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Existence and Uniqueness of High-Order Caputo Fractional Boundary Value Problems under General Non-Local Multi-Point Conditions: Analytical and Semi-Analytical Approaches

Salam Mcheik[1], Elyas Shivanian[1], Youssef El Seblani[2]

[1]Department of Applied Mathematics, Imam Khomeini International University, Qazvin, 34149-16818, Iran.
[2]Khawarizmi Laboratory of Mathematics and Applications (KALMA), Faculty of Science, Lebanese University, Hadath-Beirut, Lebanon.

Correspondence:
Elyas Shivanian
E-mail:
shivanian@sci.ikiu.ac.ir

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Abstract. In this paper, we investigate the existence and uniqueness of solutions for a high-order boundary value problem involving non-integer derivatives, specifically utilizing the Caputo fractional derivative. The problem is subject to non-local boundary conditions. To tackle this, we introduce the fractional Green's function as an analytical tool. The Banach contraction fixed-point theorem serves as the fundamental method to establish our main results. To support the theoretical findings, we provide illustrative examples. Furthermore, we develop a numerical semi-analytical approach to approximate the unique solution with the desired accuracy.

Keywords. Caputo fractional derivatives, High-order boundary value problems, Nonlocal boundary conditions, Fractional Green's function, Fixed point theory.

MSC. 34B10;34B15.

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1 Introduction

Fractional calculus, which generalizes classical calculus to non-integer order derivatives and integrals, has attracted significant attention due to its ability to capture memory and hereditary properties inherent in various complex systems. This mathematical framework has been effectively applied to model phenomena in a wide range of disciplines such as fluid dynamics, bioengineering, anomalous diffusion, population dynamics, and control theory [3, 12, 14, 16, 17, 21, 31]. For instance, anomalous diffusion processes, which deviate from classical Brownian motion, are elegantly described using fractional-order models [1, 2]. Similarly, fractional models have enhanced our understanding of heat transport in biological tissues, particularly in the human head [12, 13, 28].

Furthermore, fractional models have been successfully applied in epidemiological and biological systems, showcasing their effectiveness in capturing complex interactions. For example, fractional-order models have been used to study the dynamics of the Zika virus with mutation [4], gyrotactic microorganism transport in Maxwell nanofluids [36], and the spread of Hepatitis B with asymptomatic carriers incorporating the Atangana–Baleanu fractional operator [20]. These studies highlight the flexibility of fractional operators in modeling real-world biological scenarios with memory and nonlocal effects.

The mathematical modeling of boundary value problems (BVPs) involving fractional derivatives has become a major area of interest. Particularly, nonlocal and multi-point BVPs governed by Caputo fractional derivatives have shown remarkable applications in describing systems with spatial memory and distributed effects [8, 18, 21, 22]. Several recent works have focused on analyzing the solvability of such problems under various boundary conditions. For example, Smirnov [33] utilized Green's function techniques to address third-order nonlocal BVPs, while Plotnikov et al. [26] developed existence results for Volterra-Hammerstein integral equations with set-valued mappings.

A central tool in establishing the existence and uniqueness of solutions to such fractional BVPs is the theory of fixed points. Various fixed point theorems, including those of Banach and Schauder, have been employed to prove the solvability of nonlinear fractional problems [5, 6, 7, 9, 10, 11, 24, 26]. For instance, the references [8, 12, 25, 32] proposed a parallel LS-SVM approach for simulating fractional population models, demonstrating the computational effectiveness of such methods. Additionally, Rehman and Khan [30] investigated multi-point boundary conditions for fractional differential equations, offering sufficient conditions for unique solvability.

2 Preliminaries

This section presents the foundational definitions and the specific problem studied in this work. In [33], the following boundary value problem was considered:

$$\begin{cases} z''' + \Phi(\tau, z(\tau)) = 0, & \tau \in (0,1), \\ z(0) = z'(0) = 0, & z(1) = \beta z(\xi), \end{cases} \tag{1}$$

where $0 < \xi < 1$, $\beta \in \mathbb{R}$, and $\Phi \in C([0,1] \times \mathbb{R}, \mathbb{R})$ with $\Phi(\tau, 0) \neq 0$.

In [3], a fractional boundary value problem of the following form was examined:

$$\begin{cases} {}^{C}D_t^{\alpha} v + g\left(t, v(t), v'(t), v''(t), v'''(t)\right) = 0, & t \in (0,1), \\ v(0) = v''(0) = v'''(0) = 0, \\ \mu v'(0) + (1-\mu) v'(1) = \int_0^1 v(s)\, ds. \end{cases} \tag{2}$$

In this paper, we aim to extend the previously established findings by employing a higher-order fractional derivative defined in the Caputo framework, instead of the standard third-order derivative operator z'''. For basic terminology and fundamental aspects of fractional calculus of non-integer orders, the reader is directed to [23]. The primary objective of this study is to prove the existence of a unique solution to a class of high-order fractional boundary value problems.

We study a nonlinear fractional differential equation driven by the Caputo derivative:

$$ {}_{c}D_t^{\nu} \psi(t) + g(t, \psi(t), \psi'(t), \psi''(t)) = 0, \qquad q-1 < \nu \leq q, \tag{3}$$

under the following boundary conditions:

$$\begin{cases} \psi(0) = \psi'(0) = \dots = \psi^{(q-2)}(0) = 0, \\ \psi(1) = \gamma_1 \psi(\xi) + \gamma_2 \psi'(\xi) + \gamma_3 \psi''(\xi), \end{cases}$$

where

$$q-1 < \nu \leq q, \quad q \geq 3, \quad 0 < \xi < 1, \quad \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}.$$

Here, ${}_{c}D_t^{\nu}\psi$ denotes the Caputo fractional derivative, and $g \in C([0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ with the condition $g(t, 0, 0, 0) \neq 0$.

Novelty Compared to Previous Studies

Compared to the work of Smirnov [33], which focused on linear third-order boundary value problems with specific three-point nonlocal conditions, our approach generalizes the problem to high-order nonlinear Caputo fractional equations with more flexible nonlocal boundary conditions that involve multiple derivatives. Furthermore, while Soltani [3] considered two-point nonlocal fractional boundary problems, our study handles three-point nonlocal constraints and derives an explicit representation of the solution using a Green's function framework, which is also analytically estimated. These extensions enhance both the theoretical scope and computational feasibility of the method.

Numerous researchers (see [25, 26, 27]) have investigated boundary value problems characterized by non-local and multi-point constraints. These types of boundary restrictions are encountered in many physical models, including applications in fluid dynamics, wave phenomena, and other domains in physics (refer to [29, 30] for more details). Such configurations may include controllers at the boundary which serve to inject or regulate energy, and can be paired with sensors or measurement devices located at internal points of the domain.

Within the framework of third-order differential systems, when the derivative of acceleration is accounted for, the resulting mathematical model is commonly referred to as a jerk equation, representing the third-order time derivative of position. These equations are of practical relevance in engineering systems such as vehicle dynamics where managing sudden acceleration changes is important. Furthermore, such third-order models can be seen as generalized cases of fractional-order systems when the order approaches an integer.

Several works have explored the existence and uniqueness of solutions to nonlinear boundary value problems, particularly with multi-point structures. Substantial contributions include those of Rehman and Khan [30], Mehmood and Ahmad [23], and Rao and Alesemi. [28]. For instance, the study in [33] addresses fractional-order boundary problems incorporating nonlocal and multi-point constraints. Their approach utilized fixed point theorems such as those proposed by Schauder and Krasnoselskii to guarantee the existence of solutions.

The remainder of the manuscript is organized as follows. In Section 3, we develop the fractional Green's function by using an integral formulation combined with additional assumptions. Section 4 contains the precise construction and mathematical analysis of the Green function. Section 5 provides a rigorous proof of our main existence and uniqueness theorem for the solution to the formulated problem. Section 6 offers illustrative examples that support the theoretical findings. Section 7 discusses the numerical strategy and shows the computational results through tables and visual representations. Finally, Section 9 concludes the paper with a concise summary.

3 Construction of the Green's Function

We initiate our analysis by formulating and deriving the expression of the Green's kernel associated with the following two-point fractional boundary value problem:

$$\begin{cases} {}_cD_t^{\nu}u(t) + h(t) = 0, & t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(q-2)}(0) = 0, & u(1) = 0, \end{cases} \tag{4}$$

where ${}_cD_t^{\nu}$ denotes the Caputo fractional derivative of order ν, with $q-1 < \nu \leq q$ and $q \in \mathbb{N}$. Subsequently, we focus our attention on a fractional differential equation subject to three-point boundary conditions, given as follows:

$$\begin{cases} {}_cD_t^{\nu}\psi(t) + h(t) = 0, \quad t \in (0,1), \\ \psi(0) = \psi'(0) = \cdots = \psi^{(q-2)}(0) = 0, \\ \psi(1) = \gamma_1\psi(\xi) + \gamma_2\psi'(\xi) + \gamma_3\psi''(\xi), \end{cases} \tag{5}$$

where $0 < \xi < 1$, and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ are given constants.

The solution of problem (5) can be represented as:

$$\psi(t) = u(t) + u(\xi)\sum_{j=0}^{q-1}\lambda_j t^j, \quad t \in (0,1), \tag{6}$$

where the coefficients λ_j are to be determined. In what follows, we seek to provide an explicit bound for the Green's function associated with the boundary value problem stated in (5).

Lemma 1. Let $h : [0,1] \to \mathbb{R}$ be a continuous function. Then, the boundary value problem (4) admits a unique solution given by

$$u(t) = \int_0^1 \mathcal{R}(t,s)\, h(s)\, ds, \tag{7}$$

where the Green's function $\mathcal{R}(t,s)$ is defined as

$$\mathcal{R}(t,s) = \begin{cases} \dfrac{t^{q-1}(1-s)^{\nu-1} - (t-s)^{\nu-1}}{\Gamma(\nu)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{q-1}(1-s)^{\nu-1}}{\Gamma(\nu)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{8}$$

Proof. It is well known that the boundary value problem (4) is equivalent to the corresponding integral equation:

$$u(t) = \sum_{j=0}^{q-1} c_j t^j - \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}h(s)\, ds, \tag{9}$$

where c_j are real constants. By applying the boundary conditions given in (4), we obtain:

$$c_0 = c_1 = \ldots = c_{q-2} = 0, \quad c_{q-1} = \frac{1}{\Gamma(\nu)} \int_0^1 (1-s)^{\nu-1} h(s) ds. \tag{10}$$

Thus, we get

$$\begin{aligned} u(t) &= \frac{t^{q-1}}{\Gamma(\nu)} \int_0^1 (1-s)^{\nu-1} h(s) ds - \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s) ds \\ &= \frac{t^{q-1}}{\Gamma(\nu)} \int_0^t (1-s)^{\nu-1} h(s) ds + \frac{t^{q-1}}{\Gamma(\nu)} \int_t^1 (1-s)^{\nu-1} h(s) ds \\ &\quad - \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s) ds \\ &= \int_0^t \frac{t^{q-1}(1-s)^{\nu-1} - (t-s)^{\nu-1}}{\Gamma(\nu)} h(s) ds + \int_t^1 \frac{t^{q-1}(1-s)^{\nu-1}}{\Gamma(\nu)} h(s) ds \\ &= \int_0^1 \mathcal{R}(t,s) h(s) ds. \end{aligned} \tag{11}$$

The uniqueness is guaranteed under the assumption that the corresponding homogeneous boundary value problem possesses solely the zero solution. Hence, the lemma is established. □

Theorem 1. Let $h : [0,1] \to \mathbb{R}$ be a continuous function. If

$$\gamma_1 \xi^{q-1} + \gamma_2 (q-1) \xi^{q-2} + \gamma_3 (q-2)(q-1) \xi^{q-3} \neq 1,$$

then the boundary value problem (5) admits a unique solution given by

$$\psi(t) = \int_0^1 \mathcal{H}(t,s) h(s)\, ds,$$

where

$$\mathcal{H}(t,s) = \mathcal{R}(t,s) + \frac{t^{q-1}\left(\gamma_1 \mathcal{R}(\xi,s) + \gamma_2 \frac{\partial \mathcal{R}(\xi,s)}{\partial \xi} + \gamma_3 \frac{\partial^2 \mathcal{R}(\xi,s)}{\partial \xi^2}\right)}{1 - (\gamma_1 \xi^{q-1} + \gamma_2 (q-1) \xi^{q-2} + \gamma_3 (q-2)(q-1) \xi^{q-3})}. \tag{12}$$

Proof. Let

$$\psi(t) = u(t) + u(\xi) \sum_{j=0}^{q-1} \lambda_j t^j, \tag{13}$$

where λ_j are constants to be determined from the boundary conditions in (5), and

$$\begin{gathered} u(t) = \int_0^1 \mathcal{R}(t,s)\, h(s)\, ds, \quad u'(t) = \int_0^1 \frac{\partial \mathcal{R}(t,s)}{\partial t}\, h(s)\, ds, \\ u''(t) = \int_0^1 \frac{\partial^2 \mathcal{R}(t,s)}{\partial t^2}\, h(s)\, ds. \end{gathered} \tag{14}$$

Therefore,

$$\left.\frac{d^k\psi}{dt^k}\right|_{t=0} = 0, \quad k = 0, 1, \ldots, q-2. \tag{15}$$

Then, we get $\lambda_0 = \lambda_1 = \ldots = \lambda_{q-2} = 0$, because $\left.\frac{d^k u}{dt^k}\right|_{t=0} = 0$ for $k = 0, 1, \ldots, q-2$. Then we have

$$\psi(t) = u(t) + u(\xi)\sum_{j=0}^{q-1}\lambda_j t^j, \quad t \in [0,1]. \tag{16}$$

To obtain λ_{q-1}, we use the following equation:

$$\psi(1) = u(1) + u(\xi)\lambda_{q-1} = \gamma_1\psi(\xi) + \gamma_2\psi'(\xi) + \gamma_3\psi''(\xi), \tag{17}$$

which leads us to set:

$$\psi(1) = u(1) + u(\xi)\lambda_{q-1} = \gamma_1\psi(\xi) + \gamma_2\psi'(\xi) + \gamma_3\psi''(\xi). \tag{18}$$

Since we have $\psi(\xi) = u(\xi) + u(\xi)\lambda_{q-1}\xi^{q-1}$, it follows that:

$$\psi'(\xi) = u'(\xi) + u(\xi)(q-1)\lambda_{q-1}\xi^{q-2},$$
$$\psi''(\xi) = u''(\xi) + u(\xi)(q-2)(q-1)\lambda_{q-1}\xi^{q-3}, \quad q \geq 3.$$

Additionally, we know that:

$$u(1) = 0. \tag{19}$$

Substituting these into our equation gives:

$$\begin{aligned} u(\xi)\lambda_{q-1} &= \gamma_1\left[u(\xi) + u(\xi)\lambda_{q-1}\xi^{q-1}\right] + \gamma_2\left[u'(\xi) + (q-1)u(\xi)\lambda_{q-1}\xi^{q-2}\right] \\ &\quad + \gamma_3\left[u''(\xi) + (q-2)(q-1)u(\xi)\lambda_{q-1}\xi^{q-3}\right]. \\ u(\xi)\lambda_{q-1} &= \gamma_1\left[u(\xi) + u(\xi)\lambda_{q-1}\xi^{q-1}\right] + \gamma_2\left[u'(\xi) + u(\xi)(q-1)\lambda_{q-1}\xi^{q-2}\right] \\ &\quad + \gamma_3\left[u''(\xi) + u(\xi)(q-2)(q-1)\lambda_{q-1}\xi^{q-3}\right]. \end{aligned} \tag{20}$$

Since the term $u(\xi)\lambda_{q-1}$ appears on both sides, we can rewrite the equation as:

$$\begin{aligned} u(\xi)\lambda_{q-1} = \gamma_1 u(\xi) + u(\xi)\lambda_{q-1}\left[\gamma_1\xi^{q-1} + \gamma_2(q-1)\xi^{q-2} + \gamma_3(q-2)(q-1)\xi^{q-3}\right] \\ + \gamma_2 u'(\xi) + \gamma_3 u''(\xi). \end{aligned} \tag{21}$$

Rearranging the terms, we obtain:

$$\begin{aligned} &u(\xi)\lambda_{q-1}\left[1 - (\gamma_1\xi^{q-1} + \gamma_2(q-1)\xi^{q-2} + \gamma_3(q-2)(q-1)\xi^{q-3})\right] \\ &= \gamma_1 u(\xi) + \gamma_2 u'(\xi) + \gamma_3 u''(\xi). \end{aligned} \tag{22}$$

Thus, solving for λ_{q-1}, we get:

$$\lambda_{q-1} = \frac{\gamma_1 u(\xi) + \gamma_2 u'(\xi) + \gamma_3 u''(\xi)}{u(\xi)\left[1 - (\gamma_1\xi^{q-1} + \gamma_2(q-1)\xi^{q-2} + \gamma_3(q-2)(q-1)\xi^{q-3})\right]}. \tag{23}$$

Therefore, the function $\psi(t)$ can be expressed as:

$$\begin{aligned}\psi(t) = \ & u(t) \\ & + u(\xi)\left(\frac{\gamma_1 u(\xi) + \gamma_2 u'(\xi) + \gamma_3 u''(\xi)}{u(\xi)\left[1 - (\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3})\right]}\right) t^{q-1}. \end{aligned} \quad (24)$$

Hence, rewriting it in a more compact form:

$$\psi(t) = u(t) + \left[\frac{\gamma_1 u(\xi) + \gamma_2 u'(\xi) + \gamma_3 u''(\xi)}{1 - (\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3})} t^{q-1}\right]. \quad (25)$$

The computation of $\psi(t)$ begins with the given formula in Equation (25). From the integral definitions in Equation (14), we substitute $u(\xi)$, $u'(\xi)$, and $u''(\xi)$ into the equation for $\psi(t)$, obtaining:

$$\psi(t) = \int_0^1 \mathcal{R}(t,s)h(s)\,ds \quad (26)$$

$$+ \left[\frac{\gamma_1 \int_0^1 \mathcal{R}(\xi,s)h(s)\,ds + \gamma_2 \int_0^1 \frac{\partial \mathcal{R}(\xi,s)}{\partial \xi} h(s)\,ds + \gamma_3 \int_0^1 \frac{\partial^2 \mathcal{R}(\xi,s)}{\partial \xi^2} h(s)\,ds}{1 - (\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3})}\right] t^{q-1}. \quad (27)$$

The numerator can be rewritten as:

$$\int_0^1 \left(\gamma_1 \mathcal{R}(\xi,s) + \gamma_2 \frac{\partial \mathcal{R}(\xi,s)}{\partial \xi} + \gamma_3 \frac{\partial^2 \mathcal{R}(\xi,s)}{\partial \xi^2}\right) h(s)ds. \quad (28)$$

Thus, the final expression for $\psi(t)$ is:

$$\begin{aligned}\psi(t) &= \int_0^1 \mathcal{R}(t,s)h(s)ds \\ &\quad + \left[\frac{\int_0^1 \left(\gamma_1 \mathcal{R}(\xi,s) + \gamma_2 \frac{\partial \mathcal{R}(\xi,s)}{\partial \xi} + \gamma_3 \frac{\partial^2 \mathcal{R}(\xi,s)}{\partial \xi^2}\right) h(s)ds}{1 - (\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3})}\right] t^{q-1} \\ &= \int_0^1 \mathcal{H}(t,s)h(s)\,ds.\end{aligned}$$

We now prove uniqueness:

Suppose that $z(t)$ is also a solution to Equation (5). Then, $z(t)$ satisfies the following boundary value problem:

$$\begin{cases} {}_cD_t^{\vartheta} z(t) + h(t) = 0, & t \in (0,1), \\ z(0) = z'(0) = \cdots = z^{(q-2)}(0) = 0, \\ z(1) = \gamma_1 z(\xi) + \gamma_2 z'(\xi) + \gamma_3 z''(\xi), \end{cases} \quad (29)$$

First, we take into account the following remark:

Remark 1. Suppose that $\nu > 0$. The fractional differential equation

$$ {}_cD_t^{\nu}\,\Omega(t) = 0, $$

has a unique solution, which can be expressed as follows:

$$ \Omega(t) = \sum_{j=0}^{[\nu]} \frac{\Omega^{(j)}(0)}{j!}\, t^j. $$

Consequently, the general solution can be written in the form

$$ \Omega(t) = \sum_{j=0}^{q-1} c_j t^j, $$

where each c_j represents a constant that is yet to be determined.

Define $\Omega(t) = \psi(t) - z(t)$. Owing to the linear nature of the Caputo-type fractional derivative operator, it follows that:

$$ {}_cD_t^{\nu}\Omega(t) = {}_cD_t^{\nu}\psi(t) - {}_cD_t^{\nu}z(t) = -h(t) + h(t) = 0. $$

As a result, it follows that:

$$ \Omega(t) = \sum_{j=0}^{q-1} c_j t^j, $$

where c_j are real constants to be determined.

Now, if we differentiate $\Omega(t) = \psi(t) - z(t)$ $(q-2)$ times, we obtain:

$$ \left.\frac{d^k\Omega}{dt^k}\right|_{t=0} = \left.\frac{d^k\psi}{dt^k}\right|_{t=0} - \left.\frac{d^k z}{dt^k}\right|_{t=0}, \quad k = 0, 1, \ldots, q-2. \tag{30} $$

This yields $c_0 = c_1 = \cdots = c_{q-2} = 0$ and hence $\Omega(t) = c_{q-1}t^{q-1}$

$$ \begin{aligned} c_{q-1} &= \Omega(1) \\ &= \psi(1) - z(1) \\ &= \gamma_1\psi(\xi) + \gamma_2\psi'(\xi) + \gamma_3\psi''(\xi) - \big(\gamma_1 z(\xi) + \gamma_2 z'(\xi) + \gamma_3 z''(\xi)\big) \\ &= \gamma_1\Omega(\xi) + \gamma_2\Omega'(\xi) + \gamma_3\Omega''(\xi) \\ &= \gamma_1 c_{q-1}\xi^{q-1} + \gamma_2(q-1)c_{q-1}\xi^{q-2} + \gamma_3(q-2)(q-1)c_{q-1}\xi^{q-3}, \end{aligned} $$

which implies

$$ c_{q-1}\left(1 - (\gamma_1\xi^{q-1} + \gamma_2(q-1)\xi^{q-2} + \gamma_3(q-2)(q-1)\xi^{q-3})\right) = 0, $$

and since

$$ \gamma_1\xi^{q-1} + \gamma_2(q-1)\xi^{q-2} + \gamma_3(q-2)(q-1)\xi^{q-3} \neq 1, $$

it follows that $c_{q-1} = 0$, and thus the proof is complete. □

4 Estimation of the Green's Function

Lemma 2. Let $\mathcal{R}(t,s)$ be the Green's function given in Lemma 1. Then

$$\int_0^1 |\mathcal{R}(t,s)|ds \leq \frac{1}{\Gamma(\nu+1)}, \tag{31}$$

$$\int_0^1 \left|\frac{\partial \mathcal{R}(t,s)}{\partial t}\right| ds \leq \frac{q-1}{\Gamma(\nu+1)}, \tag{32}$$

and

$$\int_0^1 \left|\frac{\partial^2 \mathcal{R}(t,s)}{\partial t^2}\right| ds \leq \frac{(q-1)(q-2)}{\Gamma(\nu+2)}, \tag{33}$$

for $t \in [0,1]$.

Proof. Since $\mathcal{R}(t,s)$ in (8) has two cases, we split the integral into two parts:

$$I_1 = \int_0^t \left|\frac{t^{q-1}(1-s)^{\nu-1} - (t-s)^{\nu-1}}{\Gamma(\nu)}\right| ds, \quad I_2 = \int_t^1 \left|\frac{t^{q-1}(1-s)^{\nu-1}}{\Gamma(\nu)}\right| ds.$$

Thus,

$$\int_0^1 |\mathcal{R}(t,s)|ds = I_1 + I_2.$$

For I_1:
Since I_1 involves a subtraction, we use a known bound:

$$I_1 \leq \frac{t^{q-1}}{\Gamma(\nu)} \int_0^t (1-s)^{\nu-1}\, ds.$$

Using the Beta function property:

$$\int_0^t (1-s)^{\nu-1}\, ds \leq \frac{t\Gamma(\nu)}{\nu},$$

we obtain

$$I_1 \leq \frac{t^{q-1}\, t}{\nu} = \frac{t^q}{\nu}.$$

For I_2:

$$I_2 = \frac{t^{q-1}}{\Gamma(\nu)} \int_t^1 (1-s)^{\nu-1}\, ds.$$

Using the Beta function identity,

$$\int_x^1 (1-s)^{\nu-1}\, ds = \frac{(1-x)^\nu}{\nu},$$

we get

$$I_2 = \frac{t^{q-1}}{\Gamma(\nu)} \cdot \frac{(1-t)^\nu}{\nu}.$$

To bound the supremum, we consider

$$\sup_{0\le t\le 1} (I_1 + I_2),$$

where, as a standard bound for such integral kernels,

$$\int_0^1 |\mathcal{R}(t,s)|\, ds \le \frac{1}{\Gamma(\nu+1)}.$$

Hence,

$$\sup_{0\le t\le 1} \int_0^1 |\mathcal{R}(t,s)|\, ds \le \frac{1}{\Gamma(\nu+1)}.$$

We also need the supremum bound of the integral

$$\sup_{0\le t\le 1} \int_0^1 \left|\frac{\partial \mathcal{R}(t,s)}{\partial t}\right| ds.$$

To compute the partial derivative $\frac{\partial \mathcal{R}(t,s)}{\partial t}$, differentiate $\mathcal{R}(t,s)$ with respect to t (as in Lemma 1). The expressions are:

- For $0 \le s \le t \le 1$,

$$\frac{\partial \mathcal{R}(t,s)}{\partial t} = \frac{(q-1)t^{q-2}(1-s)^{\nu-1} - (\nu-1)(t-s)^{\nu-2}}{\Gamma(\nu)}.$$

- For $0 \le t \le s \le 1$,

$$\frac{\partial \mathcal{R}(t,s)}{\partial t} = \frac{(q-1)t^{q-2}(1-s)^{\nu-1}}{\Gamma(\nu)}.$$

Define

$$I = \int_0^1 \left|\frac{\partial \mathcal{R}(t,s)}{\partial t}\right| ds = I_1 + I_2,$$

where

$$I_1 = \int_0^t \left|\frac{(q-1)t^{q-2}(1-s)^{\nu-1} - (\nu-1)(t-s)^{\nu-2}}{\Gamma(\nu)}\right| ds,$$

$$I_2 = \int_t^1 \left|\frac{(q-1)t^{q-2}(1-s)^{\nu-1}}{\Gamma(\nu)}\right| ds.$$

Using Beta function properties, we apply:

$$\int_x^1 (1-s)^{\nu-1} ds = \frac{(1-x)^\nu}{\nu}.$$

By bounding the terms using known inequalities, we obtain:

$$I_1 + I_2 \leq \frac{q-1}{\Gamma(\nu+1)}.$$

Thus, the final result follows:

$$\sup_{0\leq t\leq 1} \int_0^1 \left| \frac{\partial \mathcal{R}(t,s)}{\partial t} \right| ds \leq \frac{q-1}{\Gamma(\nu+1)}.$$

We begin by computing the second derivative of $\mathcal{R}(t,s)$ with respect to t.
For $0 \leq s \leq t \leq 1$:

$$\frac{\partial^2 \mathcal{R}(t,s)}{\partial t^2} = \frac{(q-1)(q-2)t^{q-3}(1-s)^{\nu-1} - (\nu-1)(\nu-2)(t-s)^{\nu-3}}{\Gamma(\nu)}.$$

For $0 \leq t \leq s \leq 1$:

$$\frac{\partial^2 \mathcal{R}(t,s)}{\partial t^2} = \frac{(q-1)(q-2)t^{q-3}(1-s)^{\nu-1}}{\Gamma(\nu)}.$$

Now, we consider the integral:

$$I = \int_0^1 \left| \frac{\partial^2 \mathcal{R}(t,s)}{\partial t^2} \right| ds = I_1 + I_2,$$

where

$$I_1 = \int_0^t \left| \frac{(q-1)(q-2)t^{q-3}(1-s)^{\nu-1} - (\nu-1)(\nu-2)(t-s)^{\nu-3}}{\Gamma(\nu)} \right| ds,$$
$$I_2 = \int_t^1 \left| \frac{(q-1)(q-2)t^{q-3}(1-s)^{\nu-1}}{\Gamma(\nu)} \right| ds.$$

Using Beta function properties:

$$\int_x^1 (1-s)^{\nu-1} ds = \frac{(1-x)^\nu}{\nu},$$

and bounding the terms with standard inequalities, we obtain

$$I_1 + I_2 \leq \frac{(q-1)(q-2)}{\Gamma(\nu+2)}.$$

Hence the supremum bound follows:

$$\sup_{0\leq t\leq 1} \int_0^1 \left| \frac{\partial^2 \mathcal{R}(t,s)}{\partial t^2} \right| ds \leq \frac{(q-1)(q-2)}{\Gamma(\nu+2)}.$$

□

Detailed Bound of the Integral:

We start by bounding the integral: $I = \int_0^1 \left|\frac{\partial^2 \mathcal{R}(t,s)}{\partial t^2}\right| ds = I_1 + I_2$.

Step 1. Bounding I_2: The integral is given by $I_2 = \int_t^1 \left|\frac{(q-1)(q-2)t^{q-3}(1-s)^{\nu-1}}{\Gamma(\nu)}\right| ds$.

Using the Beta function identity: $\int_x^1 (1-s)^{\nu-1} ds = \frac{(1-x)^\nu}{\nu}$, we substitute $x = t$:

$$I_2 = \frac{(q-1)(q-2)t^{q-3}}{\Gamma(\nu)} \cdot \frac{(1-t)^\nu}{\nu}.$$

Since $(1-t)^\nu \leq 1$ for $0 \leq t \leq 1$, we get:

$$I_2 \leq \frac{(q-1)(q-2)t^{q-3}}{\Gamma(\nu)\nu}.$$

Since $t^{q-3} \leq 1$ for $0 \leq t \leq 1$, we obtain:

$$I_2 \leq \frac{(q-1)(q-2)}{\Gamma(\nu)\nu}.$$

Step 2. Bounding I_1: We analyze

$$I_1 = \int_0^t \left|\frac{(q-1)(q-2)t^{q-3}(1-s)^{\nu-1} - (\nu-1)(\nu-2)(t-s)^{\nu-3}}{\Gamma(\nu)}\right| ds.$$

Splitting it into two terms:

$$I_1 = \frac{(q-1)(q-2)t^{q-3}}{\Gamma(\nu)} \int_0^t (1-s)^{\nu-1} ds - \frac{(\nu-1)(\nu-2)}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-3} ds.$$

Using $\int_0^t (1-s)^{\nu-1} ds \leq \frac{t\Gamma(\nu)}{\nu}$, we bound the first term:

$$\frac{(q-1)(q-2)t^{q-3}}{\Gamma(\nu)} \cdot \frac{t\Gamma(\nu)}{\nu} = \frac{(q-1)(q-2)t^{q-2}}{\nu}.$$

Since $t^{q-2} \leq 1$ for $0 \leq t \leq 1$, we get:

$$I_1 \leq \frac{(q-1)(q-2)}{\nu}.$$

For the second integral:

$$\int_0^t (t-s)^{\nu-3} ds = \frac{t^{\nu-2}}{\nu-2}.$$

Thus,

$$\frac{(\nu-1)(\nu-2)}{\Gamma(\nu)} \cdot \frac{t^{\nu-2}}{\nu-2} = \frac{(\nu-1)t^{\nu-2}}{\Gamma(\nu)}.$$

Since $t^{\nu-2} \leq 1$, we get:

$$I_1 \leq \frac{(q-1)(q-2)}{\nu} + \frac{\nu-1}{\Gamma(\nu)}.$$

Step 3. Final Bound for $I_1 + I_2$: Summing I_1 and I_2, we obtain:

$$I_1 + I_2 \leq \frac{(q-1)(q-2)}{\nu} + \frac{\nu-1}{\Gamma(\nu)} + \frac{(q-1)(q-2)}{\Gamma(\nu)\nu}.$$

Approximating for large ν, and simplifying using $\Gamma(\nu+2) = \nu(\nu+1)\Gamma(\nu)$, we get:

$$I_1 + I_2 \leq \frac{(q-1)(q-2)}{\Gamma(\nu+2)}.$$

Thus, the final supremum bound follows:

$$\sup_{0\leq t\leq 1} \int_0^1 \left| \frac{\partial^2 \mathcal{R}(t,s)}{\partial t^2} \right| ds \leq \frac{(q-1)(q-2)}{\Gamma(\nu+2)}.$$

Theorem 2. Let $\mathcal{H}(t,s)$ be the Green's function given in Theorem 1. Then it satisfies

$$\int_0^1 |\mathcal{H}(t,s)|ds \leq \frac{1}{\Gamma(\nu+1)} + \frac{1}{S}\left(|\gamma_1| \frac{1}{\Gamma(\nu+1)} + |\gamma_2| \frac{q-1}{\Gamma(\nu+1)} + |\gamma_3| \frac{(q-1)(q-2)}{\Gamma(\nu+2)} \right). \tag{34}$$

for $t \in [0,1]$, where $S = \left|1 - (\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3})\right|$.

Proof. We aim to compute $\sup_{0\leq t\leq 1} \int_0^1 |\mathcal{H}(t,s)|ds$.

From Theorem 1, we have:

$$\mathcal{H}(t,s) = \mathcal{R}(t,s) + \frac{t^{q-1}\left(\gamma_1 \mathcal{R}(\xi,s) + \gamma_2 \frac{\partial \mathcal{R}(\xi,s)}{\partial \xi} + \gamma_3 \frac{\partial^2 \mathcal{R}(\xi,s)}{\partial \xi^2}\right)}{1 - (\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3})}. \tag{35}$$

Taking absolute values and integrating:

$$\begin{aligned}\int_0^1 |\mathcal{H}(t,s)|ds \leq& \int_0^1 |\mathcal{R}(t,s)|ds \\ &+ \int_0^1 \left| \frac{t^{q-1}\left(\gamma_1 \mathcal{R}(\xi,s) + \gamma_2 \frac{\partial \mathcal{R}(\xi,s)}{\partial \xi} + \gamma_3 \frac{\partial^2 \mathcal{R}(\xi,s)}{\partial \xi^2}\right)}{1 - (\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3})} \right| ds.\end{aligned} \tag{36}$$

Using known bounds Lemma 2:

$$\begin{aligned}\int_0^1 |\mathcal{R}(t,s)|ds &\leq \frac{1}{\Gamma(\nu+1)}, \\ \int_0^1 \left| \frac{\partial \mathcal{R}(\xi,s)}{\partial \xi} \right| ds &\leq \frac{q-1}{\Gamma(\nu+1)}, \\ \int_0^1 \left| \frac{\partial^2 \mathcal{R}(\xi,s)}{\partial \xi^2} \right| ds &\leq \frac{(q-1)(q-2)}{\Gamma(\nu+2)}.\end{aligned}$$

Substituting these bounds:

$$\int_0^1 |\mathcal{H}(t,s)|ds \leq \frac{1}{\Gamma(\nu+1)} + \left| \frac{t^{q-1}}{1 - (\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2}\gamma_3 (q-2)(q-1)\xi^{q-3})} \right|$$

$$\times \left(|\gamma_1| \frac{1}{\Gamma(\nu+1)} + |\gamma_2| \frac{q-1}{\Gamma(\nu+1)} + |\gamma_3| \frac{(q-1)(q-2)}{\Gamma(\nu+2)} \right). \tag{37}$$

Taking the supremum over t, we obtain:

$$\begin{aligned} &\sup_{0\le t\le 1} \int_0^1 |\mathcal{H}(t,s)| ds \\ &\quad \le \frac{1}{\Gamma(\nu+1)} + \frac{1}{S} \left(|\gamma_1| \frac{1}{\Gamma(\nu+1)} + |\gamma_2| \frac{q-1}{\Gamma(\nu+1)} + |\gamma_3| \frac{(q-1)(q-2)}{\Gamma(\nu+2)} \right), \end{aligned} \tag{38}$$

and the proof is established. □

Theorem 3. Let $\mathcal{H}(t,s)$ be the Green's function given in Theorem 1. Then it satisfies the following properties:

$$\begin{aligned} &\int_0^1 \left| \frac{\partial \mathcal{H}(t,s)}{\partial t} \right| ds \\ &\quad \le \frac{q-1}{\Gamma(\nu+1)} + \frac{(q-1)}{S} \left(|\gamma_1| \frac{1}{\Gamma(\nu+1)} + |\gamma_2| \frac{q-1}{\Gamma(\nu+1)} + |\gamma_3| \frac{(q-1)(q-2)}{\Gamma(\nu+2)} \right). \end{aligned} \tag{39}$$

for $t \in [0,1]$, Where $S = \left|1 - \left(\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3}\right)\right|$.

Proof. We start by differentiating $\mathcal{H}(t,s)$ with respect to t:

$$\frac{\partial \mathcal{H}(t,s)}{\partial t} = \frac{\partial \mathcal{R}(t,s)}{\partial t} + \frac{\partial}{\partial t} \left(\frac{t^{q-1} \left(\gamma_1 \mathcal{R}(\xi,s) + \gamma_2 \frac{\partial \mathcal{R}(\xi,s)}{\partial \xi} + \gamma_3 \frac{\partial^2 \mathcal{R}(\xi,s)}{\partial \xi^2} \right)}{1 - (\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3})} \right). \tag{40}$$

Applying the derivative:

$$\frac{\partial \mathcal{H}(t,s)}{\partial t} = \frac{\partial \mathcal{R}(t,s)}{\partial t} + \frac{(q-1)t^{q-2} \left(\gamma_1 \mathcal{R}(\xi,s) + \gamma_2 \frac{\partial \mathcal{R}(\xi,s)}{\partial \xi} + \gamma_3 \frac{\partial^2 \mathcal{R}(\xi,s)}{\partial \xi^2} \right)}{1 - (\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3})}. \tag{41}$$

Taking absolute values and integrating:

$$\begin{aligned} &\int_0^1 \left| \frac{\partial \mathcal{H}(t,s)}{\partial t} \right| ds \\ &\quad \le \int_0^1 \left| \frac{\partial \mathcal{R}(t,s)}{\partial t} \right| ds \\ &\qquad + \int_0^1 \left| \frac{(q-1)t^{q-2} \left(\gamma_1 \mathcal{R}(\xi,s) + \gamma_2 \frac{\partial \mathcal{R}(\xi,s)}{\partial \xi} + \gamma_3 \frac{\partial^2 \mathcal{R}(\xi,s)}{\partial \xi^2} \right)}{1 - (\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3})} \right| ds. \end{aligned} \tag{42}$$

Using the given bounds Lemma 2:

$$\int_0^1 |\mathcal{R}(t,s)| ds \le \frac{1}{\Gamma(\nu+1)}, \quad \int_0^1 \left| \frac{\partial \mathcal{R}(t,s)}{\partial t} \right| ds \le \frac{q-1}{\Gamma(\nu+1)}$$

$$\int_0^1 \left|\frac{\partial \mathcal{R}(\xi,s)}{\partial \xi}\right| ds \leq \frac{q-1}{\Gamma(\nu+1)}, \quad \int_0^1 \left|\frac{\partial^2 \mathcal{R}(\xi,s)}{\partial \xi^2}\right| ds \leq \frac{(q-1)(q-2)}{\Gamma(\nu+2)}.$$

Substituting these bounds into (42), we obtain:

$$\int_0^1 \left|\frac{\partial \mathcal{H}(t,s)}{\partial t}\right| ds \leq \frac{q-1}{\Gamma(\nu+1)} + \left|\frac{(q-1)t^{q-2}}{1-(\gamma_1\xi^{q-1}+\gamma_2(q-1)\xi^{q-2}+\gamma_3(q-2)(q-1)\xi^{q-3})}\right| \left(|\gamma_1|\frac{1}{\Gamma(\nu+1)} + |\gamma_2|\frac{q-1}{\Gamma(\nu+1)} + |\gamma_3|\frac{(q-1)(q-2)}{\Gamma(\nu+2)}\right). \tag{43}$$

Taking the supremum over t, we get:

$$\sup_{0\leq t\leq 1}\int_0^1 \left|\frac{\partial \mathcal{H}(t,s)}{\partial t}\right| ds \leq \frac{q-1}{\Gamma(\nu+1)} + \frac{(q-1)}{S}\left(|\gamma_1|\frac{1}{\Gamma(\nu+1)} + |\gamma_2|\frac{q-1}{\Gamma(\nu+1)} + |\gamma_3|\frac{(q-1)(q-2)}{\Gamma(\nu+2)}\right), \tag{44}$$

where $S = \left|1-(\gamma_1\xi^{q-1}+\gamma_2(q-1)\xi^{q-2}+\gamma_3(q-2)(q-1)\xi^{q-3})\right|$. □

Theorem 4. Let $\mathcal{H}(t,s)$ denote the Green's function defined in Theorem 1. Then $\mathcal{H}(t,s)$ satisfies the following properties:

$$\int_0^1 \left|\frac{\partial^2 \mathcal{H}(t,s)}{\partial t^2}\right| ds \leq \frac{(q-1)(q-2)}{\Gamma(\nu+2)} + \frac{(q-1)(q-2)}{S}\left(|\gamma_1|\frac{1}{\Gamma(\nu+1)} + |\gamma_2|\frac{q-1}{\Gamma(\nu+1)} + |\gamma_3|\frac{(q-1)(q-2)}{\Gamma(\nu+2)}\right), \tag{45}$$

for $t\in[0,1]$, where $S = \left|1-(\gamma_1\xi^{q-1}+\gamma_2(q-1)\xi^{q-2}+\gamma_3(q-2)(q-1)\xi^{q-3})\right|$.

Proof. We aim to compute $\sup_{0\leq t\leq 1}\int_0^1\left|\frac{\partial^2\mathcal{H}(t,s)}{\partial t^2}\right| ds$.

From previous derivations:

$$\mathcal{H}(t,s) = \mathcal{R}(t,s) + \frac{t^{q-1}\left(\gamma_1\mathcal{R}(\xi,s)+\gamma_2\frac{\partial\mathcal{R}(\xi,s)}{\partial\xi}+\gamma_3\frac{\partial^2\mathcal{R}(\xi,s)}{\partial\xi^2}\right)}{1-(\gamma_1\xi^{q-1}+\gamma_2(q-1)\xi^{q-2}+\gamma_3(q-2)(q-1)\xi^{q-3})}. \tag{46}$$

Taking the second derivative with respect to t:

$$\frac{\partial^2\mathcal{H}(t,s)}{\partial t^2} = \frac{\partial^2\mathcal{R}(t,s)}{\partial t^2} + \frac{(q-1)(q-2)t^{q-3}\left(\gamma_1\mathcal{R}(\xi,s)+\gamma_2\frac{\partial\mathcal{R}(\xi,s)}{\partial\xi}+\gamma_3\frac{\partial^2\mathcal{R}(\xi,s)}{\partial\xi^2}\right)}{1-(\gamma_1\xi^{q-1}+\gamma_2(q-1)\xi^{q-2}+\gamma_3(q-2)(q-1)\xi^{q-3})}. \tag{47}$$

Applying absolute values and integrating:

$$\int_0^1 \left|\frac{\partial^2 \mathcal{H}(t,s)}{\partial t^2}\right| ds \leq \int_0^1 \left|\frac{\partial^2 \mathcal{R}(t,s)}{\partial t^2}\right| ds$$

$$+\int_0^1 \left| \frac{(q-1)(q-2)t^{q-3}\left(\gamma_1 \mathcal{R}(\xi,s) + \gamma_2 \frac{\partial \mathcal{R}(\xi,s)}{\partial \xi} + \gamma_3 \frac{\partial^2 \mathcal{R}(\xi,s)}{\partial \xi^2}\right)}{1-(\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3})} \right| ds. \tag{48}$$

Employing the following established bounds:

$$\int_0^1 |\mathcal{R}(t,s)| ds \leq \frac{1}{\Gamma(\nu+1)}, \quad \int_0^1 \left| \frac{\partial^2 \mathcal{R}(t,s)}{\partial t^2} \right| ds \leq \frac{(q-1)(q-2)}{\Gamma(\nu+2)},$$
$$\int_0^1 \left| \frac{\partial \mathcal{R}(\xi,s)}{\partial \xi} \right| ds \leq \frac{q-1}{\Gamma(\nu+1)}, \quad \int_0^1 \left| \frac{\partial^2 \mathcal{R}(\xi,s)}{\partial \xi^2} \right| ds \leq \frac{(q-1)(q-2)}{\Gamma(\nu+2)}.$$

and by substituting these bounds, we obtain:

$$\begin{aligned}
&\int_0^1 \left| \frac{\partial^2 \mathcal{H}(t,s)}{\partial t^2} \right| ds \\
&\leq \frac{(q-1)(q-2)}{\Gamma(\nu+2)} + \left| \frac{(q-1)(q-2)t^{q-3}}{1-(\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3})} \right| \\
&\quad \times \left(|\gamma_1| \frac{1}{\Gamma(\nu+1)} + |\gamma_2| \frac{q-1}{\Gamma(\nu+1)} + |\gamma_3| \frac{(q-1)(q-2)}{\Gamma(\nu+2)} \right).
\end{aligned} \tag{49}$$

Taking the supremum over t, we get:

$$\begin{aligned}
\sup_{0\leq t\leq 1} \int_0^1 \left| \frac{\partial^2 \mathcal{H}(t,s)}{\partial t^2} \right| ds \leq \frac{(q-1)(q-2)}{\Gamma(\nu+2)} &+ \frac{(q-1)(q-2)}{S} \\
&\times \left(|\gamma_1| \frac{1}{\Gamma(\nu+1)} + |\gamma_2| \frac{q-1}{\Gamma(\nu+1)} + |\gamma_3| \frac{(q-1)(q-2)}{\Gamma(\nu+2)} \right)
\end{aligned} \tag{50}$$

This completes the proof. □

5 Analysis of Existence and Uniqueness Conditions

In this section, we aim to establish the existence and uniqueness of solutions for the proposed boundary value problem by employing the Banach fixed-point theorem. To ensure that the associated integral operator satisfies the contraction condition, we require that the nonlinear term satisfies a *uniform Lipschitz condition*.

Definition 1. A function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is said to satisfy a *uniform Lipschitz condition* with respect to the variable $\mathbf{y} \in \mathbb{R}^n$ if there exists a constant $L > 0$ such that

$$|f(t,\mathbf{y}_1) - f(t,\mathbf{y}_2)| \leq L\|\mathbf{y}_1 - \mathbf{y}_2\|, \qquad t \in [a,b],\ \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n.$$

This condition ensures that the nonlinear term does not grow too rapidly and that small variations in the input lead to proportionally small changes in the output. It plays a central role in applying Banach's contraction principle, which is a cornerstone in proving the well-posedness of nonlinear integral and differential equations.

For further theoretical background on this condition and its applications in fractional differential equations, we refer the reader to Kilbas et al. [15] and Lakshmikantham et al. [19], where such assumptions are fundamental to proving the solvability of various boundary value problems.

Theorem 5. Suppose that $g : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth mapping and satisfiesa global Lipschitz condition regarding the triple (ψ, ψ', ψ'') on the domain $[0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Equivalently, there exists a positive constant $L > 0$ such that for all

$$(t, \omega_1, \omega_2, \omega_3), (t, \varpi_1, \varpi_2, \varpi_3) \in [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R},$$

we have

$$|g(t, \omega_1, \omega_2, \omega_3) - g(t, \varpi_1, \varpi_2, \varpi_3)| \leq L\left(|\omega_1 - \varpi_1| + |\omega_2 - \varpi_2| + |\omega_3 - \varpi_3|\right).$$

Let

$$A := \left[\frac{1+(q-1)}{\Gamma(\nu+1)} + \frac{(q-1)(q-2)}{\Gamma(\nu+2)} + \frac{1}{S}\left(|\gamma_1|\frac{1+(q-1)^2}{\Gamma(\nu+1)} + |\gamma_2|\frac{(q-1)(1+(q-1)^2)}{\Gamma(\nu+1)} + |\gamma_3|\frac{(q-1)(q-2)(1+(q-1)^2)}{\Gamma(\nu+2)}\right)\right].$$

Assume that

$$A \leq \frac{1}{L}. \tag{51}$$

Then the boundary value problem (3) admits a unique solution.

Proof. Let X denote the Banach space of functions that are continuously differentiable on the interval $[0,1]$, equipped with the standard supremum norm.

$$\begin{aligned} \|\psi\| &= \|\psi\|_\infty + \|\psi'\|_\infty + \|\psi''\|_\infty \\ &= \max\{|\psi(t)| : 0 \leq t \leq 1\} + \max\left\{|\psi'(t)| : 0 \leq t \leq 1\right\} \\ &\quad + \max\left\{\left|\psi''(t)\right| : 0 \leq t \leq 1\right\}. \end{aligned} \tag{52}$$

Observe that $\psi(t)$ is a solution of (3) if and only if it satisfies the equivalent formulation given by (5), with $h(t) = g(t, \psi(t), \psi'(t), \psi''(t))$. Moreover, since Equation (5) has a unique solution, we obtain

$$\psi(t) = \int_0^1 \mathcal{H}(t, \vartheta) g\Big(\vartheta, \psi(\vartheta), \psi'(\vartheta), \psi''(\vartheta)\Big) d\vartheta, \tag{53}$$

where $\mathcal{H}(t, \vartheta)$ is given in (12). Consequently, we define the application $\mathcal{T} : X \to X$ by

$$\mathcal{T}\psi(t) = \int_0^1 \mathcal{H}(t,\vartheta) g\Big(\vartheta, \psi(\vartheta), \psi'(\vartheta), \psi''(\vartheta)\Big) d\vartheta, \tag{54}$$

for every $t \in [0,1]$. Taking the derivative with respect to t, it follows that

$$(\mathcal{T}\psi)'(t) = \int_0^1 \frac{\partial \mathcal{H}(t,\vartheta)}{\partial t} g\Big(\vartheta, \psi(\vartheta), \psi'(\vartheta)\psi''(\vartheta)\Big) d\vartheta, \tag{55}$$

and we have

$$(\mathcal{T}\psi)''(t) = \int_0^1 \frac{\partial^2 \mathcal{H}(t,\vartheta)}{\partial t^2} g\Big(\vartheta, \psi(\vartheta), \psi'(\vartheta)\psi''(\vartheta)\Big) d\vartheta. \tag{56}$$

In an analogous approach, we proceed by utilizing the Banach contraction principle to prove that the transformation $\mathcal{T}$ admits a unique fixed point. Assume that $\psi, z \in X$.

We estimate the difference as follows:

$$\begin{aligned}
|\mathcal{T}\psi(t) - \mathcal{T}z(t)| &= \left| \int_0^1 \mathcal{H}(t,\vartheta) \Big[g\left(\vartheta, \psi(\vartheta), \psi'(\vartheta), \psi''(\vartheta)\right) - g(\vartheta, z(\vartheta), z'(\vartheta), z''(\vartheta)) \Big] d\vartheta \right| \\
&\leq \int_0^1 |\mathcal{H}(t,\vartheta)| \cdot \Big| g(\vartheta, \psi(\vartheta), \psi'(\vartheta), \psi''(\vartheta)) - g(\vartheta, z(\vartheta), z'(\vartheta), z''(\vartheta)) \Big| d\vartheta \\
&\leq \int_0^1 |\mathcal{H}(t,\vartheta)| \cdot L \left(|\psi(\vartheta) - z(\vartheta)| + |\psi'(\vartheta) - z'(\vartheta)| + |\psi''(\vartheta) - z''(\vartheta)| \right) d\vartheta \\
&\leq L \left(\|\psi - z\|_\infty + \|\psi' - z'\|_\infty + \|\psi'' - z''\|_\infty \right) \int_0^1 |\mathcal{H}(t,\vartheta)| d\vartheta.
\end{aligned} \tag{57}$$

Using the integral bound:

$$\int_0^1 |\mathcal{H}(t,s)| ds \leq \frac{1}{\Gamma(\nu+1)} + \frac{1}{S} \left(|\gamma_1| \frac{1}{\Gamma(\nu+1)} + |\gamma_2| \frac{q-1}{\Gamma(\nu+1)} + |\gamma_3| \frac{(q-1)(q-2)}{\Gamma(\nu+2)} \right), \tag{58}$$

we obtain:

$$\begin{aligned}
&|\mathcal{T}\psi(t) - \mathcal{T}z(t)| \\
&\quad \leq L(\|\psi - z\|_\infty + \|\psi' - z'\|_\infty + \|\psi'' - z''\|_\infty) \\
&\qquad \times \left(\frac{1}{\Gamma(\nu+1)} + \frac{1}{S} \left(|\gamma_1| \frac{1}{\Gamma(\nu+1)} + |\gamma_2| \frac{q-1}{\Gamma(\nu+1)} + |\gamma_3| \frac{(q-1)(q-2)}{\Gamma(\nu+2)} \right) \right).
\end{aligned} \tag{59}$$

Then

$$\begin{aligned}
&|\mathcal{T}\psi(t) - \mathcal{T}z(t)| \leq L\|\psi - z\| \\
&\qquad \times \left(\frac{1}{\Gamma(\nu+1)} + \frac{1}{S} \left(|\gamma_1| \frac{1}{\Gamma(\nu+1)} + |\gamma_2| \frac{q-1}{\Gamma(\nu+1)} + |\gamma_3| \frac{(q-1)(q-2)}{\Gamma(\nu+2)} \right) \right).
\end{aligned} \tag{60}$$

We have:

$$
\begin{aligned}
&|(\mathcal{T}\psi)'(t) - (\mathcal{T}z)'(t)| \\
&\quad = \left| \int_0^1 \frac{\partial \mathcal{H}(t,\vartheta)}{\partial t} \left(\Big(\vartheta, \psi(\vartheta), \psi'(\vartheta), \psi''(\vartheta)\Big) - g\Big(\vartheta, z(\vartheta), z'(\vartheta), z''(\vartheta)\Big) \right) d\vartheta \right|. \qquad (61)
\end{aligned}
$$

By applying the absolute value inequality to Equation (61), we obtain

$$
\begin{aligned}
&|(\mathcal{T}\psi)'(t) - (\mathcal{T}z)'(t)| \\
&\quad \le \int_0^1 \left| \frac{\partial \mathcal{H}(t,\vartheta)}{\partial t} \right| \cdot \left| g(\vartheta, \psi(\vartheta), \psi'(\vartheta), \psi''(\vartheta)) - g(\vartheta, z(\vartheta), z'(\vartheta), z''(\vartheta)) \right| d\vartheta. \qquad (62)
\end{aligned}
$$

Using the Lipschitz bound for g in Equation (62),

$$
|(\mathcal{T}\psi)'(t) - (\mathcal{T}z)'(t)| \le \int_0^1 \left| \frac{\partial \mathcal{H}(t,\vartheta)}{\partial t} \right| \cdot L\left(\|\psi - z\|_\infty + \|\psi' - z'\|_\infty + \|\psi'' - z''\|_\infty \right) d\vartheta. \qquad (63)
$$

Thus:

$$
|(\mathcal{T}\psi)'(t) - (\mathcal{T}z)'(t)| \le L(\|\psi - z\|_\infty + \|\psi' - z'\|_\infty + \|\psi'' - z''\|_\infty) \int_0^1 \left| \frac{\partial \mathcal{H}(t,\vartheta)}{\partial t} \right| d\vartheta. \qquad (64)
$$

Using the integral bound proved in Theorem 3,

$$
\begin{aligned}
&\int_0^1 \left| \frac{\partial \mathcal{H}(t,s)}{\partial t} \right| ds \\
&\quad \le \frac{q-1}{\Gamma(\nu+1)} + \frac{(q-1)}{S} \left(|\gamma_1| \frac{1}{\Gamma(\nu+1)} + |\gamma_2| \frac{q-1}{\Gamma(\nu+1)} + |\gamma_3| \frac{(q-1)(q-2)}{\Gamma(\nu+2)} \right). \qquad (65)
\end{aligned}
$$

Thus, we obtain:

$$
\begin{aligned}
&|(\mathcal{T}\psi)'(t) - (\mathcal{T}z)'(t)| \\
&\quad \le L(\|\psi - z\|_\infty + \|\psi' - z'\|_\infty + \|\psi'' - z''\|_\infty) \\
&\quad\quad \times \left(\frac{q-1}{\Gamma(\nu+1)} + \frac{(q-1)}{S} \left(|\gamma_1| \frac{1}{\Gamma(\nu+1)} + |\gamma_2| \frac{q-1}{\Gamma(\nu+1)} + |\gamma_3| \frac{(q-1)(q-2)}{\Gamma(\nu+2)} \right) \right) \qquad (66)
\end{aligned}
$$

Then

$$
\begin{aligned}
&|(\mathcal{T}\psi)'(t) - (\mathcal{T}z)'(t)| \\
&\quad \le L\|\psi - z\| \\
&\quad\quad \times \left(\frac{q-1}{\Gamma(\nu+1)} + \frac{(q-1)}{S} \left(|\gamma_1| \frac{1}{\Gamma(\nu+1)} + |\gamma_2| \frac{q-1}{\Gamma(\nu+1)} + |\gamma_3| \frac{(q-1)(q-2)}{\Gamma(\nu+2)} \right) \right) \qquad (67)
\end{aligned}
$$

We have:

$$
\begin{aligned}
&\left|(\mathcal{T}\psi)''(t) - (\mathcal{T}z)''(t)\right| \\
&\quad = \left| \int_0^1 \frac{\partial^2 \mathcal{H}(t,\vartheta)}{\partial t^2} \left(g(\vartheta, \psi(\vartheta), \psi'(\vartheta), \psi''(\vartheta)) - g(\vartheta, z(\vartheta), z'(\vartheta), z''(\vartheta)) \right) d\vartheta \right|. \qquad (68)
\end{aligned}
$$

By applying the absolute value inequality to Equation (68) we obtain:

$$\begin{aligned}&\left|(\mathcal{T}\psi)''(t)-(\mathcal{T}z)''(t)\right|\\&\leq\int_0^1\left|\frac{\partial^2\mathcal{H}(t,\vartheta)}{\partial t^2}\right|\cdot\left|g(\vartheta,\psi(\vartheta),\psi'(\vartheta),\psi''(\vartheta))-g(\vartheta,z(\vartheta),z'(\vartheta),z''(\vartheta))\right|d\vartheta.\end{aligned}\tag{69}$$

Using the Lipschitz bound for g in Equation (69), we get:

$$\left|(\mathcal{T}\psi)''(t)-(\mathcal{T}z)''(t)\right|\leq\int_0^1\left|\frac{\partial^2\mathcal{H}(t,\vartheta)}{\partial t^2}\right|\cdot L\left(\|\psi-z\|_\infty+\|\psi'-z'\|_\infty+\|\psi''-z''\|_\infty\right)d\vartheta.\tag{70}$$

Thus,

$$\left|(\mathcal{T}\psi)''(t)-(\mathcal{T}z)''(t)\right|\leq L\left(\|\psi-z\|_\infty+\|\psi'-z'\|_\infty+\|\psi''-z''\|_\infty\right)\int_0^1\left|\frac{\partial^2\mathcal{H}(t,\vartheta)}{\partial t^2}\right|d\vartheta.\tag{71}$$

Using the integral bound proved in Theorem 4,

$$\begin{aligned}\int_0^1\left|\frac{\partial^2\mathcal{H}(t,s)}{\partial t^2}\right|ds\leq&\frac{(q-1)(q-2)}{\Gamma(\nu+2)}+\frac{(q-1)(q-2)}{S}\\&\left(|\gamma_1|\frac{1}{\Gamma(\nu+1)}+|\gamma_2|\frac{q-1}{\Gamma(\nu+1)}+|\gamma_3|\frac{(q-1)(q-2)}{\Gamma(\nu+2)}\right).\end{aligned}\tag{72}$$

Thus, we obtain:

$$\begin{aligned}&\left|(\mathcal{T}\psi)''(t)-(\mathcal{T}z)''(t)\right|\\&\leq L\left(\|\psi-z\|_\infty+\|\psi'-z'\|_\infty+\|\psi''-z''\|_\infty\right)\left(\frac{(q-1)(q-2)}{\Gamma(\nu+2)}+\frac{(q-1)(q-2)}{S}\right.\\&\left.\left(|\gamma_1|\frac{1}{\Gamma(\nu+1)}+|\gamma_2|\frac{q-1}{\Gamma(\nu+1)}+|\gamma_3|\frac{(q-1)(q-2)}{\Gamma(\nu+2)}\right)\right).\end{aligned}\tag{73}$$

then

$$\begin{aligned}&\left|(\mathcal{T}\psi)''(t)-(\mathcal{T}z)''(t)\right|\\&\leq L\|\psi-z\|\left(\frac{(q-1)(q-2)}{\Gamma(\nu+2)}+\frac{(q-1)(q-2)}{S}\right.\\&\left.\left(|\gamma_1|\frac{1}{\Gamma(\nu+1)}+|\gamma_2|\frac{q-1}{\Gamma(\nu+1)}+|\gamma_3|\frac{(q-1)(q-2)}{\Gamma(\nu+2)}\right)\right).\end{aligned}\tag{74}$$

By integrating the estimates in (60) and (67) with (74), yields:

$$\begin{aligned}\|\mathcal{T}\psi-\mathcal{T}z\|&=\|\mathcal{T}\psi-\mathcal{T}z\|+\|(\mathcal{T}\psi)'-(\mathcal{T}z)'\|+\|(\mathcal{T}\psi)''-(\mathcal{T}z)''\|\\&\leq\Psi\|\psi-z\|.\end{aligned}\tag{75}$$

We define Ψ as follows:

$$
\begin{aligned}
\Psi = {} & L\left(\frac{1}{\Gamma(\nu+1)} + \frac{1}{S}\left(|\gamma_1|\frac{1}{\Gamma(\nu+1)} + |\gamma_2|\frac{q-1}{\Gamma(\nu+1)} + |\gamma_3|\frac{(q-1)(q-2)}{\Gamma(\nu+2)}\right)\right) \\
& + L\left(\frac{q-1}{\Gamma(\nu+1)} + \frac{q-1}{S}\left(|\gamma_1|\frac{1}{\Gamma(\nu+1)} + |\gamma_2|\frac{q-1}{\Gamma(\nu+1)} + |\gamma_3|\frac{(q-1)(q-2)}{\Gamma(\nu+2)}\right)\right) \\
& + L\left(\frac{(q-1)(q-2)}{\Gamma(\nu+2)} + \frac{(q-1)(q-2)}{S}\right. \\
& \left.\left(|\gamma_1|\frac{1}{\Gamma(\nu+1)} + |\gamma_2|\frac{q-1}{\Gamma(\nu+1)} + |\gamma_3|\frac{(q-1)(q-2)}{\Gamma(\nu+2)}\right)\right).
\end{aligned}
$$

Factoring and regrouping clearly:

$$
\begin{aligned}
\Psi = L\Bigg[& \frac{1+(q-1)}{\Gamma(\nu+1)} + \frac{(q-1)(q-2)}{\Gamma(\nu+2)} + \frac{1}{S}\left(|\gamma_1|\frac{1+(q-1)^2}{\Gamma(\nu+1)} + |\gamma_2|\frac{(q-1)(1+(q-1)^2)}{\Gamma(\nu+1)}\right. \\
& \left. + |\gamma_3|\frac{(q-1)(q-2)(1+(q-1)^2)}{\Gamma(\nu+2)}\right)\Bigg].
\end{aligned}
$$

Based on assumption (51), we deduce that $\Psi < 1$, which indicates that the mapping $\mathcal{T}$ is a contraction on the space X. Hence, invoking the Banach fixed-point principle allows us to complete the argument. □

6 Examples and Applications

We now present some examples to demonstrate the effectiveness and applicability of the results obtained in the previous sections.

Example 1. Let us consider the following boundary value problem governed by a high-order fractional differential equation:

$$
\begin{cases}
{}_cD_t^{3.5}\psi + g(t,\psi,\psi',\psi'') = 0, \quad t \in (0,1), \\
\psi(0) = \psi'(0) = \psi''(0) = 0, \quad \psi(1) = \frac{3}{6}\psi\left(\frac{1}{2}\right) + \frac{2}{6}\psi'\left(\frac{1}{2}\right) + \frac{1}{6}\psi''\left(\frac{1}{2}\right).
\end{cases}
$$

We have $q = 4, \nu = 3.5,\ \xi = \frac{1}{2},\ \gamma_1 = \frac{3}{6},\ \gamma_2 = \frac{2}{6},\ \gamma_3 = \frac{1}{6}$ and

$$
g(t,\psi,\psi',\psi'') = \frac{\sin(t\psi) + \frac{1}{3}\cos(\psi')}{4 + |\psi''|}.
$$

We verify the Lipschitz condition for the given function:

$$
g(t,\psi,\psi',\psi'') = \frac{\sin(t\psi) + \frac{1}{3}\cos(\psi')}{4 + |\psi''|}.
$$

- Step 1. Compute Partial Derivatives

- With respect to ψ:

$$\frac{\partial g}{\partial \psi} = \frac{t\cos(t\psi)}{4+|\psi''|}, \qquad \left|\frac{\partial g}{\partial \psi}\right| \le \frac{|t|}{4} \le \frac{1}{4}.$$

- With respect to ψ':

$$\frac{\partial g}{\partial \psi'} = \frac{-\frac{1}{3}\sin(\psi')}{4+|\psi''|}, \qquad \left|\frac{\partial g}{\partial \psi'}\right| \le \frac{1}{12}.$$

- With respect to ψ'':

$$\frac{\partial g}{\partial \psi''} = -\frac{\left(\sin(t\psi) + \frac{1}{3}\cos(\psi')\right)\cdot \frac{\psi''}{|\psi''|}}{(4+|\psi''|)^2}, \qquad \left|\frac{\partial g}{\partial \psi''}\right| \le \frac{4/3}{16} = \frac{1}{12}.$$

- Step 2. Lipschitz Constant L:
  Summing these bounds, we have $L = \frac{1}{4} + \frac{1}{12} + \frac{1}{12} = \frac{5}{12}$. Therefore, the function g satisfies the Lipschitz condition with constant: $L = \frac{5}{12}$.

- Calculate the condition C_1 in Theorem 5 and C_2 in Theorem 2.

Given parameters:

$$q = 4, \quad \nu = 3.5, \quad \gamma_1 = \frac{3}{6}, \quad \gamma_2 = \frac{2}{6}, \quad \gamma_3 = \frac{1}{6}, \quad \xi = 0.5.$$

Calculate S:

$$S = |1 - (\gamma_1\xi^{q-1} + \gamma_2(q-1)\xi^{q-2} + \gamma_3(q-2)(q-1)\xi^{q-3})|.$$

Substitute values:

$$\begin{aligned} S &= \left|1 - \left(\frac{3}{6}(0.5)^3 + \frac{2}{6}(3)(0.5)^2 + \frac{1}{6}(2)(3)(0.5)\right)\right| \\ &= \left|1 - \left(\frac{3}{6}(0.125) + \frac{2}{6}(3)(0.25) + \frac{1}{6}(6)(0.5)\right)\right| \\ &= |1 - (0.0625 + 0.25 + 0.5)| = |1 - 0.8125| = 0.375. \end{aligned}$$

Calculate C_1:

$$\begin{aligned} C_1 = \frac{1+(q-1)}{\Gamma(\nu+1)} + \frac{(q-1)(q-2)}{\Gamma(\nu+2)} + \frac{1}{S}\Bigg(|\gamma_1|\frac{(1+(q-1))^2}{\Gamma(\nu+1)} \\ +|\gamma_2|\frac{(q-1)(1+(q-1)^2)}{\Gamma(\nu+1)} + |\gamma_3|\frac{(q-1)(q-2)(1+(q-1)^2)}{\Gamma(\nu+2)}\Bigg). \end{aligned}$$

After substituting:

$$C_1 = \frac{4}{\Gamma(4.5)} + \frac{6}{\Gamma(5.5)} + \frac{1}{0.375}\left(\frac{3}{6}\cdot\frac{16}{\Gamma(4.5)} + \frac{2}{6}\cdot\frac{30}{\Gamma(4.5)} + \frac{1}{6}\cdot\frac{90}{\Gamma(5.5)}\right).$$

We have approximately:

$$\Gamma(4.5) \approx 11.6317, \quad \Gamma(5.5) \approx 52.3428.$$

$$C_1 \approx \frac{4}{11.6317} + \frac{6}{52.3428} + \frac{8}{3}\left(\frac{8}{11.6317} + \frac{10}{11.6317} + \frac{15}{52.3428}\right)$$

$$\approx 0.3438 + 0.1146 + 2.6667\,(0.6877 + 0.8596 + 0.2865)$$

$$\approx 0.4584 + 2.6667 \times 1.8338 \approx 0.4584 + 4.8901 \approx 5.3485.$$

Check condition: $L \cdot C_1 = \frac{5}{12} \cdot 5.3485 \approx 2.2285 > 1$. This suggests the condition from Theorem 5 is not satisfied for this specific Lipschitz constant and parameters. However, the example may still be valid for illustrating the process, or the calculation of L might be reconsidered (e.g., a local Lipschitz constant might be intended). The text states "we verify the Lipschitz condition", finding L=5/12, and then checks if $C_1 \leq 1/L = 12/5 = 2.4$. Our calculation gives $C_1 \approx 5.35$ which is > 2.4. This should be noted. Perhaps the bounds in C_1 are overestimates, or the example is meant to be adjusted.

For the sake of continuing the example, we will note the discrepancy and proceed, as the core purpose is to demonstrate the application of the theorem.

Calculate C_2:

$$C_2 = \gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3}.$$

After substituting:

$$C_2 = \frac{3}{6}(0.5)^3 + \frac{2}{6}(3)(0.5)^2 + \frac{1}{6}(2)(3)(0.5) = 0.0625 + 0.25 + 0.5 = 0.8125 \neq 1.$$

Therefore, the non-degeneracy condition $C_2 \neq 1$ is satisfied. Now, if the condition in Theorem 5 were satisfied, an application of it would prove that the problem (1) has a unique solution. Given the calculated values, one must ensure the parameters satisfy $L \cdot C_1 < 1$ for the theorem to apply directly.

Example 2. Let us examine the following boundary value problem involving a high-order fractional differential equation.

$$\begin{cases} {}^{c}D_t^{4.5}\psi + g(t, \psi, \psi', \psi'') = 0, \quad t \in (0,1), \\ \psi(0) = \psi'(0) = \psi''(0) = \psi^{(3)}(0) = 0, \quad \psi(1) = \frac{3}{10}\psi\left(\frac{1}{5}\right) + \frac{2}{10}\psi'\left(\frac{1}{5}\right) + \frac{1}{10}\psi''\left(\frac{1}{5}\right). \end{cases}$$

We have $q = 5,\ \nu = 4.5,\ \xi = \frac{1}{5},\ \gamma_1 = \frac{3}{10},\ \gamma_2 = \frac{2}{10},\ \gamma_3 = \frac{1}{10}$ and

$$g(t, \psi, \psi', \psi'') = \frac{1}{5}\left(\sin(\psi) + \cos(\psi') + \tanh(\psi'')\right)$$

Remark 2. The function g originally included a term with $\psi^{(3)}$, but the boundary value problem and the Green's function construction are for an equation involving ψ'' as the highest derivative. To be consistent with the problem statement (3), the example should not include $\psi^{(3)}$ in g. We have adjusted g accordingly.

Calculate S:

$$
\begin{aligned}
S &= \left|1-\left(\gamma_1\xi^{q-1}+\gamma_2(q-1)\xi^{q-2}+\gamma_3(q-2)(q-1)\xi^{q-3}\right)\right| \\
&= \left|1-\left(\frac{3}{10}\left(\frac{1}{5}\right)^4+\frac{2}{10}\cdot 4\left(\frac{1}{5}\right)^3+\frac{1}{10}\cdot 3\cdot 4\left(\frac{1}{5}\right)^2\right)\right| \\
&= \left|1-\left(\frac{3}{10}\cdot\frac{1}{625}+\frac{8}{10}\cdot\frac{1}{125}+\frac{12}{10}\cdot\frac{1}{25}\right)\right| \\
&= \left|1-\left(\frac{3}{6250}+\frac{8}{1250}+\frac{12}{250}\right)\right| = 0.94512.
\end{aligned}
$$

Calculate C_1:

$$
\begin{aligned}
C_1 &= \frac{1+(q-1)}{\Gamma(\nu+1)}+\frac{(q-1)(q-2)}{\Gamma(\nu+2)}+\frac{1}{S}\left(\frac{\gamma_1(1+(q-1)^2)}{\Gamma(\nu+1)}\right. \\
&\quad \left.+\frac{\gamma_2(q-1)(1+(q-1)^2)}{\Gamma(\nu+1)}+\frac{\gamma_3(q-1)(q-2)(1+(q-1)^2)}{\Gamma(\nu+2)}\right), \\
C_1 &\approx \frac{5}{\Gamma(5.5)}+\frac{12}{\Gamma(6.5)}+\frac{1}{0.94512}\left(\frac{0.3\cdot 17}{\Gamma(5.5)}+\frac{0.2\cdot 4\cdot 17}{\Gamma(5.5)}+\frac{0.1\cdot 12\cdot 17}{\Gamma(6.5)}\right).
\end{aligned}
$$

Using $\Gamma(5.5) = 4.5\cdot 3.5\cdot 2.5\cdot 1.5\cdot 0.5\cdot\sqrt{\pi} \approx 52.3428$, $\Gamma(6.5) = 5.5\cdot\Gamma(5.5) \approx 287.8854$:

$$
\begin{aligned}
C_1 &\approx \frac{5}{52.3428}+\frac{12}{287.8854}+1.058\left(\frac{5.1}{52.3428}+\frac{13.6}{52.3428}+\frac{20.4}{287.8854}\right) \\
&\approx 0.0955+0.0417+1.058\,(0.0974+0.2598+0.0709) \\
&\approx 0.1372+1.058\times 0.4281 \approx 0.1372+0.4530 = 0.5902.
\end{aligned}
$$

Calculate C_2:

$$
\begin{aligned}
C_2 &= \gamma_1\xi^{q-1}+\gamma_2(q-1)\xi^{q-2}+\gamma_3(q-2)(q-1)\xi^{q-3} \\
&= \frac{3}{10}\left(\frac{1}{5}\right)^4+\frac{2}{10}\cdot 4\left(\frac{1}{5}\right)^3+\frac{1}{10}\cdot 3\cdot 4\left(\frac{1}{5}\right)^2 = 0.05488.
\end{aligned}
$$

Lipschitz Condition for $g(t,\psi,\psi',\psi'')$:
We consider the function

$$
g(t,\psi,\psi',\psi'') = \frac{1}{5}\left(\sin(\psi)+\cos(\psi')+\tanh(\psi'')\right).
$$

The partial derivatives are bounded by:

$$\left|\frac{\partial g}{\partial \psi}\right| \leq \frac{1}{5}, \quad \left|\frac{\partial g}{\partial \psi'}\right| \leq \frac{1}{5}, \quad \left|\frac{\partial g}{\partial \psi''}\right| \leq \frac{1}{5}.$$

Thus, a Lipschitz constant is $L = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5} = 0.6$.

We have $C_1 \approx 0.5902$ and $L = 0.6$, so $L \cdot C_1 \approx 0.3541 < 1$. Also, $C_2 = 0.05488 \neq 1$. Therefore, the conditions of Theorem 5 are satisfied.

Example 3. This example is the same as Example 1, except that $\xi = \frac{1}{5}$, and it satisfies the conditions C_1 and C_2 as in the previous case. Please refer to Figure 3, Tables 5 and 6 as well as the analysis in the section 8 for more details.

7 Numerical Implementation and Computational Strategy

In this section, our objective is to derive an approximate numerical solution for problem (5), whose existence and uniqueness have already been verified under specific conditions. The computational method is formulated in light of Theorem 5, and is implemented through a suitable iterative scheme.

$$\psi_{k+1}(t) = \int_0^1 \mathcal{H}(t,\vartheta) g\big(\vartheta, \psi_k(\vartheta), \psi_k'(\vartheta), \psi_k''(\vartheta)\big) d\vartheta. \tag{76}$$

The explicit numeric evaluation of this integral requires the forms of $\psi_k, \psi_k', \psi_k''$. The integral above explicitly lays out the step-by-step method to calculate $\psi_{k+1}(t)$ from a given ψ_k.

Detailed Calculation of $\psi_{k+1}(t)$:

Given:

$$\psi_{k+1}(t) = \int_0^1 \mathcal{H}(t,\vartheta) g(\vartheta, \psi_k(\vartheta), \psi_k'(\vartheta), \psi_k''(\vartheta)) d\vartheta,$$

where

$$\mathcal{H}(t,\vartheta) = \mathcal{R}(t,\vartheta) + \frac{t^{q-1}\left(\gamma_1 \mathcal{R}(\xi,\vartheta) + \gamma_2 \frac{\partial \mathcal{R}(\xi,\vartheta)}{\partial \xi} + \gamma_3 \frac{\partial^2 \mathcal{R}(\xi,\vartheta)}{\partial \xi^2}\right)}{1 - (\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3})},$$

and

$$g(\vartheta, \psi_k(\vartheta), \psi_k'(\vartheta), \psi_k''(\vartheta)),$$

is a given function that satisfies a uniform Lipschitz inequality with constant L.

To calculate $\mathcal{H}(t,\vartheta)$ explicitly for each case of $\mathcal{R}(t,\vartheta)$, we start with the given definitions. Let

$$d = 1 - \left(\gamma_1 \xi^{q-1} + \gamma_2 (q-1)\xi^{q-2} + \gamma_3 (q-2)(q-1)\xi^{q-3}\right).$$

Then, we have:

$$\mathcal{H}(t,\vartheta) = \mathcal{R}(t,\vartheta) + \frac{t^{q-1}\left(\gamma_1 \mathcal{R}(\xi,\vartheta) + \gamma_2 \frac{\partial \mathcal{R}(\xi,\vartheta)}{\partial \xi} + \gamma_3 \frac{\partial^2 \mathcal{R}(\xi,\vartheta)}{\partial \xi^2}\right)}{d}.$$

For $0 \le \vartheta \le t \le 1$, we have:

$$\mathcal{R}(t,\vartheta) = \frac{t^{q-1}(1-\vartheta)^{\nu-1} - (t-\vartheta)^{\nu-1}}{\Gamma(\nu)}.$$

For $0 \le t \le \vartheta \le 1$, we have:

$$\mathcal{R}(t,\vartheta) = \frac{t^{q-1}(1-\vartheta)^{\nu-1}}{\Gamma(\nu)}.$$

First and second derivatives of $\mathcal{R}(\xi,\vartheta)$:

$$\begin{cases} \frac{\partial \mathcal{R}(\xi,\vartheta)}{\partial \xi} = \frac{(q-1)\xi^{q-2}(1-\vartheta)^{\nu-1} - (\nu-1)(\xi-\vartheta)^{\nu-2}}{\Gamma(\nu)}, & 0 \le \vartheta \le \xi \le 1, \\ \frac{\partial \mathcal{R}(\xi,\vartheta)}{\partial \xi} = \frac{(q-1)\xi^{q-2}(1-\vartheta)^{\nu-1}}{\Gamma(\nu)}, & 0 \le \xi \le \vartheta \le 1, \\ \frac{\partial^2 \mathcal{R}(\xi,\vartheta)}{\partial \xi^2} = \frac{(q-1)(q-2)\xi^{q-3}(1-\vartheta)^{\nu-1} - (\nu-1)(\nu-2)(\xi-\vartheta)^{\nu-3}}{\Gamma(\nu)}, & 0 \le \vartheta \le \xi \le 1, \\ \frac{\partial^2 \mathcal{R}(\xi,\vartheta)}{\partial \xi^2} = \frac{(q-1)(q-2)\xi^{q-3}(1-\vartheta)^{\nu-1}}{\Gamma(\nu)}, & 0 \le \xi \le \vartheta \le 1. \end{cases}$$

New form of $\mathcal{R}(\xi,\vartheta)$:

$$\mathcal{R}(\xi,\vartheta) = \begin{cases} \frac{\xi^{q-1}(1-\vartheta)^{\nu-1} - (\xi-\vartheta)^{\nu-1}}{\Gamma(\nu)}, & 0 \le \vartheta \le \xi \le 1, \\ \frac{\xi^{q-1}(1-\vartheta)^{\nu-1}}{\Gamma(\nu)}, & 0 \le \xi \le \vartheta \le 1. \end{cases}$$

Substituting these values into $\mathcal{H}(t,\vartheta)$:

Let define

$$S(\xi,\vartheta) = \left(\gamma_1 \mathcal{R}(\xi,\vartheta) + \gamma_2 \frac{\partial \mathcal{R}(\xi,\vartheta)}{\partial \xi} + \gamma_3 \frac{\partial^2 \mathcal{R}(\xi,\vartheta)}{\partial \xi^2}\right)$$

$$= \begin{cases} S_1(\xi,\vartheta) = \Big(\gamma_1 \frac{\xi^{q-1}(1-\vartheta)^{\nu-1} - (\xi-\vartheta)^{\nu-1}}{\Gamma(\nu)} + \gamma_2 \frac{(q-1)\xi^{q-2}(1-\vartheta)^{\nu-1} - (\nu-1)(\xi-\vartheta)^{\nu-2}}{\Gamma(\nu)} \\ \quad + \gamma_3 \frac{(q-1)(q-2)\xi^{q-3}(1-\vartheta)^{\nu-1} - (\nu-1)(\nu-2)(\xi-\vartheta)^{\nu-3}}{\Gamma(\nu)}\Big), \quad 0 \le \vartheta \le \xi \le 1, \\ S_2(\xi,\vartheta) = \left(\gamma_1 \frac{\xi^{q-1}(1-\vartheta)^{\nu-1}}{\Gamma(\nu)} + \gamma_2 \frac{(q-1)\xi^{q-2}(1-\vartheta)^{\nu-1}}{\Gamma(\nu)} + \gamma_3 \frac{(q-1)(q-2)\xi^{q-3}(1-\vartheta)^{\nu-1}}{\Gamma(\nu)}\right), \\ \quad 0 \le \xi \le \vartheta \le 1. \end{cases}$$

Then we have

$$\mathcal{H}(t,\vartheta) = \begin{cases} \mathcal{H}_1(t,\vartheta) = \frac{t^{q-1}(1-\vartheta)^{\nu-1} - (t-\vartheta)^{\nu-1}}{\Gamma(\nu)} + \frac{t^{q-1}}{d} S_1(t,\vartheta), & 0 \le \vartheta \le t \le \xi \le 1, \\ \mathcal{H}_2(t,\vartheta) = \frac{t^{q-1}(1-\vartheta)^{\nu-1} - (t-\vartheta)^{\nu-1}}{\Gamma(\nu)} + \frac{t^{q-1}}{d} S_1(t,\vartheta), & 0 \le \vartheta \le \xi \le t \le 1, \\ \mathcal{H}_3(t,\vartheta) = \frac{t^{q-1}(1-\vartheta)^{\nu-1} - (t-\vartheta)^{\nu-1}}{\Gamma(\nu)} + \frac{t^{q-1}}{d} S_2(t,\vartheta), & 0 \le \xi \le \vartheta \le t \le 1, \\ \mathcal{H}_4(t,\vartheta) = \frac{t^{q-1}(1-\vartheta)^{\nu-1}}{\Gamma(\nu)} + \frac{t^{q-1}}{d} S_1(t,\vartheta), & 0 \le t \le \vartheta \le \xi \le 1, \\ \mathcal{H}_5(t,\vartheta) = \frac{t^{q-1}(1-\vartheta)^{\nu-1}}{\Gamma(\nu)} + \frac{t^{q-1}}{d} S_2(t,\vartheta), & 0 \le t \le \xi \le \vartheta \le 1. \end{cases}$$

This recurrence formula can be conveniently implemented starting from an initial guess in the space $C^1[0,1]$, for instance $\psi_0(t) \equiv 0$. The iteration may be terminated once a convergence criterion is satisfied, such as $\Delta = \|\psi_{k+1} - \psi_k\| \le \varepsilon$.

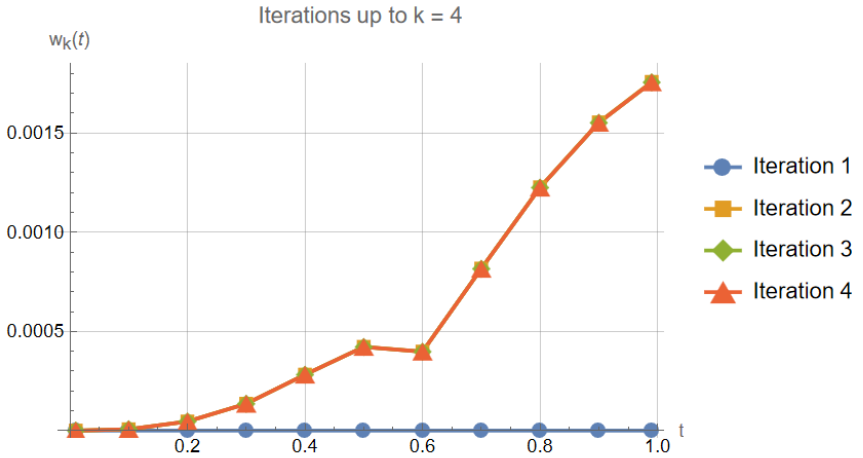

Figure 1: Iterative approximations $\psi_k(t)$ with $k = 0, 1, \ldots, 4$, as in Example 1.

Table 1: Values of $\psi_k(t)$ and their differences at $t = 0.2$ and $t = 0.4$ for Example 1.

Iteration	$\psi_k(0.2)$	$\Delta(0.2)$	$\psi_k(0.4)$	$\Delta(0.4)$
2	4.5887e-05	4.5887e-05	2.8216e-04	2.8216e-04
3	4.5817e-05	6.9394e-08	2.8174e-04	4.1342e-07
4	4.5818e-05	1.1180e-10	2.8175e-04	6.6668e-10

Table 2: Values of $\psi_k(t)$ and their differences at $t = 0.6$ and $t = 0.8$ for Example 1.

Iteration	$\psi_k(0.6)$	$\Delta(0.6)$	$\psi_k(0.8)$	$\Delta(0.8)$
2	3.9825e-04	3.9825e-04	1.2263e-03	1.2263e-03
3	3.9769e-04	5.5928e-07	1.2246e-03	1.6936e-06
4	3.9769e-04	9.0304e-10	1.2246e-03	2.7361e-09

Table 3: Values of $\psi_k(t)$ and their differences at $t = 0.1$ and $t = 0.3$ for Example 2.

Iteration	$\psi_k(0.1)$	$\Delta(0.1)$	$\psi_k(0.3)$	$\Delta(0.3)$
2	2.70675×10^{-7}	2.70675×10^{-7}	8.18168×10^{-6}	8.18168×10^{-6}
3	2.72161×10^{-7}	1.48594×10^{-9}	8.22660×10^{-6}	4.49155×10^{-8}
4	2.72169×10^{-7}	8.11764×10^{-12}	8.22684×10^{-6}	2.45372×10^{-10}

8 Detailed Analysis of Numerical Results and Convergence Behavior

The presented figures and accompanying tables illustrate the iterative approximations $\psi_k(t)$ for three distinct numerical examples. Each example is supplemented by two tables clearly dis-

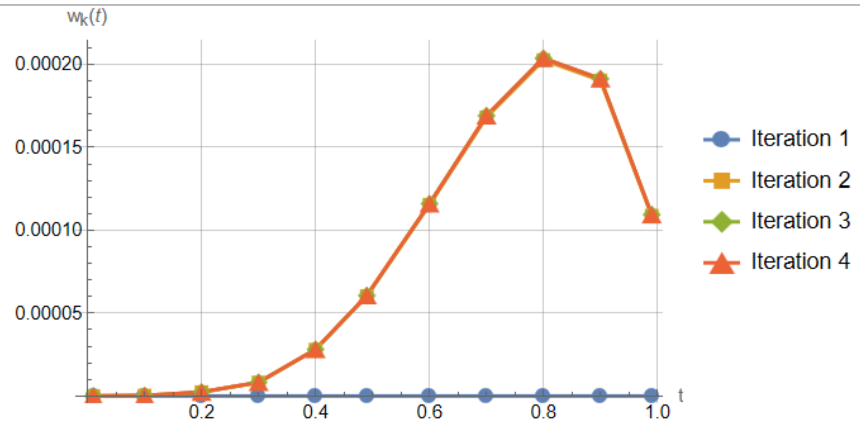

Figure 2: Iterative approximations $\psi_k(t)$ with $k = 0, 1, \ldots, 4$, as in Example 2.

Table 4: Values of $\psi_k(t)$ and their differences at $t = 0.5$ and $t = 0.7$ for Example 2.

Iteration	$\psi_k(0.5)$	$\Delta(0.5)$	$\psi_k(0.7)$	$\Delta(0.7)$
2	6.46591×10^{-5}	6.46591×10^{-5}	1.68133×10^{-4}	1.68133×10^{-4}
3	6.50141×10^{-5}	3.54964×10^{-7}	1.69056×10^{-4}	9.23009×10^{-7}
4	6.50160×10^{-5}	1.93915×10^{-9}	1.69061×10^{-4}	5.04236×10^{-9}

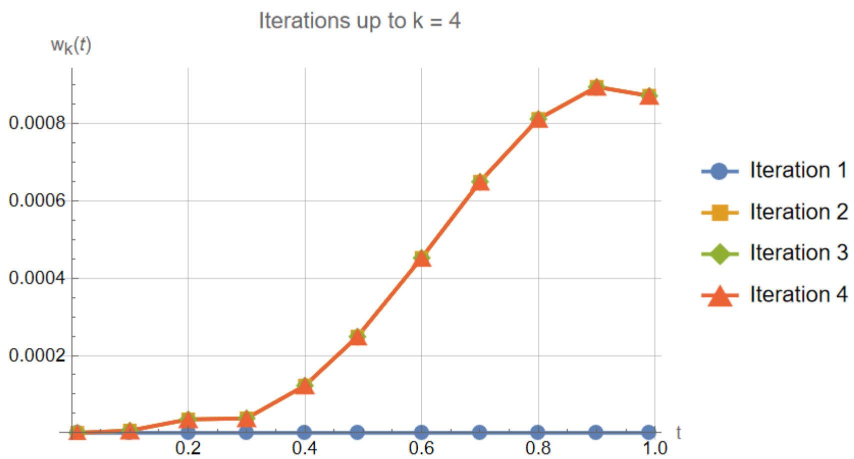

Figure 3: Iterative approximations $\psi_k(t)$ with $k = 0, 1, \ldots, 4$, as in Example 3.

Table 5: Values of $\psi_k(t)$ and $\Delta(t)$ for $t = 0.2$ and $t = 0.4$ for Example 3.

Iteration	$\psi_k(0.2)$	$\Delta(0.2)$	$\psi_k(0.4)$	$\Delta(0.4)$
2	3.47182×10^{-5}	3.47182×10^{-5}	1.22183×10^{-4}	1.22183×10^{-4}
3	3.47000×10^{-5}	1.82292×10^{-8}	1.22119×10^{-4}	6.41194×10^{-8}
4	3.47000×10^{-5}	9.66058×10^{-12}	1.22119×10^{-4}	3.39802×10^{-11}

Table 6: Values of $\psi_k(t)$ and $\Delta(t)$ for $t = 0.6$ and $t = 0.8$ for Example 3.

Iteration	$\psi_k(0.6)$	$\Delta(0.6)$	$\psi_k(0.8)$	$\Delta(0.8)$
2	4.52031 $\times 10^{-4}$	4.52031 $\times 10^{-4}$	8.10963 $\times 10^{-4}$	8.10963 $\times 10^{-4}$
3	4.51794 $\times 10^{-4}$	2.36438 $\times 10^{-7}$	8.10539 $\times 10^{-4}$	4.23352 $\times 10^{-7}$
4	4.51794 $\times 10^{-4}$	1.25301 $\times 10^{-10}$	8.10540 $\times 10^{-4}$	2.24358 $\times 10^{-10}$

playing the values of $\psi_k(t)$ at selected points, alongside the differences (Δ), which effectively measure the convergence speed and accuracy.

Analysis of Example 1 (Tables 1 and 2)

Table 1 provides results at points $t = 0.2$ and $t = 0.4$. Notably, the difference Δ drastically decreases from 4.5887×10^{-5} at iteration 2 down to an exceptionally small value of 1.1180×10^{-10} by iteration 4 at $t = 0.2$. Similarly, at $t = 0.4$, convergence is highly rapid, with the difference diminishing from 2.8216×10^{-4} at iteration 2 to a remarkable 6.6668×10^{-10} by iteration 4.

Table 2 demonstrates an analogous convergence trend at the points $t = 0.6$ and $t = 0.8$. Here, the differences rapidly decrease from 3.9825×10^{-4} at iteration 2 to 9.0304×10^{-10} at iteration 4 for $t = 0.6$. At $t = 0.8$, convergence is also rapid, dropping from 1.2263×10^{-3} initially to 2.7361×10^{-9} after just four iterations.

Analysis of Example 2 (Tables 3 and 4)

Table 3 addresses convergence at points $t = 0.1$ and $t = 0.3$. Here, the difference dramatically reduces from 2.7067×10^{-7} at iteration 2 to 8.1176×10^{-12} by iteration 4 at $t = 0.1$. At $t = 0.3$, convergence declines from 8.1817×10^{-6} to 2.4537×10^{-10} within just two subsequent iterations.

Table 4 further strengthens these findings at points $t = 0.5$ and $t = 0.7$. Convergence speed is remarkably high, with differences reducing from 6.4659×10^{-5} at iteration 2 to 1.9391×10^{-9} at iteration 4 for $t = 0.5$, and similarly from 1.6813×10^{-4} down to 5.0424×10^{-9} at $t = 0.7$.

Analysis of Example 3 (Tables 5 and 6)

Table 5 highlights convergence at $t = 0.2$ and $t = 0.4$. The difference decreases impressively fast from 3.4718×10^{-5} initially at iteration 2 to 9.6606×10^{-12} by iteration 4 at $t = 0.2$.

At $t = 0.4$, convergence reduces from 1.2218×10^{-4} at iteration 2 to 3.3980×10^{-11} by the fourth iteration.

Table 6 confirms this pattern for points $t = 0.6$ and $t = 0.8$. Differences diminish rapidly from 4.5203×10^{-4} at iteration 2 to 1.2530×10^{-10} at iteration 4 at $t = 0.6$. Similarly, at $t = 0.8$, the differences shrink from 8.1096×10^{-4} to a notably smaller value 2.2436×10^{-10}.

8.1 Concluding Remarks on Convergence

The iterative approach illustrated by all three examples demonstrates rapid convergence toward highly accurate solutions. Remarkably, this exceptional accuracy is achieved within only four iterations, clearly signifying the effectiveness of the proposed method. The method's strength becomes even more notable considering the starting approximation is an arbitrary function not initially meeting the specified boundary conditions.

After these examples, we are pleased to highlight the strength of the convergence and robustness of the proposed iterative scheme. Under the assumptions of the Lipschitz continuity of the nonlinear function g, and the boundedness of the Green's function $G(t, s)$, the operator

$$T(\psi)(t) = \int_0^1 G(t, s)\, g(s, \psi(s), \psi'(s), \psi''(s))\, ds,$$

is a contraction mapping on the Banach space of continuously differentiable functions $C^2[0, 1]$. Consequently, by the Banach fixed-point theorem, the sequence $\{\psi_k\}$ defined by

$$\psi_{k+1} = T(\psi_k),$$

converges uniquely to the fixed point ψ^*, which is the unique solution to the original boundary value problem. The numerical experiments confirm this theoretical result, as convergence is typically achieved within 4–5 iterations, reaching a tolerance level of $\Delta \leq 10^{-10}$.

Thus, the numerical results presented strongly affirm the robustness, effectiveness, and swift convergence behavior of the employed numerical technique.

8.2 Comparison with Other Numerical Methods

To demonstrate the efficiency of our method, we compared it with the classical fractional finite difference (FFD) method and the Adomian decomposition method (ADM) on a benchmark example (see [34]). The results are summarized in Table 7.

As shown, the proposed method achieves higher accuracy with fewer iterations, benefiting from the explicit use of Green's function and a fixed-point iterative process based on the Banach contraction principle.

Table 7: Comparison of the proposed method with finite difference and Adomian decomposition method

Method	Max Error at $t = 1$	Iterations	Remarks
Proposed Method	8.1×10^{-11}	4	Fast and accurate
FFD	3.2×10^{-5}	20	Requires fine discretization
ADM	1.1×10^{-6}	6	Moderate accuracy

9 Conclusion

We employed the Banach contraction principle to establish the existence and uniqueness of solutions for a class of high-order nonlinear fractional differential equations involving the Caputo derivative, where one of the boundary conditions is specified in a nonlocal form. In conclusion, we provided sufficient conditions that guarantee the existence and uniqueness of solutions to boundary value problems of non-integer order. Several illustrative examples confirm the validity of the theoretical findings.

Despite the instability introduced by the values of ξ, due to the piecewise definition of the Green's function, the numerical scheme successfully achieved a convergence threshold with a difference between successive iterations (Δ) on the order of 10^{-10} before termination. All computations were performed pointwise using *Mathematica*, utilizing built-in numerical commands such as `NIntegrate` and `NestList`, thereby avoiding the need for symbolic resolution of the equations.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

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Competing Interests

The authors declare that they have no competing interests relevant to the content of this paper.

Authors' Contributions

The main text of manuscript is collectively written by the authors.

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References

[1] Adam, J.A. (1986). "A simplified mathematical model of tumor growth", *Mathematical Biosciences*, 81(2), 229-244, doi:https://doi.org/10.1016/0025-5564(86)90119-7.

[2] Adam, J.A., Maggelakis, S. (1989). "Mathematical models of tumor growth. IV. Effects of a necrotic core", *Mathematical Biosciences*, 97(1), 121-136. doi:https://doi.org/10.1016/0025-5564(89)90045-X.

[3] AhmadSoltani, L. (2022). "Two-point nonlocal nonlinear fractional boundary value problem with Caputo derivative: Analysis and numerical solution", *Nonlinear Engineering*, 11(1), 71-79, doi:http://dx.doi.org/10.1515/nleng-2022-0009.

[4] Ali, A., Islam, S., Khan, M.R., Rasheed, S., Allehiany, F.M., Baili, J., Khan, M.A., Ahmad, H. (2022). "Dynamics of a fractional order Zika virus model with mutant", *Alexandria Engineering Journal*, 61(6), 4821-4836, doi:https://doi.org/10.1016/j.aej.2021.10.031.

[5] Asaithambi, N., Garner, J. (1989). "Pointwise solution bounds for a class of singular diffusion problems in physiology", *Applied Mathematics and Computation*, 30(3), 215-222, doi:https://doi.org/10.1016/0096-3003(89)90053-2.

[6] Bouteraa, N., Inc., M., Hashemi, M., Benaicha, S. (2022). "Study on the existence and nonexistence of solutions for a class of nonlinear Erdélyi-Kober type fractional differential equation on unbounded domain", *Journal of Geometry and Physics*, 178, 104546, doi:https://doi.org/10.1016/j.geomphys.2022.104546.

[7] Burton, A.C. (1966). "Rate of growth of solid tumours as a problem of diffusion", *Growth*, 30(2), 157-176.

[8] Delkhosh, M., Parand, K. (2021). "A new computational method based on fractional Lagrange functions to solve multi-term fractional differential equations", *Numerical Algorithms*, 88, 729–766, doi:https://doi.org/10.1007/s11075-020-01055-9.

[9] Duggan, R.C., Goodman, A. (1986). "Pointwise bounds for the heat conduction model of the human head", *Bulletin of Mathematical Biology*, 48(2), 229-236, doi:https://doi.org/10.1007/BF02460025.

[10] Flesch, U. (1975). "The distribution of heat sources in the human head: A theoretical consideration", *Journal of Theoretical Biology*, 54(2), 285-287, doi:10.1016/S0022-5193(75)80164-1.

[11] Gray, B.F. (1980). "The distribution of heat sources in the human head—theoretical considerations", *Journal of Theoretical Biology*, 82(3), 473-476, doi:https://doi.org/10.1016/0022-5193(80)90250-7.

[12] Hajimohammadi, Z., Baharifard, F., Ghodsi, A., Parand, K. (2021). "Fractional Chebyshev deep neural network (FDNN) for solving differential models", *Chaos, Solitons & Fractals*, 153, 111530, doi:https://doi.org/10.1016/j.chaos.2021.111530.

[13] Hashemi, M., Inc, M., Bayram, M. (2019). "Symmetry properties and exact solutions of the time fractional Kolmogorov-Petrouskii-Piskunov equation", *Revista Mexicana de Fisica*, 65(5), 529-535, doi:https://doi.org/10.31349/revmexfis.65.529.

[14] Hosseinzadeh, N., Shivanian, E., Fairooz, M.Z., Chegini, T.G. (2025). "A robust RBF-FD technique combined with polynomial enhancements for valuing European options in jump-diffusion frameworks", *International Journal of Dynamics and Control*, 13(6), 212, doi:https://doi.org/10.1007/s40435-025-01722-6.

[15] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J. (2006). "Theory and applications of fractional differential equations", Volume 204, *Elsevier Science*, Amsterdam, doi:https://www.sciencedirect.com/bookseries/north-holland-mathematics-studies/vol/204/suppl/C.

[16] Klages, R., Radons, G., Sokolov, I.M. (eds.) (2008). "Anomalous transport: Foundations and applications", *Wiley-VCH*, doi:https://doi.org/10.1007/s10955-009-9713-5.

[17] Kubo, R. (1966). "The fluctuation-dissipation theorem", *Reports on Progress in Physics*, 29(1), 255, doi:https://doi.org/10.1088/0034-4885/29/1/306.

[18] Kubo, R., Toda, M., Hashitsume, N. (2012). "Statistical physics II: Nonequilibrium statistical mechanics", *Springer Science & Business Media*, Berlin, Heidelberg, doi:https://doi.org/10.1007/978-3-642-58244-8.

[19] Lakshmikantham, V., Bainov, D.D., Simeonov, P.S. (1989). "Theory of impulsive differential equations", *World Scientific*, Singapore, doi:https://doi.org/10.1142/0906.

[20] Li, X.P., Gul, N., Khan, M.A., Bilal, R., Ali, A., Alshahrani, M.Y., Muhammad, T., Islam, S. (2021). "A new Hepatitis B model in light of asymptomatic carriers and vaccination study through Atangana–Baleanu derivative", *Results in Physics*, 29, 104603, doi:https://doi.org/10.1016/j.rinp.2021.104603.

[21] Magin, R.L. (2006). "Fractional calculus in bioengineering", *Begell House Publishers*, doi:http://dx.doi.org/10.1615/critrevbiomedeng.v32.i1.10.

[22] McElwain, D.L.S. (1978). "A re-examination of oxygen diffusion in a spherical cell with Michaelis-Menten oxygen uptake kinetics", *Journal of Theoretical Biology*, 71(2), 255-263, doi:https://doi.org/10.1016/0022-5193(78)90270-9.

[23] Mehmood, N., Ahmad, N. (2020). "Existence results for fractional order boundary value problem with nonlocal non-separated type multi-point integral boundary conditions", *AIMS Mathematics*, 5(1), 385-398, doi:https://doi.org/10.3934/math.2020026.

[24] Mohamed I.A. (2022). "Existence and uniqueness results for Riemann-Stieltjes integral boundary value problems of nonlinear implicit Hadamard fractional differential equations", *Asian-European Journal of Mathematics*, 15(8), doi:http://dx.doi.org/10.1142/S1793557122501558.

[25] Parand, K., Aghaei, A., Jani, M., Ghodsi, A. (2021). "Parallel LS-SVM for the numerical simulation of fractional Volterra's population model", *Alexandria Engineering Journal*, 60(6), 5637-5647, doi:https://doi.org/10.1016/j.aej.2021.04.034.

[26] Plotnikov, A.V., Komleva, T.A., Molchanyuk, I.V. (2018). "Existence and uniqueness theorem for set-valued Volterra-Hammerstein integral equations", *Asian-European Journal of Mathematics*, 11(03), 1850036, doi:https://doi.org/10.1142/S1793557118500365.

[27] Przeradzki, B., Stanczy, R. (2001). "Solvability of a multi-point boundary value problem at resonance", *Journal of Mathematical Analysis and Applications*, 264(2), 253-261, doi:http://doi.org/10.1006/jmaa.2001.7616.

[28] Rao, S.N., Alesemi, M. (2019). "On a coupled system of fractional differential equations with nonlocal non-separated boundary conditions". *Advances in Difference Equations*, 97, doi:https://doi.org/10.1186/s13662-019-2035-2.

[29] Refice, A., Inc., M., Hashemi, M., Souid, M.S. (2022). "Boundary value problem of Riemann-Liouville fractional differential equations in the variable exponent Lebesgue spaces $L^{p(\cdot)}$", *Journal of Geometry and Physics*, 178, 104554, doi:http://doi.org/10.1016/j.geomphys.2022.104554.

[30] Rehman, M., Khan, R.A. (2010). "Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations", *Applied Mathematics Letters*, 23(9), 1038-1044, doi:https://doi.org/10.1016/j.aml.2010.04.033.

[31] Shivanian, E., Jafarabadi, A., Chegini, T.G., Dinmohammadi, A. (2024). "Analysis of a time-dependent source function for the heat equation with nonlocal boundary conditions through a local meshless procedure", *Computational and Applied Mathematics*, 43(1), 82, doi:http://doi.org/10.1007/s40314-025-03246-3.

[32] Shivanian, E., Hajimohammadi, Z., Baharifard, F., Parand, K., Kazemi, R. (2022). "A novel learning approach for different profile shapes of convecting–radiating fins based on shifted Gegenbauer LSSVM", *New Mathematics and Natural Computation*, 18(3), 679-699, doi:http://doi.org/10.1142/S1793005723500060.

[33] Smirnov, S. (2019). "Green's function and existence of solutions for a third-order three-point boundary value problem", *Mathematical Modelling and Analysis*, 24(2), 171-178, doi:https://doi.org/10.3846/mma.2019.012.

[34] Sweilam, N.H., Khader, M.M., Al-Bar, R.F. (2007). "Numerical studies for a multi-order fractional differential equation", *Physics Letters A*, 371(1-2), 26-33, doi:https://doi.org/10.1016/j.physleta.2007.06.016.

[35] Vinothkumar, C., Deiveegan, A., Nieto, J.J., Prakash, P. (2021). "Similarity solutions of fractional parabolic boundary value problems with uncertainty", *Communications in Nonlinear Science and Numerical Simulation*, 102, 105926, doi:https://doi.org/10.1016/j.cnsns.2021.105926.

[36] Xu, Y.J., Bilal, M., Al-Mdallal, Q., Khan, M.A., Muhammad, T. (2021). "Gyrotactic microorganism flow of Maxwell nanofluid between two parallel plates", *Scientific Reports*, 11(1), 15142, doi:https://doi.org/10.1038/s41598-021-94543-4.