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Ritz-Approximation Method for Solving Variable-Order Fractional Mobile-Immobile Advection-Dispersion Equations

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Abstract. The advection-dispersion, variable-order differential equations have a vast application in fluid physics and energy systems. In this study, we propose a Ritz-approximation method using shifted Legendre polynomials to construct approximate numerical solutions for these equations. The proposed method discretizes the original problem, converting it into a system of nonlinear algebraic equations that can be solved numerically at selected points. We discuss the error characteristics of the proposed method. For validation, the presented examples are compared with exact solutions and with prior results. The results indicate that the proposed method is highly effective.

Keywords. Caputo fractional derivative, Mobile-immobile advection-dispersion, Ritz-approximation method, Time variable fractional order, Satisfiers function.

MSC. 35R11.

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1 Introduction

Integer-order differential equations fail to model numerous phenomena, including aspects of statistical mechanics, nonlinear earthquake vibrations, and continuum and fluid-dynamic transport, e.g., [2, 8, 19]. Although the concept of fractional calculus dates back to Newton and Leibniz, it has attracted substantial renewed attention in recent years. A notable advantage of fractional derivatives in dynamic systems is their nonlocality, meaning that the current state depends on the entire history of the system [9].

Transport dynamics and anomalous diffusion are ubiquitous. In nature, many complex-system phenomena have been effectively described using enhanced fractional differential equations. Samko and Ross introduced the theory of variable-order operators through Fourier-transform methods and the notion of variable order [22]. The variable-order fractional (VOF) derivative offers a robust mathematical framework for modeling complex dynamics in porous and heterogeneous environments and encompasses multiple definitions depending on the context [13, 18]. Research on VOF partial differential equations is still in its early stages, and numerical approximation methods for these equations are actively being developed.

A new set of equations, the advection-dispersion equations, is obtained by combining the advection and diffusion processes. The mobile–immobile model is a special case of this formulation. These equations are used to model pollutant transport, energy transfer, and subsurface/river flows in the subsurface and in deep river systems [4, 6, 7, 15].

Significant progress has been made in approximating VOF mobile-immobile advection-dispersion equations. For example, some numerical methods for two-dimensional arbitrary domains have been introduced, including reproducing kernel theory and collocation method (RKM) [14], the approximate implicit Euler method [27], the Chebyshev wavelets method [11], a meshless MLS-based approach [24], and the shifted Jacobi Gauss-Radau spectral method expressed in the Coimbra sense for time-variant fractional derivatives [17].

In the present study, we apply the Ritz-approximation method using Shifted Legendre polynomials to solve the VOF mobile-immobile advection-dispersion equations. These equations are formed by incorporating the time VOF derivative in the Caputo sense, with $0 < \alpha(x,t) \leq 1$, into the standard advection-dispersion equation [20], which models solute transport and total concentration in watershed catchments and rivers. The governing equation under consideration has the following form:

$$\eta_1\, v_t(x,t) + \eta_2\, {}^{c}\mathcal{D}_t^{\alpha(x,t)} v(x,t) + B\, v_x(x,t) - C\, v_{xx}(x,t) = g(x,t), \quad (x,t) \in [0,a] \times [0,b], \tag{1}$$

under the following initial and boundary conditions:

$$\begin{cases} v(x,0) = \phi(x), & x \in [0,a], \\ v(0,t) = \psi_1(t), \;\; v(b,t) = \psi_2(t), & t \in [0,b], \end{cases}$$

where $\eta_1, \eta_2 \geq 0, B > 0, C > 0$, and ${}^{c}\mathcal{D}_t^{\alpha(x,t)}$ denotes the Caputo variable-order fractional derivative with respect to time. $\phi(x), \psi_1(t), \psi_2(t)$ are enough smooth functions that are given and $v(x,t)$, is the unknown function to be determind.

This paper is organized as follows: in Section 2 several preliminaries of the VOF derivative and SLP are represented. In Section 3, function approximation is described. In Section 4, the error bound is estimated and in Section 5, the Ritz method is defined. In Section 6, several numerical examples have been solved by the stated method, and results are shown. Finally, conclusion is made in Section 7.

2 Preliminaries and Definitions

In this section, we present several essential preliminaries and definitions related to the VOF derivative and SLPs, which provide the foundational tools for the proposed method.

2.1 Fractional Variable-Order Derivative

The definitions of left and right VO Riemann-Liouville integrals with hiding memory are then proposed as [3]

$${}_aI_t^{\alpha(x,t)}f(t) = \int_a^t \frac{1}{\Gamma(\alpha(x,t))}(t-s)^{\alpha(x,t)-1}f(s)ds,$$

and

$${}_tI_b^{\alpha(x,t)}f(t) = \int_t^b \frac{1}{\Gamma(\alpha(x,t))}(s-t)^{\alpha(x,t)-1}f(s)ds.$$

The left-side and right-side VO Riemann-Liouville fractional derivatives are stated as [5]

$${}_a^{RL}D_t^{\alpha(t)}f(t) = \frac{1}{\Gamma(n-\alpha(t))}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha(s)-1}f(s)ds, \quad n-1<\alpha(t)<n,$$

and

$${}_t^{RL}D_b^{\alpha(t)}f(t) = \frac{(-1)^n}{\Gamma(n-\alpha(t))}\frac{d^n}{dt^n}\int_t^b (s-t)^{n-\alpha(s)-1}f(s)ds, \quad n-1<\alpha(t)<n.$$

Meanwhile, the left Riemann-Liouville fractional derivative of order $\alpha(s,t)$ is defined as [16].

$${}_a^{RL}D_t^{\alpha(x,t)}f(t) = \frac{d^n}{dt^n}\left(\frac{1}{\Gamma(n-\alpha(s,t))}\int_a^t (t-s)^{n-\alpha(x,t)-1}f(s)ds\right).$$

The right Riemann-Liouville fractional derivative of order $\alpha(x,t)$ is stated as

$${}_t^{RL}D_b^{\alpha(x,t)}f(t) = \frac{d^n}{dt^n}\left(\frac{(-1)^n}{\Gamma(n-\alpha(s,t))}\int_t^b (s-t)^{n-\alpha(x,t)-1}f(s)ds\right).$$

Because the initial conditions for the FDEs with the Caputo derivatives are the same as the integer order differential equations, Caputo type definition is extremely useful in many application fields.

Definition 1. The Caputo VOF derivative of order $\alpha(x,t)$ concerning to t for the assumed function $v(x,t)$ can be specified as follows [23].

$$ {}^{c}\mathcal{D}_t^{\alpha(x,t)} v(x,t) = \begin{cases} \frac{1}{\Gamma(k-\alpha(x,t))} \int_0^t \frac{v_\mu^{(k)}(x,\mu)}{(t-\mu)^{\alpha(x,t)-k+1}} \mathrm{d}\mu, & k-1 < \alpha(x,t) < k, \\ v_t^{(k)}(x,t), & \alpha(x,t) = k, \end{cases} \tag{2} $$

that $k \in \mathbb{N}$ and $\Gamma(\cdot)$, is the Gamma function. If the value of $\alpha(x,t)$ is an integer, the VOF Caputo derivative can be defined identically with the integer-order derivative.

In the following, we represent some general properties of the VOF Caputo derivative. The linearity is the common property between the fractional derivative and the integer-order derivative.

$$ \mathcal{D}^{\alpha(x,t)}(\lambda_1\, v(x,t) + \lambda_2\, v(x,t)) = \lambda_1\, \mathcal{D}^{\alpha(x,t)} v(x,t) + \lambda_2\, \mathcal{D}^{\alpha(x,t)} v(x,t), $$

Where the value of λ_1, and λ_2, are constant and $\mathcal{D}^{\alpha(x,t)}$ is in the range $k-1 < \alpha(x,,t) \leq k$, meets the property given below:

$$ \mathcal{D}_t^{\alpha(x,t)} C = 0 \quad (C \text{ is a constant}), $$

$$ \mathcal{D}_t^{\alpha(x,t)} t^m = \begin{cases} 0, & m = 0, \cdots, k-1, \\ \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha(x,t))} t^{m-\alpha(x,t)}, & m = k, k+1, \ldots. \end{cases} $$

2.2 Notation for Two Dimensional Shifted Legendre Polynomials

The Legendre polynomials are a well-known family of orthogonal polynomials on the interval $[-1, 1]$. They can be defined using Rodrigues' formula as follows:

$$ P_m(y) = \frac{1}{(2^m m!)} \cdot \frac{\mathrm{d}^m (y^2-1)^m}{\mathrm{d}y^m}, \quad m = 0, 1, 2, \cdots. $$

The first few Legendre polynomials are: $P_0(y) = 1$, $P_1(y) = y$. To define these polynomials on $x \in [c, d]$, we perform a change of variable $y = \frac{1}{d-c}[2x - (d+c)]$, the SLPs $P_m^*(x)$ are obtained.

The Two dimensional Shifted Legendre polynomials constructed by taking the product of one-dimensional SLPs in each direction. For $\Omega = [0, a] \times [0, b]$, we define:

$$ \begin{aligned} P_{mn}^*(x,t) &= P_m^*(x) \cdot P_n^*(t) \\ &= P_m(\frac{a}{2}(y+1)).P_n(\frac{b}{2}(s+1)), \quad m, n = 0, 1, \ldots. \end{aligned} $$

We cosider these polynomials in the space $L^2(\Omega)$ equipped with the following inner product and norm:

$$ \langle v(x,t), u(x,t) \rangle = \int_0^b \int_0^a v(x,t).u(x,t) \mathrm{d}x\mathrm{d}t, \tag{3} $$

$$ \|v(x,t)\|^2 = \langle v(x,t), v(x,t) \rangle. \tag{4} $$

These polynomials in $L^2(\Omega)$ form a complete system with the orthogonality property

$$ \int_0^a \int_0^b P_{mn}^*(x,t).P_{ij}^*(x,t) dt dx = \frac{a.b}{(2m+1)(2n+1)} \delta_{mn}, $$

for $m = i, n = j$ and δ_{mn} indicate the Kronecker delta function.

3 Function Approximation

Any function $v(x,t) \in L^2(\Omega)$ admits an infinite expansion in terms of two-dimensional SLPs:

$$v(x,t) \cong \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn} P^*_{mn}(x,t),$$

where the coefficients c_{mn} are uniquely determined by the inner product:

$$c_{mn} = (\frac{2}{a}n-1)(\frac{2}{b}m-1)\int_0^a\int_0^b P^*_{mn}(x,t).v(x,t)dtdx.$$

Let $H = span\{P^*_{mn}\}_{m,n=0}^r \subset L^2(\Omega)$. For $v(x,t) \in L^2(\Omega)$, the best approximation from H is $\tilde{v}(x,t)$ [12], such that for each $u \in H$,

$$\|v-\tilde{v}\| \le \|v-u\|.$$

Since H is finite dimensional, the best approximation $\tilde{v}(x,t)$ can be expressed as a finite sum

$$\tilde{v}(x,t) = \sum_{m=0}^{r}\sum_{n=0}^{r} c_{mn} P^*_{mn}(x,t), \tag{5}$$

and the coefficients are given by the orthogonality:

$$c_{mn} = \frac{\langle \tilde{v}(x,t), P^*_{mn}(x,t)\rangle}{\|P^*_{mn}(x,t)\|^2}.$$

4 Estimate the Error Bound

Theorem 1. Let $v(x) \in C^{m+1}[0,L]$ and let $X = span\{P^*_0(x), \cdots, P^*_m(x)\}$ be a finite-dimensional space. If $\tilde{v}(x)$ is the best approximation to $v(x)$ in X, then

$$\|v(x)-\tilde{v}(x)\|_2 \le \frac{U_m.R^{\frac{2m+3}{2}}}{(m+1)!\sqrt{2m+3}}, \quad x \in [x_i, x_i+1] \subseteq [0,L],$$

where $R = \max[x_i, x_i+1]$, $U_m = \max_{x\in[0,L]}|v^{(m+1)}(x)|$.

Proof. See [1]. □

Theorem 2. Let $v(x,t) \in C^{m+1}(\Omega)$ and let $\tilde{v}(x,t) \in H$ be the best approximation defined in Equation (5). Then the error satisfies the following bound.

$$\|v(x,t)-\tilde{v}(x,t)\|_2 \le \frac{2.M(K_x+K_t)^{m+2}}{(m+1)!\sqrt{(2m+3)(2m+4)}}.$$

Proof. We recall the two variable Taylor's series expansion of $v(x,t)$

$$v(x,t) = \sum_{p=0}^{m+1}\frac{1}{p!}\sum_{r=0}^{p}\binom{p}{r}\Delta x^r.\Delta t^{p-r}\frac{\partial^p v(x,t)}{\partial x^r \partial t^{p-r}}. \tag{6}$$

Since $\tilde{v}(x,t) \in H$, its Taylor's series expansion is

$$\tilde{v}(x,t) = \sum_{p=0}^{m+1} \frac{1}{p!} \sum_{r=0}^{p} \binom{p}{r} \Delta x^r . \Delta t^{p-r} \frac{\partial^p v(x,t)}{\partial x^r \partial t^{p-r}}. \tag{7}$$

By considering that the all of partial derivatives of $v(x,t)$ up to order $m+1$ are bounded by M, thus the difference of two equations (6) and (7) give the following result:

$$\begin{aligned} |v(x,t) - \tilde{v}(x,t)| &= \frac{1}{(m+1)!} \left| \sum_{r=0}^{m+1} \binom{m+1}{r} \Delta x^r . \Delta t^{m+1-r} \frac{\partial^{m+1} v(x,t)}{\partial x^r \partial t^{m+1-r}} \right| \\ &\leq \frac{M}{(m+1)!} \left(|\Delta x| + |\Delta t|\right)^{m+1}. \end{aligned}$$

Regarding the norm as defined in Equation (3), we deduce that

$$\begin{aligned} \|v(x,t) - \tilde{v}(x,t)\|_2^2 &= \int_0^a \int_0^b |v(x,t) - \tilde{v}(x,t)|^2 \mathrm{d}t\mathrm{d}x \\ &\leq \int_0^a \int_0^b \left(\frac{M}{(m+1)!} \left(|\Delta x| + |\Delta t|\right)^{m+1} \right)^2 \mathrm{d}t\mathrm{d}x \\ &= \frac{M^2}{((m+1)!)^2} \int_0^a \int_0^b \left(|\Delta x| + |\Delta t|\right)^{2m+2} \mathrm{d}t\mathrm{d}x \\ &\leq \frac{4M^2 (K_x + K_t)^{2m+4}}{((m+1)!)^2 (2m+3)(2m+4)}, \end{aligned}$$

where $K_x = \max|\Delta x|$, $K_t = \max|\Delta t|$. By taking square root from two side the result is obtained. $\square$

5 Description of the Method

In 1908, Ritz introduced a simple and effective scheme for solving initial and boundary value problems. In this method, the approximate solution is expressed as a truncated series, and the continuous problem is converted into a discrete algebraic system. Since the resulting system is relatively small, the computational effort is significantly reduced.

We estimate the $v(x,t)$ in Equation (1) as the following form

$$v(x,t) \cong \tilde{v}(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} \kappa(x,t) P_{ij}^*(x,t) + w(x,t), \quad (x,t) \in \Omega, \tag{8}$$

which P_{ij}^* is the SLPs used as basis functions and $\kappa(x,t)$ is a function that satisfies the homogeneous part of the initial and boundary conditions.

$$\kappa(x,t) = (x-1)xt.$$

The function $w(x,t)$, called the satisfier function,is chosen to satisfy initial and boundary conditions [26]. It is usually constructed based on the known data of the problem. Experience shows that selecting $w(x,t)$ to be close to the exact solution improves the efficiency of the computation [25]. A practical and effective choice for $w(x,t)$

$$w(x,t) = \phi(x) + (1-x)(\psi_1(t) - \psi_1(0)) + x(\psi_2(t) - \psi_2(0)). \tag{9}$$

We use the Equations (8) - (9) and substituting them in Equation (1)

$$\eta_1\ \tilde{v}_t(x,t) + \eta_2\ {}^cD_t^{\alpha(x,t)}\tilde{v}(x,t) + B\ \tilde{v}_x(x,t) - C\ \tilde{v}_{xx}(x,t) = g(x,t). \tag{10}$$

To approximate $\tilde{v}(x,t)$, we evaluate Equation (10) at the roots of Legendre polynomials, leading to a nonlinear algebraic system. This system is solved numerically using Mathematica 10, and the unknown coefficients c_{ij} are determined accordingly.

6 Illustrative Examples

In this section, the suggested method is employed for solving several test problems. The outcomes showed that this method is precise and efficient. On the other hand, since the small number terms of the series are used to approximate the solution, this method has high computational value. The error in the examples is calculated as follow

$$E_n = |u(x_i,b) - u_n(x_i,b)|,$$

where n is degree of SLPs. In all examples, for computing c_{ij} we consider $n=2$. The figures and tables presented below correspond to Theorems 1 and 2 in the examples section. According to these results, the error generated by this method decreases progressively as the number of sentences increases, stabilizes at a negligible level, and ultimately approaches a well-defined bound.

Example 1. Consider the VOF mobile-immobile equation such as was defined in Equation (1) [21]:

$$\eta_1\ v_t(x,t) + \eta_2\ {}^c\mathcal{D}_t^{\alpha(x,t)}v(x,t) + B\ v_x(x,t) - C\ v(x,t) = g(x,t), \tag{11}$$

regarding the following initial and boundary conditions

$$\begin{cases} v(x,0) = 10x^2(1-x)^2, & x \in [0,1], \\ v(0,t) = v(1,t) = 0, & t \in [0,b], \end{cases}$$

where $(x,t) \in [0,1] \times [0,b]$ and

$$\begin{aligned} g(x,t) = {} & 10(1-x)^2x^2 + \frac{10x^2(1-x)^2t^{1-\alpha(x,t)}}{\Gamma(2-\alpha(x,t))} + 10(t+1)(2x - 6x^2 + 4x^3) \\ & - 10(1+t)(12x^2 - 12x + 2), \end{aligned}$$

and its exact solution is $v(x,t) = 10x^2(1-x)^2(1+t)$.
The values of parameters are $\eta_1 = \eta_2 = B = C = 1$ and $\alpha(x,t) = 1 - \frac{1}{2}e^{-xt}$. The satisfier function regarding Equation (9) is calculated as

$$w(x,t) = 10x^2(1-x)^2.$$

Substituting this into Equation (11) and solving the resulting algebraic system, the coefficients c_{ij} are obtained as

$$c_{00} = -1.6667, \qquad c_{01} = 4.0214 \times 10^{-15}, \qquad c_{02} = -1.373 \times 10^{-15},$$
$$c_{10} = -8.1824 \times 10^{-15}, \qquad c_{11} = 7.9319 \times 10^{-15}, \qquad c_{12} = -4.1486 \times 10^{-15},$$
$$c_{20} = 1.6667, \qquad c_{21} = 2.7589 \times 10^{-15}, \qquad c_{22} = -1.5775 \times 10^{-15}.$$

The absolute error of the presented method (PM) is shown in Figure 1. We compare our numerical results with (i) finite differences with Haar wavelets from [10] and (ii) Hahn polynomials with an operational matrix (Hahn) from [21] at $b = 1$, as reported in Table 1. Furthermore, Figure 2 presents both the approximate and exact solutions. The comparison in Figure 2 indicates that the approximate solution matches the exact solution very close almost everywhere in the interval and for any x and t.

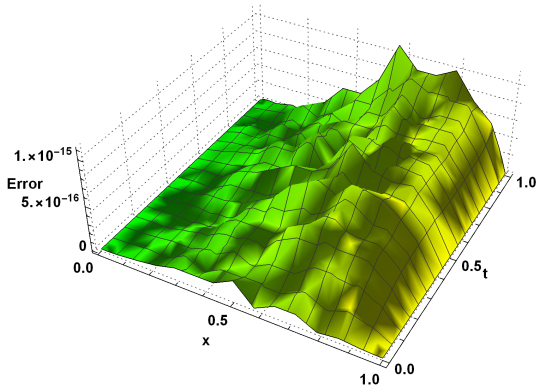

Figure 1: The absolute error E_n for Equation (11).

Table 1: Comparison of absolute errors at $b = 1$ for Equation (11).

x	PM ($n = 2$)	Ref. [10]	Hahn ($n = 5$) [21]
0.1	1.39×10^{-17}	3.19×10^{-5}	3.75×10^{-13}
0.2	3.30×10^{-17}	5.92×10^{-5}	4.41×10^{-13}
0.3	1.25×10^{-16}	8.22×10^{-5}	4.42×10^{-13}
0.4	2.22×10^{-16}	1.00×10^{-4}	3.40×10^{-14}
0.5	4.16×10^{-16}	1.12×10^{-4}	3.35×10^{-13}
0.6	7.77×10^{-16}	1.15×10^{-4}	8.70×10^{-14}
0.7	8.19×10^{-16}	1.08×10^{-4}	1.05×10^{-12}
0.8	8.62×10^{-16}	8.94×10^{-5}	1.36×10^{-10}
0.9	6.25×10^{-16}	5.50×10^{-5}	6.40×10^{-13}

Table 1 highlights the advantages of the proposed scheme, as it achieves high accuracy using significantly fewer points compared to previous methods.

Example 2. Consider another VOF advection-dispersion in mobile-immobile cases [21]

$$\eta_1\, v_t(x,t) + \eta_2\, {}^{c}\mathcal{D}_t^{\alpha(x,t)} v(x,t) + B\, v_x(x,t) - C\, v(x,t) = g(x,t), \quad (x,t) \in \Omega, \tag{12}$$

where

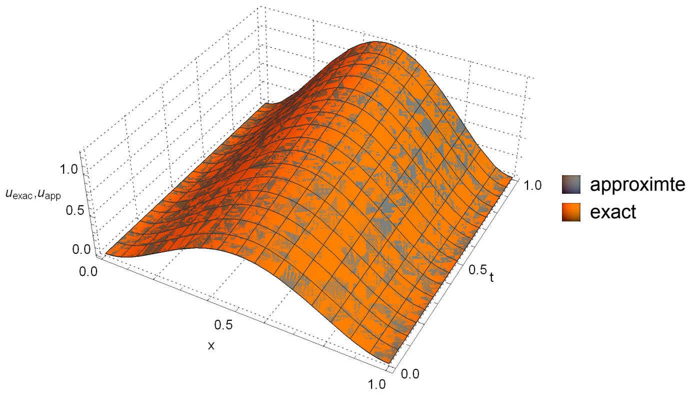

Figure 2: The approximate and the exact solution of Equation (11).

$$\begin{cases} v(x,0) = 5x(1-x), & x \in [0,1], \\ v(0,t) = v(1,t) = 0, & t \in [0,b], \end{cases}$$

that $\Omega \in [0,1]^2$ and

$$g(x,t) = 5x(1-x) + \frac{5x(1-x)t^{1-\alpha(x,t)}}{\Gamma(2-\alpha(x,t))} + 5(1+t)(1-2x) + 10(1+t),$$

and

$$v(x,t) = 5x(1-x)(1+t),$$

is the exact solution.
The parameters of Equation (12) are taken as

$$\eta_1 = \eta_2 = B = C = 1, \quad \text{and} \quad \alpha(x,t) = 0.8 + 0.005 \sin x \cos tx.$$

The $w(x,t) = 5x(1-x)$ is satisfier function, and the unknown c_{ij} are

$$\begin{array}{lll} c_{00} = -5, & c_{01} = 2.4157 \times 10^{-15}, & c_{02} = -1.2195 \times 10^{-15}, \\ c_{10} = 7.0168 \times 10^{-16}, & c_{11} = -5.7017 \times 10^{-16}, & c_{12} = 6.1121 \times 10^{-16}, \\ c_{20} = 5.1988 \times 10^{-17}, & c_{21} = 1.0543 \times 10^{-16}, & c_{22} = -1.6736 \times 10^{-16}. \end{array}$$

The absolute error is shown in Figure 3, and the exact versus approximation solutions at $t = 1$ are drawn in Figure 4. Additionally, the numerical performance of the current method is compared with reproducing kernel with collocation method (RKM) from [14] and Hahn polynomials from [21] in Table 2.

Example 3. Consider the following equation which has an exact solution $v(x,t) = t^3 e^x$ [21]:

$$\eta_1 \, v_t(x,t) + \eta_2 \, {}^c\mathcal{D}_t^{\alpha(x,t)} v(x,t) + B \, v_x(x,t) - C \, v_{xx}(x,t) = g(x,t), \quad (x,t) \in \Omega, \tag{13}$$

subject to the following conditions

$$\begin{cases} v(x,0) = 0, & & x \in [0,1], \\ v(0,t) = t^3, & v(1,t) = et^3, & t \in [0,b], \end{cases}$$

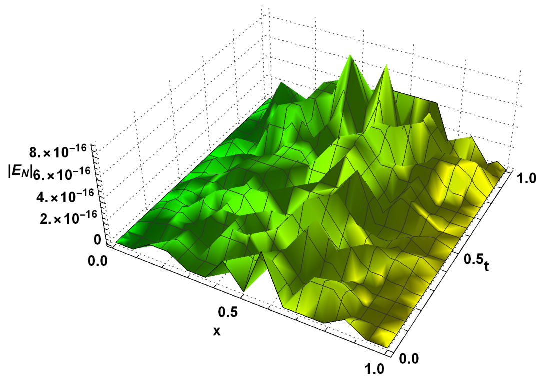

Figure 3: The absolute error E_n for Equation (12).

Table 2: Comparison of absolute errors at $b = 1$ for Equation (12).

x	PM ($n = 2$)	Hahn ($n = 10$) [21]	RKM ($n = 13$) [27]
0.1	3.33×10^{-17}	7.77×10^{-16}	0
0.2	0	8.88×10^{-16}	2.22×10^{-16}
0.3	6.12×10^{-17}	8.88×10^{-16}	4.44×10^{-16}
0.4	0	4.44×10^{-16}	0
0.5	6.77×10^{-17}	0	0
0.6	0	0	4.44×10^{-16}
0.7	2.02×10^{-16}	4.44×10^{-16}	0
0.8	2.09×10^{-16}	0	0
0.9	2.03×10^{-16}	5.55×10^{-16}	6.66×10^{-16}

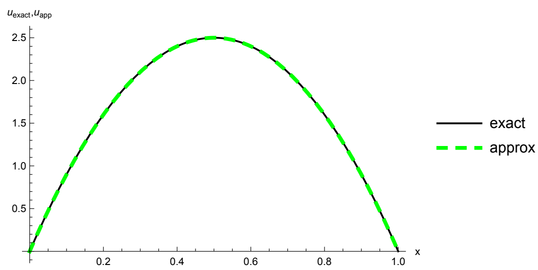

Figure 4: The exact and approximate result of Equation (12) at $b = 1$.

where $g(x,t) = e^x(3t^2 + \frac{3t^{3-\alpha(x,t)}}{\Gamma(4-\alpha(x,t))} - t^3)$ and the parameters of the equation are considered $\eta_1 = 2\eta_2 = B = \frac{1}{2}C = 1$ and $\alpha(x,t) = 0.8 + 0.02e^{-x}\sin t$. The $w(x,t) = t^3(1 - x + ex)$, is the satisfier function and the unknown c_{ij} are obtained as

$$c_{00} = 0.2825, \quad c_{01} = 0.4237, \quad c_{02} = 0.1413,$$
$$c_{10} = 0.0469, \quad c_{11} = 0.0704, \quad c_{12} = 0.0235,$$
$$c_{20} = 0.0039, \quad c_{21} = 0.0058, \quad c_{22} = 0.0019.$$

The exact and approximation solution in Figure 5 is drawn when $t = 1$ and our numerical results are compared with those obtained using Hann polynomials [21], as reported in Table 3.

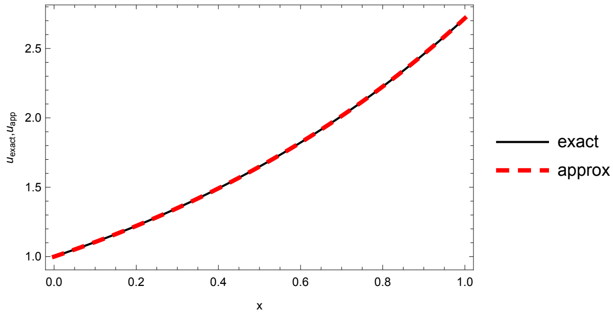

Figure 5: The approximate and exact solutions of Equation (13) at $b = 1$.

Table 3: Comparison of absolute errors at $b = 1$ for Eq. (13).

x	PM ($n = 2$)	Hann ($n = 2$) [21]
0.1	4.10×10^{-5}	4.54×10^{-2}
0.2	1.02×10^{-4}	7.70×10^{-2}
0.3	1.21×10^{-4}	9.59×10^{-2}
0.4	8.41×10^{-5}	1.03×10^{-1}
0.5	7.29×10^{-6}	1.01×10^{-1}
0.6	7.34×10^{-5}	9.01×10^{-2}
0.7	1.19×10^{-4}	7.25×10^{-2}
0.8	1.07×10^{-4}	5.02×10^{-2}
0.9	4.54×10^{-5}	4.54×10^{-2}

7 Conclusion

In this paper, we introduce a simple and effective numerical technique that combines the Ritz approximation with Shifted Legendre Polynomials (SLPs) to solve advection-dispersion equations with variable-order fractional operators in a velocity–obsoluteation framework (VOF) featuring mobile-immobile phases. The approach discretizes the domain using a small set of basis functions, transforming the original problem into a system of algebraic equations. This yields a substantial reduction in computational cost while preserving high accuracy. Several numerical examples are presented to validate the method and illustrate its efficiency. Although the current study concentrates on Caputo variable-order derivatives, the framework can be extended to other VOF operators or alternative basis-function sets.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

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Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Author Contributions

All authors contributed equally to the design of the study, data analysis, and writing of the manuscript, and share equal responsibility for the content of the paper.

Artificial Intelligence Statement

Artificial intelligence (AI) tools, including large language models, were used solely for language editing and improving readability. AI tools were not used for generating ideas, performing analyses, interpreting results, or writing the scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

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References

[1] Adel El-Sayed, A., Agarwal, P. (2019). "Numerical solution of multi-term variable-order fractional differential equations via shifted Legendre polynomials". *Mathematical Methods in the Applied Sciences*, 11, 3978-3991, doi:https://doi.org/10.1002/mma.5627.

[2] Akhavan Ghassabzade, F., Bagherpoorfard, M. (2024). "Mathematical modeling and optimal control of carbon dioxide emissions". *Control and Optimization in Applied Mathematics*, 9(1), 195-202, doi:https://doi.org/10.30473/coam.2023.67777.1233.

[3] Atangana, A. (2015). "On the stability and convergence of the time-fractional variable order telegraph equation". *Journal of Computational Physics*, 293, 104-114, doi:https://doi.org/10.1016/j.jcp.2014.12.043.

[4] Benson, D.A., Schumer, R., Meerschaert, M.M., Wheatcraft, S.W. (2001). "Fractional dispersion, le´vy motion, and the MADE tracer tests", *Transport in Porous Media*, 42, 211-240, doi:https://doi.org/10.1023/A:1006733002131.

[5] Chechkin, A.V., Gorenflo, R., Sokolov, I.M. (2005). "Fractional diffusion in inhomogeneous media". *Journal of Physics A: Mathematical and General*, 38(42), 679-684, doi:https://doi.org/10.1088/0305-4470/38/42/L03.

[6] Deng, Z., Bengtsson, L., Singh, V.P. (2006). "Parameter estimation for fractional dispersion model for rivers". *Environmental Fluid Mechanics*, 6, 451-475, doi:https://doi.org/10.1007/s10652-006-9004-5.

[7] Ebadi, G., Krishnan, E.V., Labidi, M., Zerrad, E., Biswas, A. (2011). "Analytical and numerical solutions to the Davey-Stewartson equation with power-law nonlinearity". *Waves in Random and Complex Media*, 21(4), 559-590, doi:https://doi.org/10.1080/17455030.2011.606853.

[8] Elgindy, K.T. (2024). "Fourier–Gegenbauer pseudospectral method for solving periodic fractional optimal control problems". *Mathematics and Computers in Simulation*, 225, 148-164, doi:https://doi.org/10.1016/j.matcom.2024.05.003.

[9] Hao, E., Zhang, J., Jin, Zh. (2024). "Dynamic analysis and optimal control of HIV/AIDS model with ideological transfer". *Mathematics and Computers in Simulation*, 226, 578-605, doi:https://doi.org/10.1016/j.matcom.2024.07.012.

[10] Haq, S., Ghafoor, A., Hussain, M. (2019). "Numerical solutions of variable order time fractional (1+1)-and (1+2)-dimensional advection dispersion and diffusion models". *Applied Mathematics and Computation*, 360, 107-121, doi:https://doi.org/10.1016/j.amc.2019.04.085.

[11] Heydari, M.H. (2016). "A new approach of the Chebyshev wavelets for the variable-order time fractional mobile-immobile advection-dispersion model". *arXiv preprint arXiv:1605.06332*, doi:https://doi.org/10.48550/arXiv.1605.06332.

[12] Holmes, R. (1970). "An Introduction to the Approximation of Functions (Theodore J. Rivlin)". *SIAM Review*, 12(2), doi:https://doi.org/10.1137/1012069.

[13] Ingman, D., Suzdalnitsky, J. (2004). "Control of damping oscilations by fractional differential operator with time-dependent order". *Computer Methods in Applied Mechanics and Engineering*, 193, 5585-5595, doi:https://doi.org/10.1016/j.cma.2004.06.029.

[14] Jiang, W., Liu, N. (2017). "A numerical method for solving the time variable fractional order mobile-immobile advection-dispersion model". *Applied Numerical Mathematics*, 119, 18-32,doi:https://doi.org/10.1016/j.apnum.2017.03.014.

[15] Kim, S., Levent Kavvas, M. (2006). "Generalized Fick's law and fractional ADE for pollution transport in a river: Detailed derivation". *Journal of Hydrologic Engineering*, 11(1), doi:https://doi.org/10.1061/(ASCE)1084-0699(2006)11:1(80).

[16] Lorenzo, C.F., Hartley, T.T. (2002). "Variable order and distributed order fractional operators". *Nonlinear Dynamics*, 29(1), 57-98, doi:https://doi.org/10.1023/A:1016586905654.

[17] Marasi, H.R., Derakhshan, M.H. (2023). "Numerical simulation of time variable fractional order mobile–immobile advection–dispersion model based on an efficient hybrid numerical method with stability and convergence analysis". *Mathematics and Computers in Simulation*, 205, 368-389, doi:https://doi.org/10.1016/j.matcom.2022.09.020.

[18] Parmar, D., Krishna Murthy, S.V.S.S.N.V.G., Rathish Kumar, B.V., Kumar, S. (2025). "Numerical simulation of fractional order double diffusive convective nanofluid flow in a wavy porous enclosure". *International Journal of Heat and Fluid Flow*, 112, doi:https://doi.org/10.1016/j.ijheatfluidflow.2025.109749.

[19] Podlubny, I. (1999). "Fractional Differential Equations, An introduction to fractional derivatives". In: Fractional Differential equations, to Methods of Their Solution and Some of Their Applications. *Technical University of Kosice, Slovak Republic*, 198, 1-340.

[20] Rajagopal, N., Balaji, S., Seethalakshmi, R., Balaji, V.S. (2020) "A new numerical method for fractional order Volterra integro-differential equations". *Ain Shams Engineering Journal*, 11(1), 171-177, doi:https://doi.org/10.1016/j.asej.2019.08.004.

[21] Salehi, F., Habibollah, S., Mohseni Moghadam, M. (2018). "A Hahn computational operational method for variable order fractional mobile-–immobile advection–dispersion equation". *Mathematical Sciences*, 12(2), 91-101, doi:https://doi.org/10.1007/s40096-018-0248-2.

[22] Samko, S.G., Ross, B. (1993). "Intergation and differentiation to a variable fractional order". *Integral Transforms and Special Functions*, 1(4), 277-300, doi:https://doi.org/10.1080/10652469308819027.

[23] Shen, S., Liu, F., Chen, J., Turner, I., Anh, V. (2012). "Numerical techniques for the variable order time fractional diffusion equation". *Applied Mathematics and Computation*, 218(22), 10861-10870, doi:https://doi.org/10.1016/j.amc.2012.04.047.

[24] Tayebi, A., Shekari, Y., Heydari, M.H. (2017). "A meshless method for solving two-dimensional variable-order time fractional advection-diffusion equation". *Journal of Computational Physics*, 340, 655-669, doi:https://doi.org/10.1016/j.jcp.2017.03.061.

[25] Yousefi, S.A., Barikbin, Z. (2011). "Ritz Legendre multiwavelet method for the damped generalized regularized long-wave equation". *Journal of Computational and Nonlinear Dynamics*, 7(1), doi:https://doi.org/10.1115/1.4004121.

[26] Yousefi, S.A., Lesnic, D., Barikbin, Z. (2012). "Satisfier function in Ritz-Galerkin method for the identification of a time-dependent diffusivity". *Journal of Inverse and Ill-Posed Problems*, 20(no. 5-6), 701-722, doi:https://doi.org/10.1515/jip-2012-0020.

[27] Zhang, H., Liu, F., Phanikumar, M.S., Meerschaert, M.M. (2013). "A novel numerical method for the time variable fractional order mobile-immobile advection-dispersion model". *Computers and Mathematics with Applications*, 66(5), 693-701, doi:https://doi.org/10.1016/j.camwa.2013.01.031.