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Multi-Attribute Group Decision Making Based on a New Ranking of Positive and Negative Interval Type-2 Fuzzy Numbers

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Abstract. This paper addresses multi-attribute group decision-making (MAGDM) where linguistic assessments are represented by both positive and negative interval type-2 fuzzy numbers (IT2FNs), capturing the intrinsic uncertainty of group evaluations more accurately. We introduce a novel ranking method for IT2FNs that simultaneously utilizes the mean and standard deviation of the upper and lower membership functions, as well as the IT2FN's height. This enhances its discriminatory capability. The theoretical foundations of this ranking—encompassing zero, unity, and symmetry properties— are rigorously established, and its superiority over existing techniques is demonstrated through comparative analyses on seven benchmark datasets. Building on this ranking, we develop an integrated fuzzy MAGDM framework that can handle both positive and negative IT2FN assessments for criteria and weights. The framework's practicality and effectiveness are validated through two case studies: one with exclusively positive linguistic terms and another with mixed positive and negative scales. Results indicate that the proposed ranking and decision framework yield more rational and robust group decisions under substantial uncertainty. They outperform conventional fuzzy methods and offer a nuanced solution for real-world MAGDM scenarios.

Keywords. Multi-attribute group decision making, Fuzzy, Ranking method, Linguistic assessment, Uncertainty.

MSC. 90B50; 03E72.

1 Introduction

Multi-Attribute Decision Making (MADM) is a structured approach used to identify the most appropriate alternative from a set by evaluating each against multiple criteria [5, 26, 31]. When this evaluative process requires the integration of judgments and preferences from several decision-makers (DMs), it becomes a Multi-Attribute Group Decision Making (MAGDM). In practice, DMs frequently provide their assessments using linguistic descriptors, such as "very high," "medium," or "poor" [21]. However, linguistic evaluations are inherently imprecise and subjective, causing terms to be interpreted differently by different DMs and thus introducing ambiguity and inconsistency [32].

Fuzzy set theory has become foundational in modeling such imprecision, refining MADM approaches by mathematically representing uncertainty. Early applications relied on Type-1 Fuzzy Sets (T1FSs), where each element is assigned a precise membership degree in $[0, 1]$ [17]. However, T1FSs are limited in their ability to capture higher-order ambiguities, particularly in the context of complex linguistic information [21, 32].

Introduced by Zadeh in 1965 [40], Type-2 Fuzzy Sets (T2FSs) provide an enhanced framework by allowing the membership function itself to be fuzzy. This added dimension enables T2FSs to better represent and manage uncertainty [24]. For practical computational purposes, Turksen [33] proposed the subclass of Interval Type-2 Fuzzy Sets (IT2FSs), which restrict the secondary membership function to an interval, making the sets more tractable yet expressive. Mendel [24] further emphasized that linguistic expressions are often too nuanced for T1FSs, highlighting the need for T2FS and IT2FS models in applications where uncertainty is pronounced. Interval Type-2 Fuzzy Numbers (IT2FNs), a common numeric form of IT2FSs, have thus emerged as powerful instruments in environments characterized by ambiguous information [19, 28, 29, 38].

A central issue in fuzzy MADM is the ability to rank alternatives modelled as fuzzy numbers—particularly IT2FNs—since ranking is fundamental to the selection of the optimal solution under uncertainty [1, 43]. Various IT2FN ranking approaches have been proposed, such as centroid-based, dominance degree-based, and rank-index techniques, and many have found application in MADM and group decision-making frameworks [12, 14, 25, 41, 42]. Nonetheless, these methodologies largely employ linguistic scales that only account for positive assessments, with normalized endpoints at zero and one [9, 10, 15, 27, 37]. This unilateral perspective neglects the inherent duality in many evaluation contexts.

According to the philosophy of Yin-Yang equilibrium [36], all phenomena exhibit both positive and negative facets, and judgment should be balanced accordingly. In MADM, this means that attribute assessments should allow for both positive ("very high") and negative ("very low") linguistic extremes, with neutral terms (e.g., "medium") serving as the equilibrium or "fuzzy zero." Recent work by Zamri et al. [41] developed such a linguistic scale, permitting DMs to express both positive and negative IT2FNs.

This research builds on these concepts by introducing a novel IT2FN ranking method specifically designed for linguistic MADM problems embracing both positive and negative assessments. This method, based on the center of gravity approach, offers the advantage of simplicity in understanding and calculation. The method incorporates the average and standard deviation of both the upper and lower membership functions, as well as each IT2FN's height, ensuring full use of their structural and dispersion properties. Theoretical analysis addresses key properties such as zero, unity, and symmetry. Extensive comparative testing with seven sets of IT2FNs demonstrates the new method's rationality and superiority over prevailing alternatives.

Based on this ranking, we further propose a new MAGDM framework that accommodates positive and negative IT2FNs in both criteria and their weights. The effectiveness and practicality of the proposed method are substantiated through two detailed numerical examples—one employing a purely positive linguistic scale, and the other using a combined positive/negative scale.

The remainder of this article is structured as follows: Section 2 reviews relevant fuzzy set concepts and details the proposed IT2FN ranking method; Section 3 outlines the new MAGDM procedure; Section 4 presents illustrative examples; and Section 5 concludes with key findings and avenues for future research.

2 Background and Related Work

The theory of fuzzy sets, originally introduced by Zadeh in 1965 [40], provides a robust mathematical framework for modelling systems characterized by vagueness, complexity, and imprecision. Type-1 Fuzzy Sets (T1FSs), wherein each element has a membership grade between zero and one, have been widely used in various decision-making problems [3, 34, 37]. However, T1FSs can be limited in their ability to fully capture the inherent uncertainty present in real-world assessments, particularly when the available information is itself imprecise or subjective [1, 19].

Type-2 fuzzy sets extend type-1 fuzzy sets by representing the membership function itself as fuzzy, which provides an additional degree of uncertainty modeling [24]. Instead of assigning a crisp degree of membership to an element, a T2FS employs a so-called Secondary Membership Function (SMF), which expresses fuzziness over each possible value of the primary membership grade. Therefore, T2FSs can model the "uncertainty about uncertainty," which is particularly valuable in fields that require nuanced treatment of ambiguity, such as pattern recognition, control systems, and decision support [24, 25].

2.1 Interval Type-2 Fuzzy Sets: Definitions and Operators

This subsection begins with a review of essential definitions and foundational concepts related to T2FSs [24]. It then details several prevalent arithmetic operations applicable to IT2FNs.

Definition 1. A type-2 fuzzy set $\tilde{A}$ on the reference set X, as defined in [24], is given by:

$$\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u)) \mid x \in X, u \in J_x \subseteq [0,1], \text{ and } 0 \leq \mu_{\tilde{A}}(x,u) \leq 1\}.$$

Definition 2. A T2FS $\tilde{A}$, as defined in [24], can be expressed as:

$$\tilde{A} = \int_{x \in X} \frac{\int_{u \in J_x} \mu_{\tilde{A}}(x,u)/u}{x}, \tag{1}$$

where $\int$ is the union of all combinations (x,u), x is the primary variable, and u is the secondary variable in $J_x \subseteq [0,1]$ with a secondary membership grade of $\mu_{\tilde{A}}(x,u)$.

Definition 3. As stated in [25], a fuzzy set $\tilde{A}$ is called an Interval Type-2 Fuzzy Set (IT2FS) if $\tilde{A}$ is a T2FS in which all secondary membership grades are equal to 1, i.e., $\mu_{\tilde{A}}(x,u) = 1$. Equivalently, IT2FS can be represented as:

$$\tilde{A} = \int_{x\in X} \frac{\int_{u\in J_x} 1/u}{x}. \tag{2}$$

Definition 4. The Footprint of Uncertainty (FOU) of an IT2FS $\tilde{A}$ is the union of all its primary memberships and is defined as follows [25]:

$$FOU(\tilde{A}) = \bigcup_{x\in X} J_x. \tag{3}$$

The FOU is bounded by two Type-1 membership functions: the Upper Membership Function (UMF), $\overline{\mu}_{\tilde{A}}(x)$, and the Lower Membership Function (LMF), $\underline{\mu}_{\tilde{A}}(x)$. For any $x \in X$, we have:

$$\overline{\mu}_{\tilde{A}}(x) = \sup(FOU(\tilde{A})), \tag{4}$$

$$\underline{\mu}_{\tilde{A}}(x) = \inf(FOU(\tilde{A})). \tag{5}$$

Therefore, the FOU of an IT2FS $\tilde{A}$ can be expressed as the region between the UMF and LMF: $FOU(\tilde{A}) = \bigcup_{x\in X}[\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]$.

Definition 5. [28]. A Trapezoidal Interval Type-2 Fuzzy Number (TIT2FN), denoted by $\tilde{A}$, can be represented by its UMF and LMF as:

$$\tilde{A} = (\tilde{A}^U, \tilde{A}^L) = ((a_1^U, a_2^U, a_3^U, a_4^U; h_1^U, h_2^U), (a_1^L, a_2^L, a_3^L, a_4^L; h_1^L, h_2^L)), \tag{6}$$

where $\tilde{A}^U$ and $\tilde{A}^L$ are Type-1 fuzzy numbers representing the UMF and LMF, respectively. As shown in Figure 1, h_1^U is the membership height of the interval $[a_2^U, a_3^U]$ and h_1^L is the membership height of $[a_2^L, a_3^L]$. We have $0 \le h_1^L \le h_1^U \le 1$. For a triangular fuzzy number, $a_2 = a_3$.

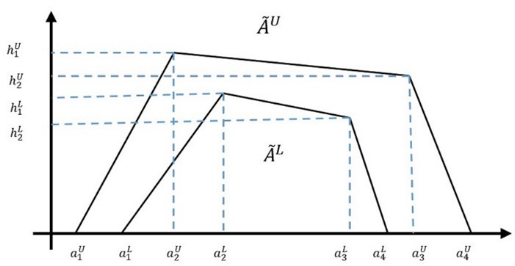

Figure 1: A trapezoidal interval type-2 fuzzy number.

Definition 6. Let $\tilde{A}_1$ and $\tilde{A}_2$ be two TIT2FNs. According to [19, 28], some common arithmetic operators can be defined as follows:

- **Addition:**

$$\tilde{A}_1 \oplus \tilde{A}_2 = \Big((a_{11}^U + a_{21}^U, a_{12}^U + a_{22}^U, a_{13}^U + a_{23}^U, a_{14}^U + a_{24}^U; \min(h_{11}^U, h_{21}^U), \min(h_{12}^U, h_{22}^U)), (a_{11}^L + a_{21}^L, a_{12}^L + a_{22}^L, a_{13}^L + a_{23}^L, a_{14}^L + a_{24}^L; \min(h_{11}^L, h_{21}^L), \min(h_{12}^L, h_{22}^L))\Big). \tag{7}$$

- **Multiplication:**

$$\tilde{A}_1 \otimes \tilde{A}_2 = \Big((a_{11}^U a_{21}^U, a_{12}^U a_{22}^U, a_{13}^U a_{23}^U, a_{14}^U a_{24}^U; \min(h_{11}^U, h_{21}^U), \min(h_{12}^U, h_{22}^U)), (a_{11}^L a_{21}^L, a_{12}^L a_{22}^L, a_{13}^L a_{23}^L, a_{14}^L a_{24}^L; \min(h_{11}^L, h_{21}^L), \min(h_{12}^L, h_{22}^L))\Big). \tag{8}$$

- **Multiplication by a crisp value λ:**

$$\lambda\tilde{A} = \begin{cases} ((\lambda a_1^U, \lambda a_2^U, \lambda a_3^U, \lambda a_4^U; h_1^U, h_2^U), (\lambda a_1^L, \lambda a_2^L, \lambda a_3^L, \lambda a_4^L; h_1^L, h_2^L)), & \lambda \geq 0, \\ ((\lambda a_4^U, \lambda a_3^U, \lambda a_2^U, \lambda a_1^U; h_1^U, h_2^U), (\lambda a_4^L, \lambda a_3^L, \lambda a_2^L, \lambda a_1^L; h_1^L, h_2^L)), & \lambda < 0. \end{cases} \tag{9}$$

- **Division by a crisp value $\lambda \neq 0$:**

$$\frac{\tilde{A}}{\lambda} = \begin{cases} ((a_1^U/\lambda, a_2^U/\lambda, a_3^U/\lambda, a_4^U/\lambda; h_1^U, h_2^U), \\ \quad (a_1^L/\lambda, a_2^L/\lambda, a_3^L/\lambda, a_4^L/\lambda; h_1^L, h_2^L)), & \lambda > 0, \\ ((a_4^U/\lambda, a_3^U/\lambda, a_2^U/\lambda, a_1^U/\lambda; h_1^U, h_2^U), \\ \quad (a_4^L/\lambda, a_3^L/\lambda, a_2^L/\lambda, a_1^L/\lambda; h_1^L, h_2^L)), & \lambda < 0. \end{cases} \tag{10}$$

2.2 Existing Methods for Ranking IT2FNs

As mentioned before, various methods have been proposed for ranking IT2FNs to address MADM problems. In this section, we will review some of these ranking methods. For clarity, the term "TIT2FN" will refer to the Trapezoidal Interval Type-2 Fuzzy Number.

2.2.1 Chen and Hong's Method

Chen and Lee [11] proposed a ranking method for $\tilde{A}_i$ denoted as $\text{Rank}(\tilde{A}_i)$ defined as follows:

$$\text{Rank}(\tilde{A}_i) = \frac{1}{8}\Bigg[\Bigg(\frac{(a_{i1}^U + k_i) + (2 - (a_{i4}^U + k_i))}{2} + \frac{(a_{i2}^U + k_i) + (2 - (a_{i3}^U + k_i))}{2} + \frac{h_{i1}^U + h_{i1}^L + h_{i2}^U + h_{i2}^L}{4}\Bigg) \times \Big((a_{i1}^U + k_i) + (a_{i2}^U + k_i) + (a_{i3}^U + k_i) + (a_{i4}^U + k_i) + (a_{i1}^L + k_i) + (a_{i2}^L + k_i) + (a_{i3}^L + k_i) + (a_{i4}^L + k_i)\Big)\Bigg] \tag{11}$$

where

$$k_i = \begin{cases} 0 & \min(a_{i1}^U, a_{i2}^U, \ldots, a_{in}^U) \geq 0 \\ |\min(a_{i1}^U, a_{i2}^U, \ldots, a_{in}^U)| & \min(a_{i1}^U, a_{i2}^U, \ldots, a_{in}^U) < 0 \end{cases}$$

and $i = 1, 2, \ldots, n$. Chen and Hong introduced this method for ranking IT2FNs to demonstrate its superiority over the approaches proposed by Cheng [14], Chen et al. [13], Wei [37], Chen and Chen [9, 10] and Murakami et al. [27].

2.2.2 Chiao's Parametric GMIR Method

Chiao [15], utilizing the Parametric Graded Mean Integration Representation (GMIR) expansion for T2FSs, proposed the following ranking criterion for IT2FNs:

$$\begin{aligned}\bar{P}_{\tilde{A}_i} &= \int_0^1 \left(\frac{1-\alpha}{6}(a_{i1}^U + a_{i4}^U) + \frac{1-\alpha}{3}(a_{i2}^U + a_{i3}^U) + \frac{\alpha}{6}(a_{i1}^L + a_{i4}^L) + \frac{\alpha}{3}(a_{i2}^L + a_{i3}^L)\right) d\alpha \\ &= \frac{1}{12}(a_{i1}^U + a_{i4}^U + a_{i1}^L + a_{i4}^L) + \frac{1}{6}(a_{i2}^U + a_{i3}^U + a_{i2}^L + a_{i3}^L),\end{aligned} \tag{12}$$

where $0 \leq \alpha \leq 1$.

2.2.3 Degree of Dominance Approach

Ghorabaee et al. [19], employing the method of Wang et al. [36] and the degree of dominance for T2FNs, proposed the rank of $\tilde{A}_i$ as follows.

$$R_{\text{value}}(\tilde{A}_i) = \frac{1}{n(n-1)}\left(\sum_{j=1}^{n} D(\tilde{A}_i > \tilde{A}_j) + \frac{n}{2} - 1\right), \tag{13}$$

where, $D(\tilde{A}_i > \tilde{A}_j)$ is the degree of dominance of $\tilde{A}_i$ over $\tilde{A}_j$ defined by

$$D(\tilde{A}_i > \tilde{A}_j) = \frac{\sum_{T\in\{U,L\}}[\omega(D_1^T) + 3\omega(D_2^T) + 3\omega(D_3^T) + \omega(D_4^T)]}{8\sum_{T\in\{U,L\}}[\max(a_{s4}^T, a_{t4}^T) - \min(a_{s1}^T, a_{t1}^T)]}, \tag{14}$$

where, for $s = 1, 2, 3, 4$ and $t = 1, 2, 3, 4$,

$$D_i^T = \begin{cases} a_{si}^T \cdot h_1(\tilde{A}_s^T) - a_{ti}^T \cdot h_1(\tilde{A}_t^T), & i = 1, 2, \\ a_{si}^T \cdot h_2(\tilde{A}_s^T) - a_{ti}^T \cdot h_2(\tilde{A}_t^T), & i = 3, 4, \end{cases}$$

and $\omega(x) = \max\{0, x\}$. Ghorabaee et al. [19] demonstrated that their method offers advantages over the approaches of Chen et al. [13], Wang et al. [35] and Balezentis and Zeng [4] in multiple MADM benchmarks.

2.2.4 Centroid and Rank Index Methods

De et al. [16] first defined the centroid value of each $\tilde{A}_i$, as follows:

$$\begin{aligned}C_{\tilde{A}_i} = \frac{1}{6}\Big[&(a_{i1}^U + b_{i1}^U + c_{i1}^U + d_{i1}^U) + (a_{i2}^L + b_{i2}^L + c_{i2}^L + d_{i2}^L) \\ &- \left(\frac{d_{i1}^U c_{i1}^U - a_{i1}^U b_{i1}^U}{(d_{i1}^U + c_{i1}^U) - (a_{i1}^U + b_{i1}^U)} + \frac{d_{i2}^L c_{i2}^L - a_{i2}^L b_{i2}^L}{(d_{i2}^L + c_{i2}^L) - (a_{i2}^L + b_{i2}^L)}\right)\Big].\end{aligned}$$

Then, they defined the rank index value as follows:

$$R(\tilde{A}_i) = \frac{1}{4}[\alpha(h_1(a_{i1}^U + b_{i1}^U) + h_2(a_{i2}^L + b_{i2}^L)) + (1-\alpha)(h_1(c_{i1}^U + d_{i1}^U) + h_2(c_{i2}^L + d_{i2}^L))]. \tag{15}$$

Now, for comparing two IT2FNs, $\tilde{A}_i$ and $\tilde{A}_j$, the following relations are used:

I. If $C_{\tilde{A}_i} > C_{\tilde{A}_j}$, then $\tilde{A}_i \succ \tilde{A}_j$.

II. If $C_{\tilde{A}_i} < C_{\tilde{A}_j}$, then $\tilde{A}_i \prec \tilde{A}_j$.

III. If $C_{\tilde{A}_i} = C_{\tilde{A}_j}$, then to compare two IT2FNs:

- If $R(\tilde{A}_i) > R(\tilde{A}_j)$, then $\tilde{A}_i \succ \tilde{A}_j$.
- If $R(\tilde{A}_i) < R(\tilde{A}_j)$, then $\tilde{A}_i \prec \tilde{A}_j$.
- If $R(\tilde{A}_i) = R(\tilde{A}_j)$, then:
  - If $(h_{i1}^U + h_{i2}^L)/2 > (h_{j1}^U + h_{j2}^L)/2$, then $\tilde{A}_i \succ \tilde{A}_j$.
  - If $(h_{i1}^U + h_{i2}^L)/2 < (h_{j1}^U + h_{j2}^L)/2$, then $\tilde{A}_i \prec \tilde{A}_j$.
  - If $(h_{i1}^U + h_{i2}^L)/2 = (h_{j1}^U + h_{j2}^L)/2$, then $\tilde{A}_i \sim \tilde{A}_j$.

Despite these advancements, most existing methods are based on a strictly positive linguistic scale, wherein normalized assessments range from zero (lowest) to one (highest). However, such frameworks may not adequately capture the nuances of neutrality, symmetry, or dual-scale reasoning reflected in real-world linguistic judgments—where both positive and negative values must be addressed, and “medium” truly represents neutral, not mid-way between two positives [41].

3 Proposed Method

This research introduces two substantive, interrelated innovations that collectively advance the state of the art in fuzzy Multi-Attribute Group Decision Making (MAGDM):

- A novel, symmetry-respecting ranking method for Interval Type-2 Fuzzy Numbers, featuring theoretical justification and designed to overcome core limitations of previous ranking indices, especially regarding the treatment of dual-scale linguistic information and “fuzzy zero” neutrality.

- A new MAGDM methodology that integrates the above ranking index to enable effective group decision making using both positive and negative linguistic scales—thereby addressing human duality in judgment and fostering a richer, more interpretable aggregation and evaluation of expert assessments.

The subsequent sections detail the construction, mathematical underpinnings, and operational steps for each contribution, underscoring their advantages and positioning relative to prevailing approaches.

3.1 Novel Ranking Method for IT2FNs

Existing IT2FN ranking functions have notable deficiencies, such as asymmetry, an inability to process negative scales, and an absence of an explicit "fuzzy zero" treatment [11, 16, 19, 37]. Most operate under the presumption of a solely positive normalized scale, limiting their fidelity in reflecting the true intent of human experts—particularly when both benefits and costs, or "good" and "bad" outcomes, require balanced evaluation.

In direct response, we propose a ranking index for IT2FNs grounded in the principles of symmetry, neutrality, and equilibrium, as inspired by Yin-Yang duality theory [41]. The methodology leverages both the mean and standard deviation of the upper and lower membership functions (UMF and LMF), as well as the core height of the IT2FN, to achieve an unbiased, theoretically sound quantification of vague expert inputs.

3.1.1 Mathematical Formulation

Let $\tilde{A}_i = ((a^U_{i1}, a^U_{i2}, a^U_{i3}, a^U_{i4}; h^U_{i1}, h^U_{i2}), (a^L_{i1}, a^L_{i2}, a^L_{i3}, a^L_{i4}; h^L_{i1}, h^L_{i2}))$ for $i = 1, 2, \ldots, n$ be n TIT2FNs, where $-\infty \leq a^L_{i1} \leq a^U_{i1} \leq a^U_{i2} \leq \ldots \leq a^U_{i4} \leq \infty$ and $0 \leq h^L_{i1}, h^L_{i2}, h^U_{i1}, h^U_{i2} \leq 1$.

A ranking criterion for $\tilde{A}_i$ is defined as:

$$RN(\tilde{A}_i) = \frac{1}{4}\left(\frac{\bar{x}^U_{\tilde{A}^S_i} \cdot (h^U_{i1} + h^U_{i2})}{1 + STD^U_{\tilde{A}^S_i}} + \frac{\bar{x}^L_{\tilde{A}^S_i} \cdot (h^L_{i1} + h^L_{i2})}{1 + STD^L_{\tilde{A}^S_i}}\right), \tag{16}$$

where $\tilde{A}^S_i$ is the normalized form of $\tilde{A}_i$. The terms $\bar{x}^U_{\tilde{A}^S_i}$ and $\bar{x}^L_{\tilde{A}^S_i}$ are the means of the UMF and LMF points, and $STD^U_{\tilde{A}^S_i}$ and $STD^L_{\tilde{A}^S_i}$ are their standard deviations, respectively.

The structure of this ranking criterion emphasizes robustness. Normalization ensures all IT2FNs are mapped onto a common scale, ensuring comparability across alternatives. The denominator $(1 + STD)$ serves as a penalty for uncertainty: larger standard deviation, reflecting greater vagueness, lowers the ranking score. Adding 1 prevents division-by-zero and stabilizes the index, a strategy analogous to constructing risk-adjusted metrics.

The rank $RN(\tilde{A}_i)$ can be calculated via the following steps:

Step 1: Normalize the IT2FNs. A normalization factor k is calculated to map all fuzzy numbers to a consistent range [19]:

$$k = \max\left(\left\{\lceil |a^U_{ij}| \rceil, \lceil |a^L_{ij}| \rceil \mid i = 1..n, j = 1..4\right\} \cup \{1\}\right). \tag{17}$$

Then, each IT2FN $\tilde{A}_i$ is normalized:

$$\tilde{A}^S_i = \left(\frac{\tilde{A}^U_i}{k}, \frac{\tilde{A}^L_i}{k}\right) = ((a^{U_S}_{i1}, \ldots, a^{U_S}_{i4}; h^U_{i1}, h^U_{i2}), (a^{L_S}_{i1}, \ldots, a^{L_S}_{i4}; h^L_{i1}, h^L_{i2})), \tag{18}$$

where $a^{U_S}_{ij} = a^U_{ij}/k$ and $a^{L_S}_{ij} = a^L_{ij}/k$.

Step 2: Calculate the mean of normalized points. For each normalized IT2FN $\tilde{A}^S_i$, calculate the mean of its UMF and LMF points:

$$\bar{x}_{\tilde{A}^S_i} = (\bar{x}^U_{\tilde{A}^S_i}, \bar{x}^L_{\tilde{A}^S_i}) = \left(\frac{\sum_{j=1}^4 a^{U_S}_{ij}}{4}, \frac{\sum_{j=1}^4 a^{L_S}_{ij}}{4}\right). \tag{19}$$

Step 3: Calculate the standard deviation of normalized points. Calculate the standard deviation for the UMF and LMF points of $\tilde{A}_i^S$:

$$STD_{\tilde{A}_i^S} = (STD^U_{\tilde{A}_i^S}, STD^L_{\tilde{A}_i^S}) = \left(\sqrt{\frac{\sum_{j=1}^4 (a_{ij}^{U_S} - \bar{x}^U_{\tilde{A}_i^S})^2}{4}}, \sqrt{\frac{\sum_{j=1}^4 (a_{ij}^{L_S} - \bar{x}^L_{\tilde{A}_i^S})^2}{4}} \right). \tag{20}$$

Step 4: Calculate the score for UMF and LMF. Calculate the individual scores for the UMF and LMF:

$$\text{score}(\tilde{A}_i^S) = (\text{score}^U, \text{score}^L) = \left(\frac{\bar{x}^U_{\tilde{A}_i^S} \cdot (h^U_{i1} + h^U_{i2})}{2(1 + STD^U_{\tilde{A}_i^S})}, \frac{\bar{x}^L_{\tilde{A}_i^S} \cdot (h^L_{i1} + h^L_{i2})}{2(1 + STD^L_{\tilde{A}_i^S})} \right). \tag{21}$$

Step 5: Calculate the final ranking criterion. The final rank is the average of the UMF and LMF scores:

$$RN(\tilde{A}_i) = \frac{\text{score}^U + \text{score}^L}{2}. \tag{22}$$

For any two IT2FNs $\tilde{A}_{i_1}$ and $\tilde{A}_{i_2}$, the order relationships are:

- If $RN(\tilde{A}_{i_1}) > RN(\tilde{A}_{i_2})$, then $\tilde{A}_{i_1} \succ \tilde{A}_{i_2}$.
- If $RN(\tilde{A}_{i_1}) < RN(\tilde{A}_{i_2})$, then $\tilde{A}_{i_1} \prec \tilde{A}_{i_2}$.
- If $RN(\tilde{A}_{i_1}) = RN(\tilde{A}_{i_2})$, then $\tilde{A}_{i_1} \sim \tilde{A}_{i_2}$.

Since the function $RN(\cdot)$ maps every IT2FN to a crisp real number, this method provides a complete and unambiguous ranking for any set of alternatives.

Example 1. Consider two IT2FNs:

- $\tilde{A}_1 = ((3, 5, 5.5, 7; 1, 1), (4, 4.5, 5, 6; 0.95, 0.95))$,
- $\tilde{A}_2 = ((5, 7, 7.5, 9; 1, 1), (6, 6.5, 7, 8; 0.95, 0.95))$.

First, find the normalization factor $k = \max(\lceil 3 \rceil, \ldots, \lceil 9 \rceil, \ldots, \lceil 8 \rceil, 1) = \lceil 9 \rceil = 9$.

For $\tilde{A}_1$:

1. $\tilde{A}_1^S = ((\frac{3}{9}, \frac{5}{9}, \frac{5.5}{9}, \frac{7}{9}; 1, 1), (\frac{4}{9}, \frac{4.5}{9}, \frac{5}{9}, \frac{6}{9}; 0.95, 0.95))$,
2. $\bar{x}_{\tilde{A}_1^S} = (0.5694, 0.5417)$,
3. $STD_{\tilde{A}_1^S} = (0.1586, 0.0817)$,
4. $\text{score}(\tilde{A}_1^S) = \left(\frac{0.5694 \cdot 2}{2(1+0.1586)}, \frac{0.5417 \cdot 1.9}{2(1+0.0817)} \right) = (0.4915, 0.4754)$,
5. $RN(\tilde{A}_1) = \frac{0.4915+0.4754}{2} = 0.4835$.

For $\tilde{A}_2$: A similar calculation yields $RN(\tilde{A}_2) = 0.6773$. Since $RN(\tilde{A}_2) > RN(\tilde{A}_1)$, we conclude that $\tilde{A}_2 \succ \tilde{A}_1$.

Lemma 1. The proposed ranking criterion $RN(\cdot)$ satisfies the following properties:

- **Property 1** (*Zero property): If* $\tilde{A}_i = ((0, 0, 0, 0; 1, 1), (0, 0, 0, 0; 1, 1))$*, then*

$$RN(\tilde{A}_i) = 0.$$

Proof. For $\tilde{A}_i = ((0,0,0,0;1,1),(0,0,0,0;1,1))$, after normalization (if needed, $k \geq 1$), the normalized points remain all zero. Thus, $\bar{x}_{\tilde{A}_i^S} = (0,0)$. From Equation (21), this leads to score$(\tilde{A}_i^S) = (0,0)$, and consequently, $RN(\tilde{A}_i) = 0$. □

- ***Property 2** (Unity property): If $\tilde{A}_1 = ((a,a,a,a;1,1),(a,a,a,a;1,1))$ and $\tilde{A}_2 = ((1-a,1-a,1-a,1-a;1,1),(1-a,1-a,1-a,1-a;1,1))$ with $0 \leq a \leq 1$, then*

$$RN(\tilde{A}_1) + RN(\tilde{A}_2) = 1.$$

Proof. For $\tilde{A}_1$ and $\tilde{A}_2$ as defined, the normalization factor $k = 1$. For $\tilde{A}_1$, we have $\bar{x}_{\tilde{A}_1^S} = (a,a)$ and $STD_{\tilde{A}_1^S} = (0,0)$. This gives score$(\tilde{A}_1^S) = (a,a)$, so $RN(\tilde{A}_1) = a$. For $\tilde{A}_2$, we have $\bar{x}_{\tilde{A}_2^S} = (1-a,1-a)$ and $STD_{\tilde{A}_2^S} = (0,0)$, leading to $RN(\tilde{A}_2) = 1-a$. Therefore, $RN(\tilde{A}_1) + RN(\tilde{A}_2) = a + (1-a) = 1$. □

- ***Property 3** (Symmetry property): If $\tilde{A}_2 = -\tilde{A}_1$, then $RN(\tilde{A}_2) = -RN(\tilde{A}_1)$.*

Proof. Let $\tilde{A}_1 = ((a_1^U,\ldots,a_4^U;h_1^U,h_2^U),(a_1^L,\ldots,a_4^L;h_1^L,h_2^L))$. Then its symmetric counterpart is $\tilde{A}_2 = -\tilde{A}_1 = ((-a_4^U,\ldots,-a_1^U;h_1^U,h_2^U),(-a_4^L,\ldots,-a_1^L;h_1^L,h_2^L))$. The normalization factor k is the same for both, as it depends on absolute values. After normalization, we have $\bar{x}^U_{\tilde{A}_2^S} = -\bar{x}^U_{\tilde{A}_1^S}$, and similarly $\bar{x}^L_{\tilde{A}_2^S} = -\bar{x}^L_{\tilde{A}_1^S}$. The standard deviation remains unchanged: $STD(\tilde{A}_2^S) = STD(\tilde{A}_1^S)$, because it is based on squared differences. From Equation (21), we see that score$(\tilde{A}_2^S) = (-\text{score}^U(\tilde{A}_1^S), -\text{score}^L(\tilde{A}_1^S))$. Therefore, $RN(\tilde{A}_2) = -RN(\tilde{A}_1)$. □

3.2 Comparison of the New Ranking Method with Existing Methods

In this section, we compare the performance of the proposed ranking method with several existing methods using seven sets of IT2FNs, which are visually depicted in Figure 2. The rank of each IT2FN is computed using the following methods:

- **Method 1:** The method proposed by Chen and Lee [11].
- **Method 2:** The method proposed by Chiao [15].
- **Method 3:** The method proposed by Ghorabaee et al. [19].
- **Method 4:** The method proposed by De et al. [16].
- **Proposed Method:** The newly developed method in this paper.

The comparative results of the ranking values and the final orderings are summarized in Tables 1 and 2, respectively.

An analysis of the data presented in Tables 1, 2 and Figure 2, allows for several key assessments of the examined ranking methods:

- **Set 1:** The proposed approach, along with Methods 3 and 4, produced coherent and distinguishable ranking outcomes ($\tilde{A}_1 \succ \tilde{A}_2$), whereas Methods 1 and 2 failed to differentiate between the two IT2FNs.

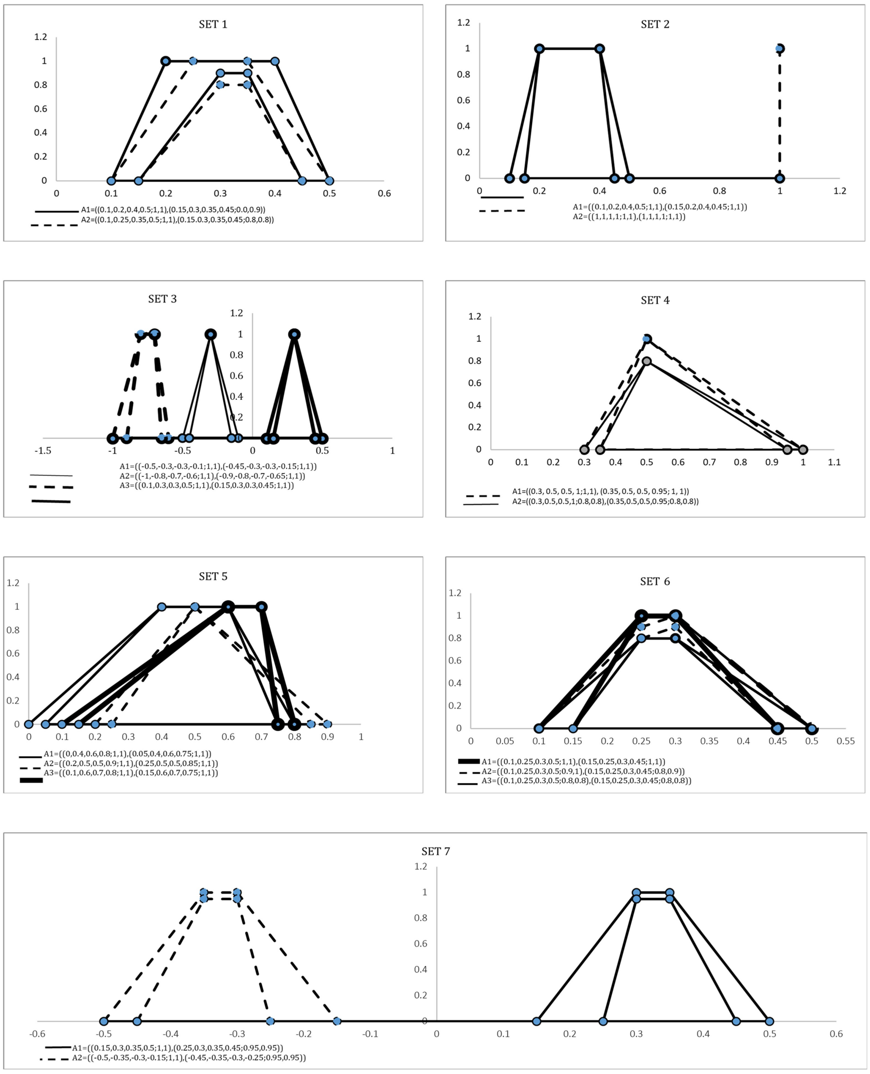

Figure 2: Seven IT2FN sets used in comparing ranking methods.

- **Set 2:** Method 4 was unable to produce a ranking for $\tilde{A}_2$. All other methods successfully ranked $\tilde{A}_1 \prec \tilde{A}_2$.
- **Set 3:** All evaluated methods performed successfully, yielding the same accurate and consistent ranking order: $\tilde{A}_2 \prec \tilde{A}_1 \prec \tilde{A}_3$.
- **Set 4:** Except for Method 2, which yielded an implausible equivalence, all methods generated the meaningful ranking $\tilde{A}_1 \succ \tilde{A}_2$.
- **Set 5:** Every method, aside from Method 4, correctly produced the ranking $\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$. Method 4 incorrectly ranked $\tilde{A}_3$ below $\tilde{A}_2$.
- **Set 6:** The results from the proposed method were in alignment with those of Methods 1 and 3 ($\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_3$), while Methods 2 and 4 were unable to distinguish between the three IT2FNs.

Table 1: Numerical ranking results for seven sets of IT2FNs.

Sets	Alternatives	Method 1	Method 2	Method 3	Method 4	Proposed Method
Set1	A1	0.8116	0.3083	0.0723	0.3094	0.2582
	A2	0.8116	0.3083	0.0268	0.3048	0.2558
Set2	A1	0.81	0.3	0	0.3	0.2626
	A2	3	1	0.4	N/A	1
Set3	A1	-0.84	-0.3	0.1761	-0.3	-0.267
	A2	-2.1141	-0.7625	0.0833	-0.7721	-0.6854
	A3	0.84	0.3	0.1979	0.3	0.267
Set4	A1	1.5238	0.55	0.1654	0.6	0.4631
	A2	1.4088	0.55	0	0.5	0.4229
Set5	A1	1.125	0.4667	0.0907	0.4404	0.3519
	A2	1.3913	0.5167	0.1036	0.5333	0.4265
	A3	1.43	0.5833	0.1361	0.5256	0.4389
Set6	A1	0.7978	0.2833	0.122	0.2906	0.2555
	A2	0.7691	0.2833	0.0982	0.2906	0.2297
	A3	0.7403	0.2833	0.0833	0.2906	0.2044
Set7	A1	0.9192	0.3292	0.1683	0.3325	0.2716
	A2	-0.9192	-0.3292	0	-0.3325	-0.2716

Table 2: Results of ranking for seven sets of IT2FNs (ranking orders).

Sets	Method 1	Method 2	Method 3	Method 4	Proposed Method
Set1	$\tilde{A}_1 \approx \tilde{A}_2$	$\tilde{A}_1 \approx \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_2$
Set2	$\tilde{A}_1 \prec \tilde{A}_2$	$\tilde{A}_1 \prec \tilde{A}_2$	$\tilde{A}_1 \prec \tilde{A}_2$	Failed	$\tilde{A}_1 \prec \tilde{A}_2$
Set3	$\tilde{A}_2 \prec \tilde{A}_1 \prec \tilde{A}_3$	$\tilde{A}_2 \prec \tilde{A}_1 \prec \tilde{A}_3$	$\tilde{A}_2 \prec \tilde{A}_1 \prec \tilde{A}_3$	$\tilde{A}_2 \prec \tilde{A}_1 \prec \tilde{A}_3$	$\tilde{A}_2 \prec \tilde{A}_1 \prec \tilde{A}_3$
Set4	$\tilde{A}_1 \succ \tilde{A}_2$	$\tilde{A}_1 \approx \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_2$
Set5	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	$\tilde{A}_1 \prec \tilde{A}_3 \prec \tilde{A}_2$	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$
Set6	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_3$	$\tilde{A}_1 \approx \tilde{A}_2 \approx \tilde{A}_3$	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_3$	$\tilde{A}_1 \approx \tilde{A}_2 \approx \tilde{A}_3$	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_3$
Set7	$\tilde{A}_1 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_2$

- **Set 7:** For the symmetric cases $\tilde{A}_1$ and $\tilde{A}_2 = -\tilde{A}_1$, every method returned the correct order $\tilde{A}_1 \succ \tilde{A}_2$. However, a closer inspection of Table 1 reveals that only Method 3 failed to maintain the symmetry property in its rank values, as $RN(\tilde{A}_1) \neq -RN(\tilde{A}_2)$.

Table 3 summarizes how each method aligns with the three fundamental ranking properties discussed previously.

In summary, the proposed method and Method 2 both consistently satisfy all three properties, reflecting their robustness in ranking IT2FNs. Conversely, Methods 1, 3, and 4 exhibit notable deficiencies,

Table 3: Comparison of properties satisfied by different ranking methods.

Methods	Zero property	One property	Symmetric property
Method 1	Yes	No	Yes
Method 2	Yes	Yes	Yes
Method 3	Yes	No	No
Method 4	Yes	No	Yes
The Proposed Method	**Yes**	**Yes**	**Yes**

either in ranking accuracy or in property compliance [11, 15, 16, 19]. These results underscore the clear advantages of the proposed approach over existing alternatives, particularly regarding its reliability and adherence to essential theoretical criteria.

4 New MAGDM Framework Integrating the Proposed Ranking Method

Building on the above ranking index, we propose a comprehensive MAGDM framework that systematically incorporates dual-scale linguistic assessment and the new IT2FN ranking throughout all procedural steps. This framework adapts the classic TOPSIS methodology to accommodate the richer semantics provided by the new ranking approach.

4.1 Procedural Steps

Consider a MAGDM problem with the following elements:

- m alternatives: $A_1, A_2, \ldots, A_m$.
- n attributes: $C_1, C_2, \ldots, C_n$.
- K decision-makers (DMs): $D_1, D_2, \ldots, D_K$, with corresponding importance weights η_l.

Assume that the weight vector of the attributes is denoted as $W^T = (w_1, w_2, \ldots, w_n)$, where $w_j \in [0, 1]$ for $j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$.

Let $\tilde{X}_l = (\tilde{x}_{ijl})_{m\times n}$ be the decision matrix from the l-th DM, where $\tilde{x}_{ijl}$ is the IT2FN representing the evaluation of alternative A_i with respect to attribute C_j. The MAGDM process consists of the following steps:

Step 1: Construct the collective decision matrix. Construct the collective decision matrix $\tilde{X} = (\tilde{x}_{ij})_{m\times n}$ by computing the weighted average of inputs from the K DMs [19, 28]:

$$\tilde{x}_{ij} = \bigoplus_{l=1}^{K} \eta_l \otimes \tilde{x}_{ijl}; \quad i = 1, \ldots, m; \quad j = 1, \ldots, n. \tag{23}$$

Note: While equal weights for DMs can be assumed for simplicity, the framework flexibly accommodates non-uniform weights to reflect varying levels of expertise. The resulting matrix is:

$$\tilde{X} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \cdots & \tilde{x}_{mn} \end{pmatrix} \end{array}. \tag{24}$$

Step 2: Construct the weighted decision matrix. Form the weighted fuzzy decision matrix $\tilde{V} = (\tilde{v}_{ij})_{m\times n}$ by multiplying each element of the collective matrix by its corresponding attribute weight w_j [19, **?**]:

$$\tilde{v}_{ij} = w_j \otimes \tilde{x}_{ij}. \tag{25}$$

Here, $\otimes$ denotes the scalar multiplication of an IT2FN by a crisp weight.

Step 3: Defuzzify the weighted matrix. Calculate the crisp rank for each element of the weighted decision matrix $\tilde{V}$ to form a real-valued matrix $R = (r_{ij})_{m\times n}$. Each element r_{ij} is computed using the proposed ranking method (by using (16)):

$$r_{ij} = RN(\tilde{v}_{ij}). \tag{26}$$

Step 4: Determine the Positive and Negative Ideal Solutions (PIS and NIS) Identify the PIS, $r^+ = (r_1^+, r_2^+, \ldots, r_n^+)$, and the NIS, $r^- = (r_1^-, r_2^-, \ldots, r_n^-)$, from the crisp matrix R. Let X_b be the set of benefit attributes and X_c be the set of cost attributes:

$$r_j^+ = \begin{cases} \max_i\{r_{ij}\}, & \text{if } C_j \in X_b \\ \min_i\{r_{ij}\}, & \text{if } C_j \in X_c \end{cases}, \quad j = 1, \ldots, n, \tag{27}$$

$$r_j^- = \begin{cases} \min_i\{r_{ij}\}, & \text{if } C_j \in X_b \\ \max_i\{r_{ij}\}, & \text{if } C_j \in X_c \end{cases}, \quad j = 1, \ldots, n. \tag{28}$$

Step 5: Calculate distances from the ideal solutions. Calculate the Euclidean distance of each alternative A_i from the PIS (d_i^+) and the NIS (d_i^-) [19, 38]:

$$d_i^+ = \sqrt{\sum_{j=1}^{n}(r_{ij} - r_j^+)^2}, \quad i = 1, \ldots, m, \tag{29}$$

$$d_i^- = \sqrt{\sum_{j=1}^{n}(r_{ij} - r_j^-)^2}, \quad i = 1, \ldots, m. \tag{30}$$

Step 6: Calculate the relative closeness coefficient and rank. Calculate the relative closeness coefficient $C(A_i)$ for each alternative A_i [38]:

$$C(A_i) = \frac{d_i^-}{d_i^- + d_i^+}, \quad i = 1, \ldots, m. \tag{31}$$

Rank the alternatives in descending order of their $C(A_i)$ values. The alternative with the highest value is deemed the most preferable solution, A^*:

$$A^* = \arg\max_i\{C(A_i)\}. \tag{32}$$

5 Numerical Examples

To demonstrate the effectiveness and practical applicability of the proposed MAGDM method, two numerical examples are presented. The first example addresses a supplier selection problem utilizing only positive linguistic IT2FN scales. The results obtained using the proposed approach are compared with alternative established methods: those of Chen and Lee [11], Chiao [15], Ghorabaee et al. [19], and De et al. [16]. The second example utilizes both positive and negative IT2FN linguistic terms, enabling a broader comparative evaluation.

5.1 Numerical Example 1: Supplier Selection Using Positive Linguistic IT2FNs

Adapted from Ghorabaee et al. [19], this example evaluates seven potential suppliers (A_1 to A_7) in a supply chain management decision. The evaluation is conducted by three DMs (D_1, D_2, D_3), members of the board of directors, who assess each supplier according to five principal attributes:

- C_1: Defect rate (*cost attribute*): Proportion of nonconforming items.
- C_2: Cost (*cost attribute*): Estimated procurement-related costs.
- C_3: Delivery reliability (*benefit attribute*): Timeliness of deliveries.
- C_4: Responsiveness (*benefit attribute*): Speed of reacting to demands.
- C_5: Flexibility (*benefit attribute*): Adaptability to customer requirements.

Decision-makers use linguistic variables corresponding to positive IT2FNs (Table 4), assigning equal weights to each DM ($\eta^T = (1/3, 1/3, 1/3)$). Table 5 presents the raw linguistic assessments, and Table 6 shows the linguistic weights assigned by the DMs to each attribute.

Table 4: Linguistic Variables and Corresponding IT2FNs [19].

Linguistic Variable	TIT2FNs
Very Low (VL)	((0,0,0,0.1;1,1), (0,0,0,0.05;0.9,0.9))
Low (L)	((0,0.1,0.15,0.3;1,1), (0.05,0.1,0.15,0.2;0.9,0.9))
Medium Low (ML)	((0.1,0.3,0.35,0.5;1,1), (0.2,0.3,0.35,0.4;0.9,0.9))
Medium (M)	((0.3,0.5,0.55,0.7;1,1), (0.4,0.5,0.55,0.6;0.9,0.9))
Medium High (MH)	((0.5,0.7,0.75,0.9;1,1), (0.6,0.7,0.75,0.8;0.9,0.9))
High (H)	((0.7,0.85,0.9,1;1,1), (0.8,0.85,0.9,0.95;0.9,0.9))
Very High (VH)	((0.9,1,1,1;1,1), (0.95,1,1,1;0.9,0.9))

The proposed method is executed through the following steps. The aggregated mean weights are shown in Table 7.

Table 5: Linguistic performance values of alternatives (Example 1).

DMs	**Alternatives**	C_1	C_2	C_3	C_4	C_5
	A_1	L	ML	VH	M	MH
	A_2	L	VL	VH	H	VH
	A_3	H	MH	M	MH	ML
DM1	A_4	MH	VH	MH	L	VL
	A_5	M	VH	M	ML	MH
	A_6	VH	M	L	MH	VH
	A_7	MH	M	VL	VH	H
	A_1	VL	L	H	MH	M
	A_2	ML	VL	VH	H	VH
	A_3	MH	M	MH	MH	M
DM2	A_4	MH	MH	H	ML	ML
	A_5	M	H	M	M	MH
	A_6	H	ML	ML	H	H
	A_7	MH	M	L	H	MH
	A_1	VL	M	H	MH	H
	A_2	VL	VL	VH	H	VH
	A_3	M	MH	M	M	M
DM3	A_4	M	VH	M	VL	L
	A_5	ML	H	MH	ML	H
	A_6	MH	MH	ML	MH	VH
	A_7	M	M	ML	MH	MH

Table 6: Linguistic weights of attributes evaluated by DMs (Example 1).

Attributes	**D1**	**D2**	**D3**
C_1	VH	VH	H
C_2	MH	MH	M
C_3	VH	H	VH
C_4	VH	MH	MH
C_5	H	H	MH

Step 1 & 2: Construct Aggregated and Weighted Decision Matrices

First, the DMs' evaluations are aggregated using Equation (23). For instance, the aggregated evaluation for A_1 under attribute C_1 is $\tilde{x}_{11} = \frac{1}{3}L \oplus \frac{1}{3}VL \oplus \frac{1}{3}VL$. Next, the aggregated attribute weights ($\bar{w}_j$)

are used to compute the weighted decision matrix $\tilde{V} = (\tilde{v}_{ij})$ using Equation (25). The full weighted decision matrix is presented in Table 8.

Table 7: IT2FNs subjective weights by DMs and aggregated mean values in Example 1.

DM	Attributes	IT2FN Weight Value
Mean	C_1	((0.833,0.95,0.967,1;1,1), (0.9,0.95,0.967,0.983;0.9,0.9))
	C_2	((0.433,0.633,0.683,0.833;1,1), (0.533,0.633,0.683,0.733;0.9,0.9))
	C_3	((0.833,0.95,0.967,1;1,1), (0.9,0.95,0.967,0.983;0.9,0.9))
	C_4	((0.567,0.75,0.8,0.933;1,1), (0.667,0.75,0.8,0.85;0.9,0.9))
	C_5	((0.633,0.8,0.85,0.967;1,1), (0.733,0.8,0.85,0.9;0.9,0.9))

Table 8: The weighted decision matrix $(\tilde{v}_{ij})$ for Example 1.

Alternatives	Attributes	Weighted IT2FN Value $\tilde{v}_{ij}$
A_1	C_1	((0,0.03,0.05,0.17;1,1), (0.02,0.03,0.05,0.1;0.9,0.9))
	C_2	((0.06,0.19,0.24,0.42;1,1), (0.12,0.19,0.24,0.29;0.9,0.9))
	C_3	((0.64,0.86,0.9,1;1,1), (0.77,0.86,0.9,0.95;0.9,0.9))
	C_4	((0.25,0.48,0.55,0.78;1,1), (0.36,0.48,0.55,0.62;0.9,0.9))
	C_5	((0.32,0.55,0.62,0.84;1,1), (0.44,0.55,0.62,0.71;0.9,0.9))
A_2	C_1	((0.03,0.13,0.16,0.3;1,1), (0.08,0.13,0.16,0.21;0.9,0.9))
	C_2	((0,0,0,0.08;1,1), (0,0,0,0.04;0.9,0.9))
	C_3	((0.75,0.95,0.97,1;1,1), (0.86,0.95,0.97,0.98;0.9,0.9))
	C_4	((0.4,0.64,0.72,0.93;1,1), (0.53,0.64,0.72,0.81;0.9,0.9))
	C_5	((0.57,0.8,0.85,0.97;1,1), (0.7,0.8,0.85,0.9;0.9,0.9))

Step 3: Rank the Weighted Decision Matrix

The new IT2FN ranking method (Equation (16)) is applied to each element $\tilde{v}_{ij}$ to obtain the crisp rank matrix $R = (r_{ij})$, shown in Table 9.

Steps 4, 5, and 6: Determine Ideal Solutions, Distances, and Final Ranking

The Positive and Negative Ideal Solutions (PIS and NIS) are determined using (27) and (28). Then, the Euclidean distances of each alternative from PIS (d_i^+) and NIS (d_i^-) are computed, followed by the relative closeness coefficient $C(A_i)$. The results are shown in Table 10, and the final ranking is

Table 9: Rank of the weighted decision matrix (r_{ij}) in Example 1.

Alternatives	C_1	C_2	C_3	C_4	C_5
A_1	0.0501	0.1893	0.7408	0.4201	0.4818
A_2	0.1318	0.0141	0.8199	0.5574	0.6878
A_3	0.1748	0.3591	0.4651	0.4201	0.3159
A_4	0.5173	0.4898	0.5613	0.1116	0.1178
A_5	0.3607	0.4953	0.4651	0.2562	0.5251
A_6	0.6951	0.2908	0.2088	0.4923	0.6570
A_7	0.5173	0.2908	0.1318	0.5518	0.5251

compared with other methods in Table 11. The optimal supplier is A_2, with the final order $A_2 \succ A_1 \succ A_3 \succ A_5 \succ A_6 \succ A_7 \succ A_4$.

Table 10: Distances from ideal solutions and relative closeness for Example 1.

Alternative	d_i^+	d_i^-	$C(A_i)$
A_1	0.313028	1.052798	0.770814
A_2	0.081689	1.243355	0.938350
A_3	0.646078	0.731224	0.530910
A_4	1.017412	0.464600	0.313492
A_5	0.696348	0.639699	0.478800
A_6	0.933250	0.694878	0.426796
A_7	0.891578	0.657707	0.424523

Table 11: Final rankings comparison for Example 1.

Alternatives	Method 1		Method 2		Method 3		Method 4		Proposed Method	
	C(A_i)	R	C(A_i)	R	C(A_i)	R	C(A_i)	R	C(A_i)	R
A_1	0.769	2	0.938	2	0.881	2	0.781	2	**0.771**	**2**
A_2	0.938	1	0.779	1	0.743	1	0.940	1	**0.938**	**1**
A_3	0.529	3	0.546	3	0.567	3	0.538	3	**0.531**	**3**
A_4	0.313	7	0.487	7	0.463	7	0.314	7	**0.313**	**7**
A_5	0.477	4	0.438	4	0.402	4	0.490	4	**0.479**	**4**
A_6	0.425	5	0.436	5	0.388	5	0.439	5	**0.427**	**5**
A_7	0.423	6	0.313	6	0.354	6	0.439	6	**0.425**	**6**

The results affirm that the proposed MAGDM method yields rankings consistent with those reported by existing frameworks. This robustness underscores the validity and reliability of the approach when restricted to positive IT2FNs.

5.2 Numerical Example 2: Car Selection with Positive and Negative IT2FNs

To further demonstrate the robustness and flexibility of the proposed MAGDM methodology, a car selection problem involving both positive and negative linguistic scales is presented. Three vehicles (A_1, A_2, A_3) are evaluated by three DMs across four attributes: C_1 (Safety), C_2 (Price), C_3 (Appearance), and C_4 (Performance). C_2 is a cost attribute, while the others are benefit attributes. The linguistic scales are defined in Tables 12 and 13, based on [41].

Table 12: Linguistic terms for performance and their corresponding IT2FNs [41].

Linguistic Variable	TIT2FNs
Very Low (VL)	((-10, -9, -8, -7; 0.8, 0.8), (-10, -10, -8, -6; 1, 1))
Low (L)	((-8, -7, -5, -4; 0.8, 0.8), (-9, -7, -5, -3; 1, 1))
Medium (M)	((-2, -1, 1, 2; 0.8, 0.8), (-3, -2, 2, 3; 1, 1))
High (H)	((4, 5, 7, 8; 0.8, 0.8), (3, 5, 7, 9; 1, 1))
Very High (VH)	((7, 8, 10, 10; 0.8, 0.8), (6, 8, 10, 10; 1, 1))

Table 13: Linguistic terms for weights and their corresponding IT2FNs [41].

Linguistic Term	TIT2FNs
Medium (M)	((-0.2, -0.1, 0.1, 0.2; 0.8, 0.8),(-0.3, -0.2, 0.2, 0.3; 1, 1))
Medium High (MH)	((0.1, 0.2, 0.4, 0.5; 0.8, 0.8),(0, 0.2, 0.4, 0.6; 1, 1))
High (H)	((0.4, 0.5, 0.7, 0.8; 0.8, 0.8),(0.3, 0.5, 0.7, 0.9; 1, 1))
Very High (VH)	((0.7, 0.8, 1, 1; 0.8, 0.8), (0.6, 0.8, 1, 1; 1, 1))

The procedural steps are analogous to the first example. The weighted decision matrix is constructed (Table 14) and then ranked (Table 15).

Finally, the PIS and NIS are determined, distances are calculated, and the final ranking is produced. The results and comparison are shown in Table 16.

According to the proposed method, alternative A_1 achieves the highest relative closeness coefficient (0.580), making it the best choice. The results produced by the proposed approach display strong consistency with some existing methods (agreeing closely with Methods 2 and 4 in selecting A_1 as optimal), but differences in ranking order for less-preferred alternatives reflect the improved sensitivity of the proposed method, especially when negative and positive linguistic terms coexist.

Table 14: The weighted decision matrix ($\tilde{v}_{ij}$) for Example 2.

Alternative	Attributes	Weighted IT2FN Value $\tilde{v}_{ij}$
A_1	C_1	((-2.8,-1.2,1.2,2.8;1,1),(-1.94,-0.9,0.9,1.94;0.8,0.8))
	C_2	((2.4,4.2,7.2,8.71;1,1),(2.5,4.2,7.2,8.41;0.8,0.8))
	C_3	((0,0.6,2.4,3.73;1,1), (-0.87,0.45,2.64,4.33;0.8,0.8))
	C_4	((2,3.6,6.4,8.09;1,1), (2,3.6,6.4,8.09;0.8,0.8))
A_2	C_1	((-8.71,-7.2,-4.2,-2.4;1,1),(-8.41,-6.9,-4.2,-2.5;0.8,0.8))
	C_2	((-4.98,-3.6,-1.4,0;1,1),(-4.53,-3.3,-1.4,-0.5;0.8,0.8))
	C_3	((-3.87,-2.7,-0.7,0;1,1),(-4.67,-2.75,-0.53,0.93;0.8,0.8))
	C_4	((-6.36,-5.07,-2.2,-1;1,1),(-6.22,-4.53,-2.4,-1.2;0.8,0.8))
A_3	C_1	((-7.78,-6.3,-3.5,-1.8;1,1),(-7.44,-6,-3.5,-2;0.8,0.8))
	C_2	((-6.53,-4.8,-1.87,-0.6;1,1),(-5.82,-4.5,-2.1,-1;0.8,0.8))
	C_3	((-3.33,-2.1,-0.5,0;1,1),(-3.83,-2.2,-0.38,0.77;0.8,0.8))
	C_4	((-8.38,-7.2,-4.2,-2.5;1,1),(-8.71,-6.67,-4.2,-2.4;0.8,0.8))

Table 15: Rank of the weighted decision matrix (r_{ij}) in Example 2.

Alternatives	C_1	C_2	C_3	C_4
A_1	0.0000	0.4420	0.1400	0.3980
A_2	-0.4400	-0.2060	-0.1490	-0.2950
A_3	-0.3840	-0.2750	-0.1240	-0.4380

Table 16: Final Rankings and Closeness Coefficients for Example 2.

Alternative	Method 1		Method 2		Method 3		Method 4		Proposed Method	
	C(A_i)	R	C(A_i)	R	C(A_i)	R	C(A_i)	R	C(A_i)	R
A_1	0.450	3	0.579	1	0.441	3	0.647	1	**0.580**	**1**
A_2	0.544	1	0.434	2	0.532	2	0.350	3	**0.432**	**2**
A_3	0.533	2	0.430	3	0.559	1	0.361	2	**0.427**	**3**

6 Discussion

Developing robust methodologies for multi-attribute group decision-making (MAGDM) under uncertainty remains a key challenge, especially when expert opinions involve imprecise linguistic terms. Fuzzy set theory—and more specifically, Type-2 Fuzzy Sets (T2FSs) and their interval forms (IT2FSs/IT2FNs)—has long been recognized as a powerful paradigm for modelling such complex, ambiguous evaluations [19, 20]. However, a persistent gap in the literature has been the asymmetrical treatment of linguistic assessment scales, which has constrained the fidelity and interpretability of group decision-making models.

This study overcomes these limitations by presenting two interconnected advancements. First, a symmetry-oriented ranking method for IT2FNs is presented. Unlike prior approaches—which either compute centroids, dominance, or similar indices over a positive-normalized scale—this method leverages the mean and standard deviation of both the upper (UMF) and lower membership functions (LMF), as well as the IT2FN height, to deliver a ranking index inherently sensitive to both positive and negative linguistic scales. Critically, this index is theoretically calibrated so that the "medium" or "neutral" linguistic value—the fuzzy equivalent of zero—serves as a precise balance point, directly reflecting the duality proposed in equilibrium/Yin-Yang theories and more accurately representing human reasoning in decision contexts. The rigorous development and validation of this ranking index go beyond numerical performance: the method aligns with the psychological and philosophical underpinnings of human judgment, rendering it theoretically robust. Key properties—including scale symmetry, zero-point neutrality, and the ability to treat mirror-opposite IT2FNs equivalently but with reversed sign—are demonstrated both formally and through comparative experiments.

Second, this ranking method becomes the cornerstone of a new MAGDM framework. By integrating dual-scale linguistic assessment—allowing both positive ("very high," "high") and negative ("very low," "low") terms—into the MADM pipeline, the methodology generalizes classical multi-attribute decision models to environments characterized by true evaluative duality. This means that decision-makers' linguistic inputs are not forcibly mapped onto a one-sided, positive-normalized scheme, but rather maintain their inherent semantic richness and interpretability throughout the evaluative process. In a practical implementation, decision-makers would continue to use familiar linguistic terms. The mapping to positive or negative IT2FNs is a back-end process, ensuring that the user interface remains intuitive while the underlying model captures the full semantic duality of their judgments.

Numerical experiments, including scenarios with both strictly positive and genuinely dual-scale linguistic assessments, confirm the practical superiority of the proposed approach. In benchmark problems, the new ranking method demonstrates enhanced discernibility between alternatives and a closer alignment with intuitive expectations of neutrality and duality. When deployed within the new MAGDM framework, group preference aggregation, attribute weighting, and alternative ranking all benefit from a more nuanced and interpretable processing of uncertainty and human linguistic judgment. Importantly, the framework facilitates more transparent and justifiable decision processes in real-world scenarios where not just positive advantages but also negative aspects and trade-offs must be considered—such as sustainability assessment, risk-benefit analysis, and social/environmental impact evaluations. The interpretability of the results, especially the explicit meaning of a "neutral" evaluation, can strengthen decision acceptance and stakeholder trust.

Compared with the state-of-the-art, these innovations substantially extend the modelling scope and accuracy of fuzzy MAGDM methods. Prior work [11, 16, 19, 20, 28], while advancing IT2FN ranking for purely positive scales or through incremental extensions, has not addressed the need for symmetric treatment encompassing both negative and positive evaluations. The present study, inspired by advanced theoretical considerations (e.g., Yin-Yang equilibrium), explicitly overcomes these gaps and empirically demonstrates why this is not merely an theoretical concern, but a critical factor for real-world decision support.

A key advantage of the proposed ranking method lies in its ability to symmetrically handle both positive and negative linguistic assessments by incorporating the mean, standard deviation, and height of the IT2FNs. This provides a more robust and nuanced evaluation compared to methods that operate only on positive scales. However, a potential trade-off is the slightly increased computational complexity due to the normalization and standard deviation calculation steps. We contend that this is a worthwhile compromise for the enhanced accuracy and applicability to real-world problems that feature inherent duality.

7 Conclusion

This study addresses the persistent challenge of effective multi-attribute group decision-making in environments marked by linguistic vagueness and subjective expert judgment. While Type-2 Fuzzy Sets have enhanced the capacity to model uncertainty in decision contexts, a major limitation of previous methods has been the asymmetrical treatment of linguistic scales. Conventional approaches have predominantly emphasized a normalized, positive scale, thereby failing to capture the natural duality of human perception and the importance of neutrality, as articulated in principles such as the Yin-Yang equilibrium theory. To overcome these limitations, this paper introduces two principal innovations: First, a novel, symmetry-oriented ranking method for IT2FNs is presented. This method uniquely incorporates both the average and standard deviation of the upper and lower membership functions, as well as the height of IT2FNs, to systematically treat positive, negative, and neutral ("fuzzy zero") linguistic evaluations in a balanced manner. Comprehensive comparative testing against state-of-the-art ranking methods confirms that the proposed approach delivers theoretically consistent and practically interpretable results, especially in contexts requiring explicit recognition of neutral and negative values.

Second, building on this ranking foundation, a new MAGDM framework is advanced, capable of integrating group expert assessments across dual-scale linguistic data. The method generalizes classical MADM techniques by enabling the aggregation and discrimination of alternatives not just on a unipolar, but on a truly bipolar (positive-negative) scale. Numerical demonstrations on benchmark problems with both positive-only and dual-scale linguistic inputs highlight the framework's superiority in terms of discernibility, interpretability, and faithfulness to human-centered reasoning.

In summary, the proposed IT2FN ranking method and the MAGDM framework together establish a new standard in fuzzy group decision modeling, respecting the complexity of expert linguistic judgments and robustly supporting real-world decisions characterized by uncertainty and duality. Future research directions include the adaptation of the method to general Type-2 or hesitant fuzzy environments, further exploration of aggregation strategies, and practical validation through large-scale, domain-specific applications.

Looking ahead, future work should focus on extending the developed ranking methodology to other classes of fuzzy sets—such as Intuitionistic, Pythagorean, and Neutrosophic sets—to further enhance its generality and applicability. Additionally, applying the proposed IT2FN ranking method across a broader range of multi-attribute decision-making problems may deepen its practical impact and demonstrate its versatility in complex real-world scenarios.

Declarations

Availability of Supporting Data

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Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Author Contributions

Ali Dehghani Filabadi: Conceptualization, Methodology, Software, Formal analysis, Investigation, Writing – Original Draft, Writing – Review and Editing. Hossein Nahid Titkanlue: Conceptualization, Methodology, Supervision, Validation, Resources, Writing – Original Draft, Writing – Review and Editing.

Artificial Intelligence Statement

Artificial intelligence (AI) tools, including large language models, were used solely for language editing and improving readability. AI tools were not used for generating ideas, performing analyses, interpreting results, or writing the scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

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