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Optimal Control of the Van der Pol Oscillator Problem by Using Orthogonal Polynomial-Based Optimization

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Abstract. The orthogonal polynomials approximation method is widely regarded as a highly effective and versatile technique for solving optimal control problems in nonlinear systems. This powerful approach has found extensive applications in both theoretical research and practical engineering, demonstrating its capability to address complex dynamical behaviors. In this paper, we thoroughly investigate the optimal control problem of the Van der Pol oscillator, a classic nonlinear system with broad scientific and engineering relevance. The proposed solution follows two distinct and systematic steps. First, the state and control functions are approximated by linear combinations of shifted Chelyshkov polynomials, whose coefficients are treated as unknown parameters to be determined. Second, the resulting transformed problem is formulated as a nonlinear optimization problem and efficiently solved using advanced numerical optimization tools implemented in Matlab. To demonstrate the accuracy and robustness of the proposed approach, we present and analyze numerical results across several representative scenarios.

Keywords. Nonlinear optimization, Numerical approximation, Nonlinear control system, Chelyshkov polynomials.

MSC. 90C30; 41A10.

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1 Introduction

The Van der Pol oscillator is a non-conservative system characterized by nonlinear damping and is widely studied in the context of dynamical systems. The Dutch electrical engineer Balthazar Van der Pol (1889–1959) introduced this model, which is governed by the second-order nonlinear differential equation:

$$\ddot{x}(t) - \mu(1 - x^2(t))\dot{x}(t) + x(t) = 0, \tag{1}$$

where μ is a scalar parameter representing the strength of the damping.

This equation serves as a fundamental model for oscillatory processes in electronics, biology, sociology, physics, and economics [10]. In this study, we focus on an optimal control formulation of the Van der Pol oscillator given by:

$$\text{Minimize} \quad J = \frac{1}{2}\int_0^T \left(x^2 + \dot{x}^2 + u^2\right) dt, \tag{2}$$

subject to

$$\ddot{x}(t) - \epsilon\mu(1 - x^2(t))\dot{x}(t) + \mu^2 x(t) = u(t), \tag{3}$$

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \tag{4}$$

$$x(T) = 0, \quad \dot{x}(T) = 0, \tag{5}$$

where constraint (3) represents the controlled Van der Pol oscillator with a perturbation parameter ϵ, and $x(t)$ and $u(t)$ denote the state and control variables, respectively.

Various numerical methods have been developed for solving such optimal control problems, including Pontryagin's maximum principle [11], Bellman's dynamic programming [2], piecewise polynomial parameterization techniques [12, 13, 14], the Chebyshev approach [16], and other computational strategies.

In this paper, we propose a direct numerical method based on orthogonal polynomial series expansions. Specifically, the state variable $x(t)$ and the control function $u(t)$ are approximated by Chelyshkov polynomial series with undetermined coefficients. By leveraging the orthogonality of Chelyshkov polynomials, the original optimal control problem is converted into a finite-dimensional nonlinear optimization problem. Solving this optimization problem yields accurate approximations of the state and control trajectories.

While our method emphasizes polynomial-based representations, it is worth noting that Adaptive Dynamic Programming (ADP) has recently emerged as a powerful alternative for tackling nonlinear optimal control problems. ADP methods—such as neural network-based approaches and policy iteration—enable adaptive control in high-dimensional or uncertain environments without requiring explicit system models. For further information on ADP applications, we refer the reader to [3, 4, 5, 17].

Although the present work focuses on the Van der Pol oscillator, optimal control theory has demonstrated effectiveness in diverse domains such as epidemic modeling [7], cholera transmission [9], and COVID-19 intervention strategies [8]. These applications highlight the broad potential of optimal control methods in addressing real-world challenges. Integrating the techniques developed here with those used in public health optimization may be a promising direction for future research.

In summary, this paper demonstrates the efficacy of a new class of orthogonal polynomials, Chelyshkov polynomials, in addressing a benchmark of nonlinear optimal control problems. The proposed approach provides a flexible and computationally efficient alternative to traditional methods.

2 The Chelyshkov Algorithm

2.1 Chelyshkov Polynomials and Properties

The Chelyshkov polynomials, introduced by Chelyshkov in [6], are a class of orthogonal polynomials defined by the following formula:

$$P_{N,n}(x) = \sum_{i=0}^{N-n} (-1)^i \binom{N-n}{i} \binom{N+n+i+1}{N-n} x^{n+i}, \quad n = 0, 1, \dots, N. \tag{6}$$

These polynomials are orthogonal on the interval $[0, 1]$ with respect to the weight function $\omega(x) = 1$, and satisfy the following orthogonality condition:

$$\int_0^1 P_{N,n}(x) P_{N,m}(x)\, dx = \begin{cases} \dfrac{1}{n+m+1}, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases} \tag{7}$$

The behavior of Chelyshkov polynomials for $N = 4$ and $n = 0, 1, \dots, 4$ is illustrated in Figure 1.

Remark 1. To apply Chelyshkov polynomials on the interval $[0, T]$, we define the shifted Chelyshkov polynomials by substituting $x = \frac{t}{T}$ into Equation (6).

Let $f \in L^2[0, T]$ and define the space $A = \text{span}\{P_{N,0}, P_{N,1}, \dots, P_{N,N}\}$, where $\{P_{N,n}\}_{n=0}^N$ are the shifted Chelyshkov polynomials of degree at most N. Since A is a finite-dimensional subspace of $L^2[0, T]$, the function f has a unique best approximation $h^\star \in A$ such that

$$\|f - h^\star\|_2 \leq \|f - h\|_2, \quad h \in A.$$

Therefore, there exist unique coefficients $\delta_0, \delta_1, \dots, \delta_N$ such that

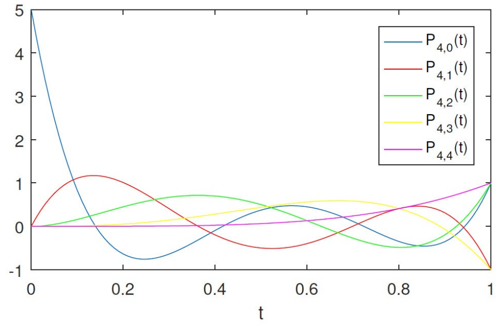

Figure 1: Chelyshkov polynomials for $N = 4$ and $n = 0, 1, 2, 3, 4$.

$$f(t) \approx h^{\star}(t) = \sum_{n=0}^{N} \delta_n P_{N,n}(t),$$

where the coefficients δ_n can be approximated by

$$\delta_n = \int_0^T f(t) P_{N,n}(t)\, dt, \quad n = 0, 1, \dots, N.$$

Lemma 1. Suppose that the function $f : [0, T] \to \mathbb{R}$ is $(N + 1)$ times continuously differentiable, i.e., $f \in C^{N+1}[0, T]$, and let $A = \text{span}\{P_{N,0}, P_{N,1}, \dots, P_{N,N}\}$. Define

$$M = \max_{t \in [0,T]} |f^{(N+1)}(t)|.$$

If $h^{\star}$ is the best approximation to f from A, then the following error bound holds:

$$\|f - h^{\star}\|_2 \leq \frac{M}{(N+1)!} \sqrt{\frac{T^{2N+3}}{2N+3}}.$$

Proof. See [1]. □

2.2 Algorithm

To obtain an approximate solution of the controlled Van der Pol oscillator optimal control problem, the Chelyshkov series of order m is used for the state and control variables as follows:

$$x_m(t) = \sum_{i=0}^{m} \eta_i P_{m,i}(t), \tag{8}$$

$$u_m(t) = \sum_{i=0}^{m} \lambda_i P_{m,i}(t), \tag{9}$$

where $\eta = (\eta_0, \eta_1, \dots, \eta_m)$ and $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_m)$ are the unknown coefficients.

Since $x_m(t)$ is a polynomial in terms of the Chelyshkov series, its first and second derivatives can be calculated as follows:

$$\dot{x}_m(t) = \sum_{i=0}^{m} \eta_i \dot{P}_{m,i}(t), \tag{10}$$

$$\ddot{x}_m(t) = \sum_{i=0}^{m} \eta_i \ddot{P}_{m,i}(t). \tag{11}$$

Substituting Equations (8)–(11) into the optimal control problem (2)-(5) yields:

$$\text{Minimize} \quad J_m = \frac{1}{2}\int_0^T \left(x_m^2(t) + \dot{x}_m^2(t) + u_m^2(t)\right) dt, \tag{12}$$

subject to the dynamic constraint:

$$\ddot{x}_m(t) - \epsilon\mu(1 - x_m^2(t))\dot{x}_m(t) + \mu^2 x_m(t) = u_m(t), \tag{13}$$

and boundary conditions:

$$x_m(0) = x_0, \tag{14}$$

$$\dot{x}_m(0) = \dot{x}_0, \tag{15}$$

$$x_m(T) = 0, \tag{16}$$

$$\dot{x}_m(T) = 0. \tag{17}$$

To compute J_m using optimization tools in Matlab, it is necessary to express the performance index and all constraints solely in terms of the coefficients η and λ. For this purpose, we multiply both sides of (13) by $P_{m,i}(t)$ for $i = 0, 1, \ldots, m$ and employ the orthogonality property to obtain:

$$F_i(\eta, \lambda) = \int_0^T P_{m,i}(t)\left(\ddot{x}_m(t) - \epsilon\mu(1 - x_m^2(t))\dot{x}_m(t) + \mu^2 x_m(t) - u_m(t)\right) dt = 0, \tag{18}$$

for $i = 0, 1, \ldots, m$.

The functions F_i are linear in λ but nonlinear in η. Consequently, the optimal control problem is reduced to the following constrained nonlinear programming problem:

$$\text{Minimize} \quad J_m(\eta, \lambda) \tag{19}$$

$$\text{subject to} \quad F_i(\eta, \lambda) = 0, \qquad i = 0, 1, \ldots, m, \tag{20}$$

$$\sum_{i=0}^{m} \eta_i P_{m,i}(0) = x_0, \tag{21}$$

$$\sum_{i=0}^{m} \eta_i \dot{P}_{m,i}(0) = \dot{x}_0, \tag{22}$$

$$\sum_{i=0}^{m} \eta_i P_{m,i}(T) = 0, \tag{23}$$

$$\sum_{i=0}^{m} \eta_i \dot{P}_{m,i}(T) = 0. \tag{24}$$

Finally, by solving this constrained nonlinear programming problem and substituting the obtained values of η and λ into Equations (8) and (9), approximate solutions for $x(t)$ and $u(t)$ are obtained.

The interior-point method is employed to solve this nonlinear optimization problem. For implementation, Matlab's `fmincon` function is used. This function is a gradient-based optimization method that requires both the objective and constraint functions to be continuous and differentiable.

3 Numerical Results

The results for Chelyshkov approximations of different orders m with

$$T = 2, \quad \epsilon = 0.15, \quad \mu = 1, \quad x(0) = 0.5, \quad \dot{x}(0) = -0.5,$$

are reported as follows.

In Table 1, the Chelyshkov solutions and J_m for $m = 4, 5, 7, 8$ are presented. It can be seen that as the value of m increases, the value of J_m decreases. In particular, the Chelyshkov approximation of order $m = 7$ compared to that of $m = 8$ yields a difference of $|J_8 - J_7| \approx 7.99 \times 10^{-12}$.

Approximated state and control variables for different cases of m are shown in Figures 2 and 3. To test the accuracy of this approximate method, additional values for the parameters T, μ, ϵ, x_0, and $\dot{x}_0$ have also been used.

The Chebyshev method, another direct method proposed in [15], is used for comparison. In this method, $x(t)$ and $u(t)$ are approximated as follows:

$$x_m(t) = \frac{1}{2}\eta_0 T_0(t) + \sum_{n=1}^{m} \eta_n T_n(t), \tag{25}$$

$$u_m(t) = \frac{1}{2}\lambda_0 T_0(t) + \sum_{n=1}^{m} \lambda_n T_n(t), \tag{26}$$

where $\{T_0, T_1, \ldots, T_m\}$ are Chebyshev polynomials. Similar to the Chelyshkov approximation, the considered nonlinear optimal control problem becomes a nonlinear optimization problem by using Equations (25) and (26). The results obtained by the Chebyshev method are shown in Table 2.

Table 1: The chelyshkov approximations of various orders for the Van der Pol oscillator.

	$m = 4$	$m = 5$	$m = 7$	$m = 8$
η_0	0.100000	0.083333	0.062500	0.055555
η_1	0.250000	0.221429	0.175595	0.158333
η_2	0.257044	0.275324	0.253311	0.236294
η_3	0.137327	0.220513	0.276080	0.274710
η_4	0.030283	0.106145	0.237504	0.264888
η_5		0.022861	0.154460	0.210147
η_6			0.066420	0.129559
η_7			0.013601	0.054012
η_8				0.010907
λ_0	0.125982	0.099747	0.074809	0.066496
λ_1	0.384209	0.313761	0.230393	0.203668
λ_2	0.602810	0.531102	0.395342	0.348713
λ_3	0.677035	0.669809	0.551474	0.493038
λ_4	0.406494	0.634328	0.659629	0.613646
λ_5		0.387107	0.676416	0.679288
λ_6			0.575041	0.662218
λ_7			0.351367	0.548488
λ_8				0.338226
J_m	0.358230043954352	0.358092136163495	0.358092104096317	0.358092104088319

A comparison of Tables 3 and 4 shows that the solution obtained by the Chelyshkov method is comparable to that obtained by the Chebyshev method. For illustration, we can express the state variable and the control function in terms of Chelyshkov and Chebyshev polynomials based on the results of Tables 1 and 2. For example, for $m = 4$:

1. $x(t)$ and $u(t)$ using Chelyshkov approximation from Table 1:

$$x_4(t) = 0.1P_{40} + 0.25P_{41} + 0.257044P_{42} + 0.137327P_{43} + 0.030283P_{44},$$

$$u_4(t) = 0.125982P_{40} + 0.384209P_{41} + 0.602810P_{42} + 0.677035P_{43} + 0.406494P_{44}.$$

2. $x(t)$ and $u(t)$ using Chebyshev approximation from Table 2:

$$x_4(t) = \frac{0.358501}{2}T_0(t) - 0.250000T_1(t) + 0.073500T_2(t) + 0.000000T_3(t) - 0.002750T_4(t),$$

Table 2: The chebyshev approximations of various orders for the Van der Pol oscillator.

	$m = 4$	$m = 7$	$m = 10$
η_0	0.358501	0.35849949	0.3584995225
η_1	-0.250000	-0.24950648	-0.2495064561
η_2	0.073500	0.07349670	0.0734966309
η_3	0.000000	-0.00074168	-0.0007416369
η_4	-0.002750	-0.00274427	-0.0027442061
η_5		0.00024910	0,0002490091
η_6		-0.00000218	-0.0000021572
η_7		-0.00000094	-0.0000009428
η_8			-0.0000000367
η_9			0.0000000266
η_{10}			0.0000000078
λ_0	0.836200	0.83620247	0.8362001016
λ_1	-0.282980	-0.27069736	-0.2706901511
λ_2	-0.063034	-0.06290773	-0.0629103261
λ_3	0.004951	0.02388213	0.0238903519
λ_4	-0.002983	-0.00360335	-0.0036078890
λ_5		-0.00001130	0.0000008078
λ_6		0.00006481	0.0000615566
λ_7		-0.00001349	-0.0000057747
η_8			0.0000021472
η_9			0.0000006810
η_{10}			-0.0000000936
J_m	0.358233	0.35809209	0.3580921041

$$u_4(t) = \frac{0.836200}{2} T_0(t) - 0.282980 T_1(t) - 0.063034 T_2(t) + 0.004951 T_3(t) - 0.002983 T_4(t).$$

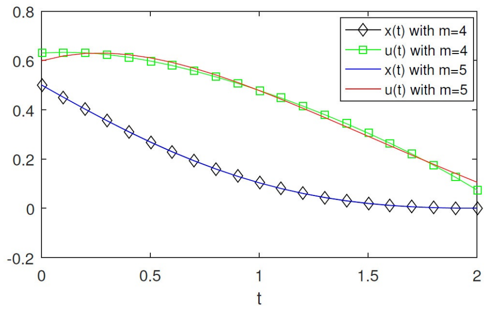

Figure 2: Approximation of state and control functions with $m = 4, 5$.

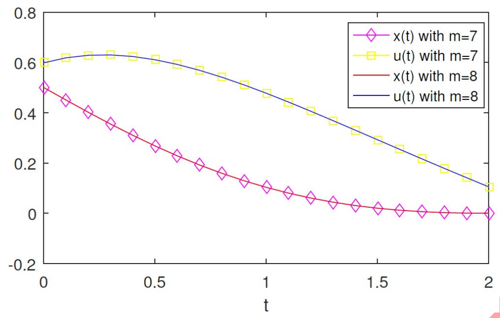

Figure 3: Approximation of state and control functions with $m = 7, 8$.

Table 3: Comparison of numerical results of two methods for state variable $x(t)$.

t	$(Chelyshkov)_{m=4}$	$(Chebyshev)_{m=4}$	$(Chelyshkov)_{m=7}$	$(Chebyshev)_{m=7}$
0.0	0.500000000000000	0.500000500000000	0.500000000000000	0.500000935000000
0.2	0.402148204004908	0.402149300000000	0.401736625218565	0.401736927602880
0.4	0.310986916361192	0.310989300000000	0.310012515446059	0.310012511025600
0.6	0.229473555137834	0.229477300000000	0.228359055029722	0.228358693286080
0.8	0.159720561812682	0.159725300000000	0.159001447881122	0.159000898683840
1.0	0.102995401272441	0.103000500000000	0.103011417094712	0.103010955000000
1.2	0.059720561812682	0.059725300000000	0.060463993745922	0.060463749522880
1.4	0.029473555137834	0.029477300000000	0.030594724540929	0.030594612216000
1.6	0.010986916361192	0.010989300000000	0.011953627990215	0.011953484345280
1.8	0.002148204004908	0.002149300000000	0.002552228778045	0.002551930882240
2.0	-0.000000000000000	0.000000500000000	0.000000000000000	-0.000000945000000

Table 4: Comparison of numerical results of two methods for control function $u(t)$.

t	$(Chelyshkov)_{m=4}$	$(Chebyshev)_{m=4}$	$(Chelyshkov)_{m=7}$	$(Chebyshev)_{m=7}$
0.0	0.629911775601662	0.630112000000000	0.598477916757448	0.598508475000000
0.2	0.631061153337303	0.631092497600000	0.628435169155285	0.628429840955200
0.4	0.612724427083472	0.612686921600000	0.623592047590547	0.623587303570880
0.6	0.579099578387002	0.579053185600000	0.592026634055332	0.592030365860160
0.8	0.533459609145372	0.533432825600000	0.541150872688548	0.541157711200960
1.0	0.478152541606711	0.478151000000000	0.477337840989421	0.477340805000000
1.2	0.414601418369792	0.414616489600000	0.405758958519094	0.405756197284800
1.4	0.343304302384040	0.343321697600000	0.330374092994909	0.330369958876480
1.6	0.263834276949525	0.263842649600000	0.254017523681979	0.254017683796160
1.8	0.174839445716966	0.174838993600000	0.178522721986642	0.178523490557760
2.0	0.074042932687728	0.074054000000000	0.104828909156403	0.104801455000000

4 Conclusion

This paper presented a direct numerical scheme for the optimal control of the Van der Pol oscillator based on finite Chelyshkov-polynomial series. By projecting the state and control trajectories onto an orthogonal Chelyshkov basis, the original two-point boundary-value problem was converted into a finite-dimensional nonlinear program that can be solved efficiently with standard interior-point solvers. The numerical experiments confirm three key findings.

- First, the Chelyshkov expansion yields highly accurate state and control approximations: for instance, raising the polynomial order from $m = 7$ to $m = 8$ changed the objective value by only 8×10^{-12}.
- Second, when benchmarked against a Chebyshev-based collocation scheme, the proposed method achieves comparable—or slightly superior—accuracy with fewer basis functions and lower computational cost (Tables 3–4).
- Third, the procedure naturally enforces both standard and multipoint boundary conditions, giving it a structural simplicity that is often lacking in indirect approaches such as Pontryagin's Maximum Principle or dynamic programming.

Owing to these advantages, the Chelyshkov framework is well suited to a broad class of nonlinear optimal-control problems. Future research will focus on

- extending the algorithm to high-dimensional systems that arise in epidemiology, power-system dynamics, and robotic motion planning;
- incorporating parameter uncertainty and measurement noise through robust or stochastic formulations; and
- combining Chelyshkov collocation with adaptive-dynamic-programming ideas to cope with partially unknown system models.

Taken together, the results demonstrate that Chelyshkov polynomials constitute a flexible, accurate, and computationally attractive alternative to more traditional polynomial bases in nonlinear optimal control.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

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Competing Interests

The author declares that there are no competing interests relevant to the content of this paper.

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