(k)}} v_i \psi_{ij}^{-1(k)}(1-\alpha) + \sum_{g=h^{(k-2)}+1}^{h^{(k-1)}} w_g \Gamma_{gj}^{-1(k-1)}(1-\alpha)\right) d\alpha \le 0.$$

Finally, for $k = p$:

$$\int_0^1 \sum_{r=s^{(p-1)}+1}^{s^{(p)}} u_r \phi_{rj}^{-1(p)}(\alpha) - \left(\sum_{i=m^{(p-1)}+1}^{m^{(p)}} v_i \psi_{ij}^{-1(p)}(1-\alpha) + \sum_{g=h^{(p-2)}+1}^{h^{(p-1)}} w_g \Gamma_{gj}^{-1(p-1)}(1-\alpha) \right) d\alpha \leq 0$$

Step 3: Non-Negativity Constraints

The decision variables are constrained to be non-negative:

$$v_i, u_r, w_g \geq \epsilon, \quad \forall i, r, g.$$

Proof of Theorem 7

Since all the uncertain variables in the model are assumed to be positive, their expected values, computed via their inverse uncertainty distributions, are also strictly positive. This positivity ensures that, for the input-oriented DEA model under uncertainty, it is possible to construct a feasible solution explicitly by selecting the weights accordingly.

Define the input weights v_i as:

$$v_i = \frac{1 - \sum_{g=1}^{h} \varepsilon Z_{g0}}{m X_{i0}}, \quad i = 1, \ldots, m,$$

where $\varepsilon > 0$ is a sufficiently small scalar, and Z_{g0} and X_{i0} are the expected values of the respective uncertain variables associated with the decision-making unit (DMU) under evaluation.

In addition, set the output and intermediate output weights uniformly as:

$$w_g = u_r = \varepsilon, \quad g = 1, \ldots, h, \ r = 1, \ldots, s.$$

With these choices, we verify the satisfaction of the model constraints:

Normalization (input) constraint: The input constraint (normalization condition) is satisfied because:

$$E\left[\sum_{i=1}^{m} \frac{1 - \sum_{g=1}^{h} \epsilon(Z_{g0})}{m X_{i0}} X_{i0} + \sum_{g=1}^{h} \epsilon Z_{g0} \right] = 1,$$

due to the specific structure of v_i absorbing the additive ε terms and by simplifying the given expressions

$$E\left[\sum_{i=1}^{m} \frac{1}{m} \right] = 1.$$

Remaining input-output constraints:

For each $p \in \{2, \ldots, P\}$, the expected value of the left-hand side (LHS) of the p-th constraint becomes:

$$E\left[\sum_{r=s^{(p-1)}+1}^{s^{(p)}} \epsilon Y_{rj}^{(p)} + \sum_{g=h^{(p-1)}+1}^{h^{(p)}} \epsilon Z_{gj}^{(p)} - \left(\sum_{i=m^{(p-1)}+1}^{m^{(p)}} \frac{1-\sum_{g=1}^{h} \epsilon(Z_{g0})}{mX_{i0}} X_{ij}^{(p)} + \sum_{g=h^{(p-2)}+1}^{h^{(p-1)}} \epsilon Z_{gj}^{(p-1)} \right)\right] \leq 0.$$

Since $u_r = w_g = \varepsilon$ and $v_i < 1$, each term involving outputs or intermediate outputs is small and the sum of weighted outputs is strictly less than the weighted inputs, making all inequalities satisfied. This logic can be extended for any $p \in \{2, \ldots, P\}$, since in each constraint group the weights remain the same, and the positivity and small magnitude of ε ensures feasibility. Thus, all constraints are satisfied under these weight assignments, and we conclude that the model has at least one feasible solution.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

Funding

The authors conducted this research without any funding, grants, or support.

Competing Interests

The authors declare that they have no competing interests relevant to the content of this paper.

Authors' Contributions

All authors contributed equally to the design of the study, data analysis, and writing of the manuscript, and share equal responsibility for the content of the paper.

References

[1] Ahmadi, O., Mortazavi, S.B., Mahabadi, H.A., Hosseinpouri, M. (2020). "Development of a dynamic quantitative risk assessment methodology using fuzzy DEMATEL-BN and leading indicators", *Process Safety and Environmental Protection*, 142, 15-44, doi:https://doi.org/10.1016/j.psep.2020.04.038.

[2] Boloori, F., Afsharian, M., Pourmahmoud, J. (2016). "Equivalent multiplier and envelopment DEA models for measuring efficiency under general network structures", *Measurement*, 80, 259-269, doi:https://doi.org/10.1016/j.measurement.2015.11.012.

[3] Charnes, A., Cooper, W.W., Rhodes, E. (1978). "Measuring the efficiency of decision making units", *European Journal of Operational Research*, 2(6), 429-444, doi:https://doi.org/10.1016/0377-2217(78)90138-8.

[4] Chen, Y., Liang, L., Zhu, J. (2009). “Equivalence in two-stage DEA approaches”, *European Journal of Operational Research*, 193(2), 600-604, doi:https://doi.org/10.1016/j.ejor.2007.11.040.

[5] Cooper, W.W., Park, K.S., Yu, G. (1999). “IDEA and AR-IDEA: Models for dealing with imprecise data in DEA”, *Management Science*, 45(4), 597-607, doi:https://doi.org/10.1287/mnsc.45.4.597.

[6] Dyson, R.G., Shale, E.A. (2010). “Data envelopment analysis, operational research and uncertainty”, *Journal of the Operational Research Society*, 61(1), 25-34, doi:https://doi.org/10.1057/jors.2009.145.

[7] Ebrahimnejad, A., Tavana, M., Lotfi, F.H., Shahverdi, R., Yousefpour, M. (2014). “A three-stage data envelopment analysis model with application to banking industry”, *Measurement*, 49, 308-319, doi:https://doi.org/10.1016/j.measurement.2013.11.043.

[8] Emrouznejad, A., Tavana, M., Hatami-Marbini, A. (2014). “The state of the art in fuzzy data envelopment analysis”, *Performance Measurement with Fuzzy Data Envelopment Analysis*, 1-45, doi:https://doi.org/10.1007/978-3-642-41372-8_1.

[9] Färe, R., Grosskopf, S., Whittaker, G. (2007). “Network dea”, *Modeling Data Irregularities and Structural Complexities in Data Envelopment analysis*, 209-240, doi:https://doi.org/10.1007/978-0-387-71607-7_12.

[10] Fried, H.O., Lovell, C.K., Schmidt, S.S. (Eds.). (2008). “The measurement of productive efficiency and productivity growth”, *Oxford University Press*, doi:https://doi.org/10.1093/acprof:oso/9780195183528.001.0001.

[11] Ghaffari-Hadigheh, A., Lio, W. (2020). “Network data envelopment analysis in uncertain environment”, *Computers & Industrial Engineering*, 148, 106657, doi:https://doi.org/10.1016/j.cie.2020.106657.

[12] Guo, P., Tanaka, H. (2001). “Fuzzy DEA: A perceptual evaluation method”, *Fuzzy Sets and Systems*, 119(1), 149-160, doi:https://doi.org/10.1016/S0165-0114(99)00106-2.

[13] Guo, P., Tanaka, H. (2008). “Decision making based on fuzzy data envelopment analysis”, *Intelligent Decision and Policy Making Support Systems*, 39-54, doi:https://doi.org/10.1007/978-3-540-78308-4_3.

[14] Hatami-Marbini, A., Emrouznejad, A., Tavana, M. (2011). “A taxonomy and review of the fuzzy data envelopment analysis literature: Two decades in the making”, *European Journal of Operational Research*, 214(3), 457-472., doi:https://doi.org/10.1016/j.ejor.2011.02.001.

[15] Henriques, I.C., Sobreiro, V.A., Kimura, H., Mariano, E.B. (2020). “Two-stage DEA in banks: Terminological controversies and future directions”, *Expert Systems with Applications*, 161, 113632, doi:https://doi.org/10.1016/j.eswa.2020.113632.

[16] Heydari, C., Omrani, H., Taghizadeh, R. (2020). “A fully fuzzy network DEA-Range Adjusted Measure model for evaluating airlines efficiency: A case of Iran”, *Journal of Air Transport Management*, 89, 101923, doi:https://doi.org/10.1016/j.jairtraman.2020.101923.

[17] Hwang, S.N., Kao, T.L. (2008). "Using two-stage DEA to measure managerial efficiency change of non-life insurance companies in Taiwan", *International Journal of Management and Decision Making*, 9(4), 377-401, doi:https://doi.org/10.1504/IJMDM.2008.019362.

[18] Jahanshahloo, G.R., Vencheh, A.H., Foroughi, A.A., Matin, R.K. (2004). "Inputs/outputs estimation in DEA when some factors are undesirable", *Applied Mathematics and Computation*, 156(1), 19-32, doi:https://doi.org/10.1016/S0096-3003(03)00814-2.

[19] Jamshidi, M., Sanei, M., Mahmoodirad, A., Lotfi, F.H., Tohidi, G. (2021). "Uncertain SBM data envelopment analysis model: A case study in Iranian banks", *International Journal of Finance & Economics*, 26(2), 2674-2689, doi:https://doi.org/10.1002/ijfe.1927.

[20] Jemric, I., Vujcic, B. (2002). "Efficiency of banks in Croatia: A DEA approach", *Comparative Economic Studies*, 44, 169-193, doi:https://doi.org/10.1057/ces.2002.13.

[21] Kao, C. (2014). "Network data envelopment analysis: A review", *European Journal of Operational Research*, 239(1), 1-16, doi:https://doi.org/10.1016/j.ejor.2014.02.039

[22] Korostelëv, A.P., Simar, L., Tsybakov, A.B. (1995). "Efficient estimation of monotone boundaries", *The Annals of Statistics*, 476-489, doi:https://doi.org/10.1214/aos/1176324531

[23] Lertworasirikul, S., Fang, S.C., Joines, J.A., Nuttle, H.L. (2003). "Fuzzy data envelopment analysis (DEA): A possibility approach", *Fuzzy Sets and Systems*, 139(2), 379-394, doi:https://doi.org/10.1016/S0165-0114(02)00484-0.

[24] Lio, W., Liu, B. (2018). "Uncertain data envelopment analysis with imprecisely observed inputs and outputs", *Fuzzy Optimization and Decision Making*, 17, 357-373, doi:https://doi.org/10.1007/s10700-017-9276-x.

[25] Liu, B. (2009). "Some research problems in uncertainty theory", *Journal of Uncertain Systems*, 3(1), 3-10.

[26] Liu, B. (2010). "Uncertainty theory: A new paradigm for handling uncertain systems", *Journal of Uncertain Systems*, 3-10, doi:https://doi.org/10.1109/TKDE.2012.179.

[27] Mergoni, A., Emrouznejad, A., De Witte, K. (2024). "Fifty years of data envelopment analysis." *European Journal of Operational Research*, 326(3), 389-412, doi:https://doi.org/10.1016/j.ejor.2024.12.049.

[28] Mohmmad Nejad, Z., Ghaffari-Hadigheh, A. (2018). "A novel DEA model based on uncertainty theory", *Annals of Operations Research*, 264, 367-389, doi:https://doi.org/10.1007/s10479-017-2652-7.

[29] Olesen, O.B., Petersen, N.C. (2016). "Stochastic data envelopment analysis—A review", *European Journal of Operational Research*, 251(1), 2-21, doi:https://doi.org/10.1016/j.ejor.2015.07.058.

[30] Paul, C.J.M., Nehring, R. (2005). "Product diversification, production systems, and economic performance in US agricultural production", *Journal of Econometrics*, 126(2), 525-548, doi:https://doi.org/10.1016/j.jeconom.2004.05.012.

[31] Pourbabagol, H., Amiri, M., Taghavifard, M.T., Hanafizadeh, P. (2023). "A new fuzzy DEA network based on possibility and necessity measures for agile supply chain performance evaluation: A case study", *Expert Systems with Applications*, 220, 119552, doi:https://doi.org/10.1016/j.eswa.2023.119552.

[32] Pourmahmoud, J., Bafekr Sharak, N. (2018). "Measuring cost efficiency with new fuzzy DEA models", *International Journal of Fuzzy Systems*, 20(1), 155-162, doi:https://doi.org/10.1007/s40815-017-0316-z.

[33] Pourmahmoud, J., Bagheri, N. (2021). "Providing an uncertain model for evaluating the performance of a basic two-stage system", *Soft Computing*, 25, 4739-4748, doi:https://doi.org/10.1007/s00500-020-05481-8.

[34] Pourmahmoud, J., Bagheri, N. (2023). "Uncertain malmquist productivity index: An application to evaluate healthcare systems during COVID-19 pandemic", *Socio-Economic Planning Sciences*, 87, 101522, doi:https://doi.org/10.1016/j.seps.2023.101522.

[35] Pourmahmoud, J., Eini, M., Darvishi, D., Mehrabian, S. (2025). "A new approach in DEA with three-parameter interval grey numbers: investigating the effect of the drug-TPA on stroke patients", *Grey Systems: Theory and Application*, 15(3), 547-573, doi:https://doi.org/10.1108/GS-08-2024-0097.

[36] Puri, J., Yadav, S.P. (2014). "A fuzzy DEA model with undesirable fuzzy outputs and its application to the banking sector in India", *Expert Systems with Applications*, 41(14), 6419-6432, doi:https://doi.org/10.1016/j.eswa.2014.04.013.

[37] Ratner, S.V., Shaposhnikov, A.M., Lychev, A.V. (2023). "Network DEA and its applications (2017–2022): A systematic literature review", *Mathematics*, 11(9), 2141, doi:https://doi.org/10.3390/math11092141.

[38] Rostamy-Malkhalifeh, M., Poudineh, E., Payan, A. (2018). "A fully fuzzy method of network data envelopment analysis for assessing revenue efficiency based on ranking functions", *Control and Optimization in Applied Mathematics*, 3(2), 77-96, doi:https://doi.org/10.30473/coam.2019.46226.1115.

[39] Shakouri, R., Salahi, M., Kordrostami, S. (2019). "Stochastic p-robust approach to two-stage network DEA model", *Quantitative Finance and Economics*, 3(2), 315-346, doi:https://doi.org/10.3934/QFE.2019.2.315.

[40] Simar, L. (1996). "Aspects of statistical analysis in DEA-type frontier models", *Journal of Productivity Analysis*, 7, 177-185, doi:https://doi.org/10.1007/BF00157040.

[41] Simar, L., Wilson, P.W. (2000). "A general methodology for bootstrapping in non-parametric frontier models", *Journal of Applied Statistics*, 27(6), 779-802, doi:https://doi.org/10.1080/02664760050081951.

[42] Simar, L., Wilson, P.W. (2011). "Two-stage DEA: Caveat emptor", *Journal of Productivity Analysis*, 36, 205-218, doi:https://doi.org/10.1007/s11123-011-0230-6.

[43] Soleimani-Damaneh, M., Jahanshahloo, G.R., Abbasbandy, S. (2006). “Computational and theoretical pitfalls in some current performance measurement techniques; and a new approach”, *Applied Mathematics and Computation*, 181(2), 1199-1207, doi:https://doi.org/10.1016/j.amc.2006.01.085.

[44] Thanassoulis, E. (1999). “Data envelopment analysis and its use in banking”, *Interfaces*, 29(3), 1-13, doi:https://doi.org/10.1287/inte.29.3.1.

[45] Wang, C.N., Nguyen, T.L., Dang, T.T. (2021). “Analyzing operational efficiency in real estate companies: An application of GM (1, 1) and dea malmquist model”, *Mathematics*, 9(3), 202, doi:https://doi.org/10.3390/math9030202.

[46] Wen, M., Guo, L., Kang, R., Yang, Y. (2014). “Data envelopment analysis with uncertain inputs and outputs”, *Journal of Applied Mathematics*, 2014(1), 307108, doi:https://doi.org/10.1155/2014/307108.

[47] Yeh, C.C., Chi, D.J., Hsu, M.F. (2010). “A hybrid approach of DEA, rough set and support vector machines for business failure prediction”, *Expert Systems with Applications*, 37(2), 1535-1541, doi:https://doi.org/10.1016/j.eswa.2009.06.088.

[48] Zadeh, L. (1965). “Fuzzy sets”, *Inform Control*, 8, 338-353, doi:https://doi.org/10.1016/S0019-9958(65)90241-X.

[49] Zadeh, L.A., Klir, G.J., Yuan, B. (1996). “Fuzzy sets, fuzzy logic, and fuzzy systems: Selected papers”, (Vol. 6). *World Scientific*, doi:https://doi:10.1142/2895doi:10.1142/2895.

[50] Zhou, Z., Lin, L., Xiao, H., Ma, C., Wu, S. (2017). “Stochastic network DEA models for two-stage systems under the centralized control organization mechanism”, *Computers & Industrial Engineering*, 110, 404-412, doi:https://doi.org/10.1016/j.cie.2017.06.005.

[51] Zhou, X., Xu, Z., Chai, J., Yao, L., Wang, S., Lev, B. (2019). “Efficiency evaluation for banking systems under uncertainty: A multi-period three-stage DEA model”, *Omega*, 85, 68-82, doi:https://doi.org/10.1016/j.omega.2018.05.012.