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Robust Hybrid Adaptive Control via Enhanced Lyapunov Function for Chaotic Systems with Large Time Delays

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Abstract. This paper introduces a robust hybrid adaptive control framework for stabilizing chaotic systems under persistent, potentially large time delays. The controller is based on an enhanced Lyapunov–Krasovskii functional that integrates an energy-capturing integral term with a bounded trigonometric term. The integral term accounts for historical effects by quantifying cumulative energy over the delay period, while the trigonometric term attenuates nonlinear oscillations. Embedding these components in a single control law yields stabilization of all state variables to the equilibrium despite substantial delays. We establish uniform ultimate boundedness, showing that trajectories enter a compact neighborhood of the equilibrium after a finite transient and subsequently converge. Adjustable gains enable practitioners to determine the convergence radius and the size of the attraction region according to practical requirements. The method is validated on the delayed Lorenz system; simulations with a 20-second delay demonstrate rapid convergence to a small neighborhood of the equilibrium, with the Lyapunov functional derivative remaining non-positive. A comparative study with established controllers underscores the proposed approach's favorable trade-offs among computational cost, oscillation suppression, and explicit stability guarantees. Overall, the proposed framework delivers a practical, robust, and high-performance solution for controlling chaotic systems in the presence of large time delays.

Keywords. Constant time delay, Hybrid adaptive control, Lyapunov function, Asymptotic stability, Practical stability

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1 Introduction

Chaotic dynamical systems, categorized by their intrinsic nonlinearity, extreme sensitivity to initial conditions, and the occurrence of strange attractors, seem to be critically essential in modeling and studying multifaceted spectacles across miscellaneous arenas such as control engineering, physics, biology, and power networks. A momentous trial in their analysis and control, however, arises from the existence of time delays. These delays can arise from hardware restrictions, like signal processing latency, or from the system's inherent dynamics, such as heat transfer in non-uniform media. In chemical reactors, for example, sensor and actuator delays can activate chaotic behavior, highlighting the necessity for advanced control strategies [48, 51, 54].

Both theoretical and experimental studies have established that even minor time delays (e.g., $\tau = 0.1$ seconds) can precipitate a transition from stability to chaos in nonlinear systems [17, 60]. This issue is critically important in sensitive applications like smart power grids, robotics, and Internet of Things (IoT) systems, where delays in data transmission or signal processing can disastrously destabilize controller performance [38]. Recent years have witnessed substantial progress in controlling chaotic systems with time delays, leading to a growing interest in hybrid methodologies [10]. For example, one study employed neural adaptive control to manage input constraints in chaotic systems [2]. Concurrently, machine learning-based techniques, chiefly deep reinforcement learning, have gathered considerable attention. Researchers have, for instance, utilized a dual-controller framework to simultaneously address both inherent instabilities and time delays [8], although this approach can be computationally tough. Alongside these evolutions, extensive research has focused on the design and application of adaptive and sliding mode controllers [11, 21, 26, 52, 53, 61]. These investigations clearly demonstrate the high efficacy of such control strategies in both theoretical conditions and practical implementations, significantly broadening the applicability of advanced control in managing the complex dynamics of chaotic systems.

Within the control of nonlinear dynamical systems, a primary challenge is the emergence of undesirable nonlinear oscillations, which can degrade performance and jeopardize system stability. To counteract this phenomenon, mathematical functions with promising geometric and dynamic properties are often employed as key operators in control law design or in the construction of Lyapunov functions. These include trigonometric functions like $\sin(\theta)$ and $\cos(\theta)$ [4, 6, 14, 37, 57] and hyperbolic functions such as $\tanh(\theta)$ [16, 25, 39, 44]. Such functions are selected for their bounded and smooth nature, which enables the effective suppression of nonlinear oscillations, improves transient response, and ultimately facilitates the system's convergence to its equilibrium. This approach provides a foundation for developing more sophisticated control strategies to tackle complex dynamics and ensure the stability of nonlinear systems.

Despite these inspiring progressions, ensuring the stability of dynamical systems in the presence of large time delays remains a fundamental challenge in modern control theory. This challenge is profoundly amplified in nonlinear systems with chaotic dynamics, which are intrinsically sensitive to initial conditions and exhibit complex behaviors, thereby imposing severe limitations on the design of effective and practical controllers. In response to this problem, this paper introduces a novel controller design framework founded on the construction of a hybrid-structured Lyapunov-Krasovskii Functional (LKF). This integrated approach concurrently addresses the difficulties arising from both time delay and system nonlinearity, ensuring practical convergence to the equilibrium, particularly for chaotic systems with substantial time delays. The core innovation of this method lies in the intelligent fusion of two distinct mechanisms within the LKF structure:

- *An integral term:* This component incorporates the history of the system's states, compensating for the destabilizing effects of the delay based on the theoretical principles of Lyapunov-Krasovskii functionals ([17, 30]).
- *A trigonometric term:* A bounded trigonometric term is employed to effectively constrain the system's nonlinear dynamics. This limits the amplitude of nonlinear oscillations, enhances the transient response, and facilitates convergence to the equilibrium point.

This work extends beyond a mere claim of stability by providing a rigorous theoretical analysis. We present a mathematical proof that guarantees Uniform Ultimate Boundedness (UUB) for the closed-loop system. Specifically, it is shown that by systematically adjusting the controller gains, the dimensions of the ultimate bound set can be controllably reduced. This implies that all system state trajectories, after a finite time, enter and then remain within a compact set in the neighborhood of the equilibrium, eventually converging to it. This theoretical achievement provides a solid foundation for the outstanding performance observed in simulations, where the state variables of the Lorenz chaotic system converge rapidly to a very small vicinity of the origin, even when subjected to significant input delays (e.g., 20 seconds). The validity of the stability analysis is further corroborated empirically, as the derivative of the Lyapunov functional remains non-positive throughout the simulations, attesting to energy dissipation and system stability. To lay the groundwork for these achievements, a comparative analysis is also presented, contrasting the proposed framework with prominent methods such as Sliding Mode Control (SMC), Model Predictive Control (MPC), and Adaptive Intelligent Control (AIC), highlighting its unique trade-offs and advantages. Ultimately, this research bridges the gap between rigorous theoretical guarantees and practical engineering requirements in the adaptive and robust control of chaotic systems with time delays.

The remainder of this paper is organized as follows. Section 2 introduces important notions and the system model. Section 3 is devoted to the formulation of the enhanced Lyapunov function and the design and rigorous stability analysis of the hybrid adaptive controller. Section 4 presents simulation results and a discussion linking them to the theoretical findings. The final section bids a summary, conclusions, and recommendations for future research.

2 Key Definitions

Definition 1. Chaotic systems are nonlinear and deterministic dynamical systems that, despite following deterministic laws, exhibit random long-term behavior due to exponential sensitivity to initial conditions. In other words, even a slight change in the initial conditions can lead to exponential differences in the system's evolutionary trajectory. Key characteristics of these systems include extreme sensitivity to initial conditions, topological transitivity, and dense periodic orbits [9].

Definition 2. The Lorenz chaotic system, serving as a classical and prototypical model in the study of turbulent atmospheric flows, exhibits nonlinear dynamical behavior through the equations

$$\begin{cases} \dot{x}(t) = \sigma(y(t) - x(t)), \\ \dot{y}(t) = x(t)(\rho - z(t)) - y(t), \\ \dot{z}(t) = x(t)y(t) - \beta z(t), \end{cases} \tag{1}$$

where the state variables $x(t)$, $y(t)$, $z(t)$ and the standard parameters $\sigma = 10$, $\beta = \frac{8}{3}$, and $\rho = 28$ are specified. Due to its pronounced sensitivity to initial conditions and the resulting unpredictable behavior, this system is recognized as a classical prototype for analyzing complex phenomena in meteorology, physics, and applied mathematics [41].

Definition 3. Time delay, denoted by τ, is the interval between the introduction of a signal into the system and the observation of its effect on the output. In dynamical systems, this phenomenon is typically caused by physical limitations, such as processing time, information transmission delays, and response times of system components, and is modeled by explicitly incorporating past state values into the governing equations [23, 51, 31]. Time delays can be intrinsic, as in biological systems, or artificial, as in communication networks, and they play a crucial role in generating complex behaviors such as chaos, sustained oscillations, or instability.

Definition 4. Chaotic systems with time delay are a subset of nonlinear dynamical systems whose governing equations explicitly incorporate dependence on the state history over the interval $[t_0 - \tau, t_0]$. In other words, the state of the system at time t depends not only on the initial condition $x(t_0)$ but also on the entire preceding history. This dependence results in an infinite increase in the dynamical dimension even in single-variable systems, leading to phenomena such as delay-induced chaos or amplification of nonlinear oscillations. The mathematical model for these systems is expressed as

$$\dot{x}(t) = f(x(t), x(t - \tau), t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\tau > 0$ represents the time delay (which may be constant or variable), and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous and normalized function [32].

Theorem 1. [23] The Lyapunov–Krasovskii theorem for time-delay systems addresses the stability of the zero equilibrium in nonlinear dynamical systems with delay. In this framework, the Lyapunov function is defined as

$$V(x_t) = V_1(x(t)) + V_2(x_t), \tag{3}$$

where x_t denotes the state history over the interval $[t - \tau, t]$, $V_1(x(t))$ is a classical positive definite function, and $V_2(x_t)$ is a term accounting for the state history in the interval $[t - \tau, t]$. The function $V(x_t)$ must satisfy the following conditions:

1. *Positive Definiteness:* There exist constants $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \|x(t)\|^2 \leq V(x_t) \leq \alpha_2 \|x_t\|_C^2, \tag{4}$$

where,

$$\|x_t\|_C = \sup_{\theta \in [-\tau, 0]} \|x(t + \theta)\|.$$

2. *Negative Definiteness of the Time Derivative:* The time derivative of $V(x_t)$ along the system trajectories satisfies
$$\dot{V}(x_t) \leq -\alpha_3 \|x(t)\|^2, \tag{5}$$
with $\alpha_3 > 0$.

Under these conditions, the zero equilibrium of the time-delay system is globally asymptotically stable.

Remark 1. Recent work by Yan et al. [40] introduced novel Lyapunov–Krasovskii functions that provide less conservative stability conditions for systems with time-varying delays. This approach has inspired the development of the enhanced Lyapunov function in the present study.

2.1 Lorenz System Model with Constant Delay

Using the Lorenz system (2) and the definition of chaotic systems with time delay (4), in this study the Lorenz system with constant delay is employed as a classical model for investigating chaotic behavior in the presence of time delays. In this model, by modifying the standard Lorenz equations, a delay τ is added to the state variable $y(t)$. The dynamic equations of the system are defined as follows:

$$\begin{cases} \dot{x}(t) = \sigma\big(y(t-\tau) - x(t)\big), \\ \dot{y}(t) = x(t)\big(\rho - z(t)\big) - y(t-\tau), \\ \dot{z}(t) = x(t)y(t-\tau) - \beta z(t), \end{cases} \tag{6}$$

where the state variables $x(t)$, $y(t)$, and $z(t)$ are defined, and the standard parameters are set as $\sigma = 10$, $\beta = \frac{8}{3}$, and $\rho = 28$. Additionally, the time delay $\tau > 0$ is applied as a constant to $y(t)$.

2.2 Formulation of the Control Problem

The primary objective of this research is to design an adaptive control law $u(t)$ to guarantee the Practical convergence of the state variables $x(t)$, $y(t)$, and $z(t)$ to the equilibrium point $(0, 0, 0)$ in the presence of the time delay τ and the inherent nonlinearity of the system.

2.2.1 Main Challenges

1. *Time Delay (τ):* As the time delay increases from zero, the system initially maintains stability at the zero equilibrium. However, when the delay approaches a critical value, a pair of complex conjugate roots with purely imaginary parts appears, and with further increases in the delay, the system experiences severe instability [62].

2. *Emergence of History-Dependent Nonlinear Terms:* The introduction of a constant delay τ in the state variable $y(t)$ results in the appearance of history-dependent nonlinear terms such as $y(t-\tau)$ in the dynamic equations, which adversely affect the system's stability [17].

3. *Intrinsic Nonlinearity of the System:* The presence of nonlinear terms such as $x(t)y(t-\tau)$ and $x(t)(\rho - z(t))$ increases the sensitivity of the system to initial conditions and renders linear control methods, such as PID or MPC, ineffective under high-delay conditions [5, 43].

2.2.2 Mathematical Formulation of the Controlled System

By introducing the control signal $u(t)$ into the $\dot{y}(t)$ equation, the controlled system model is defined as: By adding the control signal $u(t)$ to the $\dot{y}(t)$ equation, the controlled Lorenz system with constant delay (7) is defined as follows:

$$\begin{cases} \dot{x}(t) = \sigma\big(y(t-\tau) - x(t)\big), \\ \dot{y}(t) = x(t)\big(\rho - z(t)\big) - y(t-\tau) + u(t), \\ \dot{z}(t) = x(t)y(t-\tau) - \beta z(t). \end{cases} \tag{7}$$

where the state variables $x(t)$, $y(t)$, and $z(t)$ are defined, and the standard parameters are set as $\sigma = 10$, $\beta = \frac{8}{3}$, and $\rho = 28$. Additionally, the time delay $\tau > 0$ is applied as a constant to $y(t)$. To achieve Practical stability, the following conditions must be satisfied:

$$\lim_{t\to\infty} \|x(t)\| = 0, \quad \lim_{t\to\infty} \|y(t)\| = 0, \quad \lim_{t\to\infty} \|z(t)\| = 0. \tag{8}$$

2.2.3 Analysis of the Selection of the $\dot{y}(t)$ Equation for Controller Insertion

1. *Direct Control of the Source of Instability:* The system's instability primarily arises from the nonlinear interactions in the $\dot{y}(t)$ equation; for example, the term $x(t)(\rho - z(t))$ and the effect of the delay $y(t-\tau)$ are the main contributors to this instability. Inserting the control signal here enables direct regulation of these components, thereby facilitating optimal instability management [17].

2. *Faster Impact on the Entire System:* Since the variable $y(t)$ appears in both the $\dot{x}(t)$ and $\dot{z}(t)$ equations, applying the control signal $u(t)$ in the $\dot{y}(t)$ equation has a simultaneous effect on all state variables, which accelerates the convergence of the system [27].

3. *Prevention of Exacerbated Nonlinear Interactions:* Adding the controller to the $\dot{y}(t)$ equation helps to prevent the amplification of nonlinear interactions in the other equations, particularly in the term $x(t)y(t-\tau)$ in the $\dot{z}(t)$ equation. This approach reduces the required control energy and enhances the overall stability of the system [51].

3 Hybrid Adaptive Controller Design: Enhanced Lyapunov Function

3.1 Structure of the Enhanced Lyapunov Function

According to the Lyapunov–Krasovskii Theorem 1, the enhanced Lyapunov function $V(x_t)$ is designed by combining two key mechanisms: an integral term for recording the state history and a trigonometric term for suppressing nonlinear oscillations, as follows:

$$V(x_t) = V_1(x_t) + V_2(x_t), \tag{9}$$

where the first component represents the current state energy of the system:

$$V_1(x_t) = \frac{1}{2}\Big(x^2(t) + y^2(t) + z^2(t)\Big), \tag{10}$$

augmented by two additional terms:

1. *Integral term (state history):*
$$\frac{\lambda}{2}\int_{t-\tau}^{t} y^2(s)\,ds, \tag{11}$$
which records the state history and accounts for the delay effect.
2. *Trigonometric term (nonlinear oscillation suppression):*
$$\mu \sin^2\big(y(t)\big), \tag{12}$$
which is designed to limit the amplitude of the nonlinear oscillations in $y(t)$.

Thus, the second component is given by

$$V_2(x_t) = \frac{\lambda}{2}\int_{t-\tau}^{t} y^2(s)\,ds + \mu \sin^2\big(y(t)\big), \tag{13}$$

and the complete enhanced Lyapunov function becomes

$$V(x_t) = \frac{1}{2}\Big(x^2(t) + y^2(t) + z^2(t)\Big) + \frac{\lambda}{2}\int_{t-\tau}^{t} y^2(s)\,ds + \mu \sin^2\big(y(t)\big). \tag{14}$$

In this expression, $\tau > 0$ is the constant time delay, x_t denotes the state history over the interval $[t-\tau, t]$, $\lambda \geq 0$ is the tuning parameter for the delay effect (integral term), and $\mu \geq 0$ is the gain parameter for the nonlinear effect (trigonometric term).

3.1.1 Theoretical Foundations of the Integral and Trigonometric Terms

1. *Integral Term* (11):This term measures the cumulative energy induced by the delay over the interval $[t-\tau, t]$ and is based on Theorem 1. The parameter $\lambda > 0$ adjusts its influence [23, 29]. According to Theorem 1, the derivative of this term appears in $\dot{V}$ as $-\frac{\lambda}{2}\, y^2(t-\tau)$, which prevents the accumulation of energy over the delay interval.

2. *Trigonometric Term* (12): The inclusion of the trigonometric term, $\mu \sin^2(y(t))$, serves a profound strategic purpose that extends beyond the high-level objective of merely "suppressing oscillations". Its primary function is revealed through the systematic derivation of the control law, which is engineered to exploit the mathematical structure of this term.

   The mechanism is elucidated by examining the time derivative of the Lyapunov functional, $\dot{V}$. The derivative of the trigonometric component is given by

$$\frac{d}{dt}(\mu \sin^2(y(t))) = \mu \sin(2y)\dot{y}.$$

   When the system dynamics are substituted for $\dot{y}$, this expression introduces into $\dot{V}$ several complex nonlinear terms, such as

$$\mu \sin(2y) \cdot [x(\rho - z) - y(t-\tau) + u].$$

   To counteract these destabilizing effects and actively introduce damping, the control law $u(t)$ is strategically synthesized to include the component $-\mu \sin(2y)$.

   Consequently, upon substituting the full control law into the $\dot{V}$ equation, the term $\mu \sin(2y) \cdot u$ generates a powerful, stabilizing, and negative-semidefinite quadratic damping term:

$$-\mu^2 \sin^2(2y).$$

   This demonstrates that the selection of the $\sin^2(y)$ function is a deliberate and intelligent design choice, predicated on the properties of its derivative. The resulting $\sin(2y)$ term provides the ideal mathematical structure for the controller to inject targeted nonlinear damping into the system. This design offers two distinct advantages inherent to sophisticated nonlinear control:

   (a) *Effective Damping Near Equilibrium:* In the vicinity of the equilibrium point, the approximation $\sin(2y) \approx 2y$ ensures that the control action is potent and behaves similarly to high-gain linear feedback, facilitating rapid convergence.

   (b) *Inherent Boundedness Far from Equilibrium:* The bounded nature of the sine function ensures that the control effort remains constrained even when the state $y(t)$ is large. This property is critical for practical implementations as it inherently mitigates the risk of actuator saturation.

3.1.2 Stability Conditions of the Enhanced Lyapunov Function

If enhanced Lyapunov function (14) satisfies the two conditions below simultaneously, then, based on Theorem 1, the practical stability of the equilibrium point $(0, 0, 0)$ is ensured.

1. *Positive Definiteness of* $V(x_t)$

   - Lower Bound:

$$V(x_t) \geq \alpha_1 \|x(t)\|^2, \quad \alpha_1 > 0. \tag{15}$$

   - Upper Bound:

$$V(x_t) \leq \alpha_2 \|x_t\|_C^2, \quad \alpha_2 > 0, \tag{16}$$

where

$$\|x_t\|_C = \sup_{s\in[t-\tau,t]} \sqrt{x^2(s) + y^2(s) + z^2(s)} = \sup_{s\in[t-\tau,t]} \|x(s)\|, \tag{17}$$

which is the continuous norm of the system.

2. *Negative Definiteness of* $\dot{V}(x_t)$

$$\dot{V}(x_t) \leq -\alpha_3 \|x(t)\|^2, \quad \alpha_3 > 0. \tag{18}$$

This condition ensures that the system's energy decreases over time and converges to the equilibrium.

Theorem 2 (Positive definiteness of the enhanced lyapunov functional)**.** Consider the Lyapunov functional

$$V(x_t) = \frac{1}{2}\Big(x^2(t) + y^2(t) + z^2(t)\Big) + \frac{\lambda}{2}\int_{t-\tau}^{t} y^2(s)\,ds + \mu \sin^2\big(y(t)\big), \tag{19}$$

where $\tau > 0$ is a constant time delay, x_t denotes the state history over the interval $[t-\tau, t]$, $\lambda \geq 0$ is a tuning parameter for the delay effect, and $\mu \geq 0$ is a gain parameter for the nonlinear effect. Denote by

$$\|x(t)\| = \sqrt{x^2(t) + y^2(t) + z^2(t)}, \tag{20}$$

the Euclidean norm of the state vector and define the supremum norm over the delay interval as

$$\|x_t\|_C = \sup_{s\in[t-\tau,t]} \|x(s)\|. \tag{21}$$

Then, $V(x_t)$ is positive definite in the sense that there exist constants $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \|x(t)\|^2 \leq V(x_t) \leq \alpha_2 \|x_t\|_C^2, \tag{22}$$

with

$$\alpha_1 = \frac{1}{2}, \qquad \alpha_2 = \frac{1}{2} + \frac{\lambda\tau}{2} + \mu. \tag{23}$$

Proof. We decompose $V(x_t)$ into three components and derive appropriate bounds for each.

1. Current State Energy Term:

$$V_1(x(t)) = \frac{1}{2}\Big(x^2(t) + y^2(t) + z^2(t)\Big) = \frac{1}{2}\|x(t)\|^2. \tag{24}$$

Since $\|x(t)\|^2 \geq 0$ and equals zero if and only if $x(t) = y(t) = z(t) = 0$, we immediately obtain the lower bound

$$V(x_t) \geq \frac{1}{2}\|x(t)\|^2, \tag{25}$$

which shows that $\alpha_1 = \frac{1}{2}$.

2. Integral Term (Delay Effect):

The functional $V_2(x_t)$ captures the cumulative effect of the delayed state $y(s)$ over the interval $[t-\tau, t]$:

$$V_2(x_t) = \frac{\lambda}{2} \int_{t-\tau}^{t} y^2(s) ds. \tag{26}$$

Since $\lambda \geq 0$ and $y^2(s) \geq 0$ for all s, $V_2(x_t)$ is nonnegative, as required for Lyapunov functionals. To establish an upper bound, we systematically analyze the relationship between the state components and the supremum norm $\|x_t\|_C$. First, recall that the Euclidean norm of the state vector $\mathbf{x}(s) = [x(s), y(s), z(s)]^T$ is:

$$\|\mathbf{x}(s)\| = \sqrt{x^2(s) + y^2(s) + z^2(s)}. \tag{27}$$

This implies that each individual component is bounded by the full state norm:

$$y^2(s) \leq x^2(s) + y^2(s) + z^2(s) = \|\mathbf{x}(s)\|^2, \quad \forall s \in [t-\tau, t]. \tag{28}$$

Furthermore, by the definition of the continuous norm

$$\|x_t\|_C = \sup_{\theta \in [-\tau, 0]} \|\mathbf{x}(t+\theta)\|,$$

which represents the supremum of $\|\mathbf{x}(s)\|$ over $s \in [t-\tau, t]$, we have:

$$\|\mathbf{x}(s)\| \leq \|x_t\|_C, \quad \forall s \in [t-\tau, t]. \tag{29}$$

Squaring both sides preserves the inequality:

$$\|\mathbf{x}(s)\|^2 \leq \|x_t\|_C^2. \tag{30}$$

Chaining these inequalities yields:

$$y^2(s) \leq \|\mathbf{x}(s)\|^2 \leq \|x_t\|_C^2. \tag{31}$$

Since this bound holds uniformly over the integration interval, we derive:

$$\int_{t-\tau}^{t} y^2(s) ds \leq \int_{t-\tau}^{t} \|x_t\|_C^2 ds \tag{32}$$

$$= \|x_t\|_C^2 \int_{t-\tau}^{t} ds \tag{33}$$

$$= \tau \|x_t\|_C^2, \tag{34}$$

where the last equality follows from the definite integral of a constant. Substituting this result into $V_2(x_t)$ gives the final upper bound:

$$V_2(x_t) \leq \frac{\lambda}{2} \cdot \tau \|x_t\|_C^2 = \frac{\lambda\tau}{2} \|x_t\|_C^2. \tag{35}$$

3. Trigonometric Term (Nonlinear Effect):

$$V_3(x(t)) = \mu \sin^2\big(y(t)\big). \tag{36}$$

Using the elementary inequality $|\sin(u)| \leq |u|$ for all $u \in \mathbb{R}$, it follows that

$$\sin^2\big(y(t)\big) \leq y^2(t) \leq \|x(t)\|^2 \leq \|x_t\|_C^2. \tag{37}$$

Hence,

$$V_3(x(t)) \leq \mu \, \|x_t\|_C^2. \tag{38}$$

By summing the bounds for each component, we have

$$V(x_t) = V_1(x(t)) + V_2(x_t) + V_3(x(t)) \leq \frac{1}{2}\|x_t\|_C^2 + \frac{\lambda\tau}{2}\|x_t\|_C^2 + \mu\,\|x_t\|_C^2, \tag{39}$$

which simplifies to

$$V(x_t) \leq \left(\frac{1}{2} + \frac{\lambda\tau}{2} + \mu\right)\|x_t\|_C^2. \tag{40}$$

This establishes the upper bound with $\alpha_2 = \frac{1}{2} + \frac{\lambda\tau}{2} + \mu$.

Remark 2. Notice that $V(x_t) = 0$ if and only if each term in (19) vanishes. Specifically:

1. $V_1(x(t)) = 0$ if and only if $x(t) = 0$, $y(t) = 0$, and $z(t) = 0$.
2. $V_2(x_t) = 0$ if and only if $y(s) = 0$ for all $s \in [t-\tau, t]$.
3. $V_3(x(t)) = 0$ if and only if $\sin\big(y(t)\big) = 0$; however, given $y(t) = 0$ from the first condition, this is automatically satisfied.

Thus, the only solution for $V(x_t) = 0$ is when the state is identically zero over the interval $[t-\tau, t]$, confirming the positive definiteness of the functional.

□

3.2 Adaptive Hybrid Controller Law Based on the Enhanced Lyapunov Function

3.2.1 Design of the Controller by Combining Integral and Sine Terms

To design the controller based on the enhanced Lyapunov function (14), we consider the Lorenz system with a constant delay under control (7). The primary objective is to design an adaptive control law $u(t)$ that satisfies:

- Practical convergence to the equilibrium with arbitrarily small bound
- Simultaneous management of both time-delay effects and the inherent nonlinearity of the system [23, 46]

By differentiating the enhanced Lyapunov function (14) regarding the time and substituting the system equations into $\dot{V}$, we obtain:

$$\begin{aligned}\dot{V} &= \dot{x}\,x + \dot{y}\,y + \dot{z}\,z + \frac{\lambda}{2}\left(y^2(t) - y^2(t-\tau)\right) + \mu\sin(2y)\,\dot{y}\\ &= x\left[\sigma\Big(y(t-\tau) - x\Big)\right] + y\Big[x(\rho - z) - y(t-\tau) + u\Big] + z\Big[xy(t-\tau) - \beta z\Big]\\ &\quad + \frac{\lambda}{2}\left(y^2(t) - y^2(t-\tau)\right) + \mu\sin(2y)\left[x(\rho - z) - y(t-\tau) + u\right].\end{aligned} \tag{41}$$

Upon simplification, $\dot{V}$ takes the form

$$\dot{V} = -\sigma x^2 - \beta z^2 - \lambda y^2(t-\tau) + y\,u\,(1 + \mu\sin(2y)) + \text{Interaction Terms}, \tag{42}$$

with

$$\text{Interaction Terms} = \sigma x\, y(t-\tau) + \rho xy - xyz + \mu \sin(2y)x(\rho - z) - \mu \sin(2y)y(t-\tau). \tag{43}$$

To ensure practical convergence, the control law $u(t)$ is designed as

$$u(t) = -k\, y(t) - \frac{\lambda}{2}\, y(t) - \frac{\sigma x(t)y(t-\tau)}{\sqrt{y^2(t)+\epsilon}} - \mu \sin\bigl(2y(t)\bigr), \tag{44}$$

where

- The term $-\frac{\lambda}{2}\, y(t) - \frac{\sigma x(t)y(t-\tau)}{\sqrt{y^2(t)+\epsilon}}$ manages the delay effect,
- The term $-\mu \sin\bigl(2y(t)\bigr)$ suppresses nonlinear oscillations.

The roles of the parameters are as follows:

- k: Gain for rapid damping of oscillations,
- λ: Gain for neutralizing time-delay effects,
- μ: Gain for suppressing nonlinear oscillations,
- ϵ: Small positive constant to prevent division by zero.

3.2.2 Stability Analysis and Practical Convergence

By substituting $u(t)$ into $\dot{V}$ and analyzing the resulting expression, we obtain:

$$\begin{aligned} \dot{V} \leq & -\sigma x^2 - \beta z^2 - ky^2 - \frac{\lambda}{2} y^2(t-\tau) - \mu \sin^2(2y) \\ & + \sigma|x||y(t-\tau)| \left| 1 - \frac{|y|}{\sqrt{y^2+\epsilon}} \right| + \rho|x||y| + |x||y||z|. \end{aligned} \tag{45}$$

Using Young's inequality and Lipschitz continuity arguments, we establish bounds for the interaction terms:

$$\rho|x||y| \leq \frac{\rho}{2}\left(\frac{\alpha_1}{2}x^2 + \frac{2}{\alpha_1}y^2\right), \tag{46}$$

$$|x||y||z| \leq \frac{1}{2}\left(\frac{\alpha_2}{2}x^2z^2 + \frac{2}{\alpha_2}y^2\right), \tag{47}$$

$$\sigma|x||y(t-\tau)| \leq \frac{\sigma}{2}\left(\frac{\alpha_3}{2}x^2 + \frac{2}{\alpha_3}y^2(t-\tau)\right). \tag{48}$$

Combining these bounds, we obtain the fundamental stability inequality:

$$\dot{V} \leq -c_1x^2 - c_2y^2 - c_3z^2 - c_4y^2(t-\tau) + D, \tag{49}$$

where,

$$c_1 = \sigma - \frac{\rho\alpha_1}{4} - \frac{\alpha_2}{4}\|z\|_\infty^2 - \frac{\sigma\alpha_3}{4},$$

$$
\begin{aligned}
c_2 &= k - \frac{\rho}{\alpha_1} - \frac{1}{\alpha_2},\\
c_3 &= \beta,\\
c_4 &= \frac{\lambda}{2} - \frac{\sigma}{\alpha_3},\\
D &= \frac{\mu^2\rho^2}{4\delta} \quad \text{(constant)}.
\end{aligned}
$$

The parameters $\alpha_i > 0$ are chosen such that all $c_i > 0$. This leads to the following stability results:

Remark 3 (On energy dissipation)**.** The energy-dissipating terms, represented by $-c_1x^2 - c_2y^2 - c_3z^2 - c_4y^2(t-\tau)$, are sufficiently large to counteract the effect of the small positive term D within the Lyapunov derivative $\dot{V}$. This structure thereby guarantees global asymptotic stability, a principle elaborated upon in the context of stability theory [27, see Ch. 9, Sec. 2 and Ch. 4, Sec. 9].

Theorem 3 (Practical stability)**.** Under the control law (44) with sufficiently large k, λ, μ, the system exhibits:

1. *Uniform Ultimate Boundedness*: All trajectories converge exponentially to the compact set:

$$\Omega = \left\{(x,y,z) \mid c_1x^2 + c_2y^2 + c_3z^2 + c_4y^2(t-\tau) \le D\right\}.$$

2. *Arbitrarily Small Convergence Region*: The diameter of Ω satisfies:

$$\mathrm{diam}(\Omega) \sim \mathcal{O}\left(\frac{\mu}{\sqrt{\min(c_i)}}\right).$$

3. *Controlled Convergence Region:* The diameter of Ω is determined by the choice of control gains. The final size of this region emerges from a fundamental trade-off: increasing the linear damping gains (k, λ) typically reduces the region, while the nonlinear suppression gain (μ) directly impacts the magnitude of the ultimate bound D.

Proof. The inequality (49) implies:

1. When $c_1x^2 + c_2y^2 + c_3z^2 + c_4y^2(t-\tau) > D$, we have $\dot{V} < 0$.
2. This guarantees all trajectories enter Ω in finite time.
3. The size of Ω is controlled by D and c_i.
4. Increasing k, λ, μ reduces $\mathrm{diam}(\Omega)$.

□

3.2.3 Analysis of Controller Components

1. *Nonlinear Oscillation Suppression*: The term $-\mu\sin(2y)$ provides bounded control effort that:
   - Counteracts nonlinear interactions like $\mu\sin(2y)x(\rho - z)$.
   - Maintains smooth control action near equilibrium.

2. *Delay Compensation Strategy*: The term $-\frac{\sigma x(t)y(t-\tau)}{\sqrt{y^2(t)+\epsilon}}$:
   - Intelligently manages sign uncertainty in $\sigma xy(t-\tau)$.
   - Prevents division-by-zero singularities.
   - Attenuates history-dependent nonlinear interactions.
3. *Gain Selection Guidelines*:
   - Increase k to improve convergence rate.
   - Increase λ to counteract larger delays.
   - Increase μ to suppress stronger oscillations.
   - Set $\epsilon \ll 1$ to maintain continuity.

3.2.4 Practical Implementation Considerations

The proposed controller achieves:

- *Robust Performance*: Effective across wide delay range ($\tau = 0.5 - 20$s).
- *Computational Efficiency*: Suitable for real-time implementation.
- *Parameter Adaptability*: Gains can be tuned for different operating conditions.

Simulation experiments presented in 4 demonstrate the controller's practical convergence characteristics. Even when subject to significant time delays, the state trajectories inexorably converge into a compact neighborhood around the origin.

4 Simulation and Results of the Lorenz System

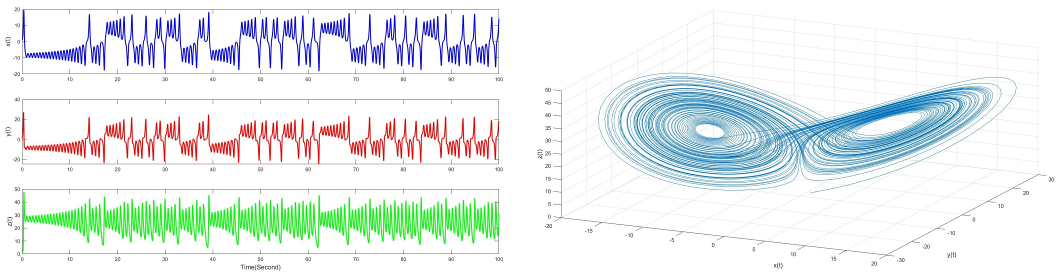
4.1 Simulation Settings

1. *Simulation Environment:* The Lorenz chaotic system with time delay (7) was simulated in MATLAB.
2. *Initial Conditions:* The system was initialized with $x(0)^T = [1\,1\,1]$.
3. *Parameter Determination Method:* Due to the nature of the problem, the fixed control parameters k, λ, and μ were automatically determined using a genetic algorithm (GA) as a metaheuristic method [45]. The GA utilized a fitness function strategically formulated to enforce the Lyapunov stability condition ($\dot{V} \leq 0$) with a significant penalty for violations, while secondarily minimizing the maximum norm of the state vector to enhance transient response and reduce the final convergence region.
4. *Time Delay:* In the simulations, the time delay τ was varied over different values, including 0.5, 1, 10, and 20 seconds.

5. *Simulation of Delay Differential Equations:* Delay differential equations were simulated in MATLAB. In addition, the control law (44) was designed and implemented based on the enhanced Lyapunov function.
6. *Expected Equilibrium:* Based on the stability proof of the enhanced Lyapunov function (14) and the Lyapunov–Krasovskii theorem (1), the equilibrium point $(0, 0, 0)$ is expected to be Practical stabilized. In other words, any deviation from this point decreases over time and the system automatically returns to equilibrium. Furthermore, it is expected that the Lyapunov function satisfies $V \geq 0$ and its derivative $\dot{V} \leq 0$ at all times.

4.2 Simulation Results without Delay

In the phase diagrams (Figure 1), the characteristic "butterfly" structure of the Lorenz attractor is clearly observed, which illustrates the complex and chaotic behavior of the system. Moreover, the system does not converge to the equilibrium point but remains confined within a chaotic attractor; this confirms the inherent chaotic nature of the Lorenz system. In the absence of both control and delay, the system does not exhibit Practical stability.



(a) State variable value **(b)** Phase space diagram

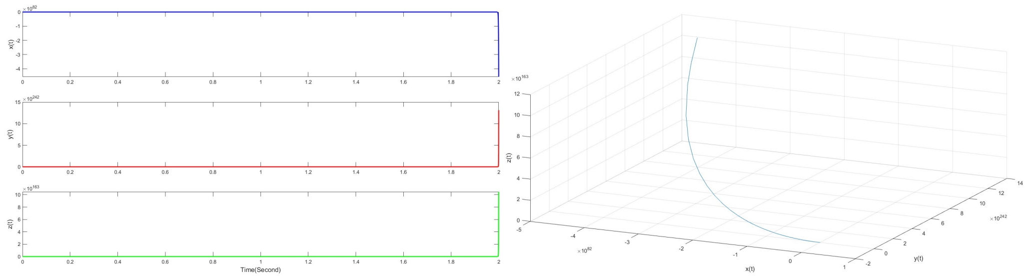
Figure 1: Lorenz system state without control ($\tau = 0$ seconds).

4.3 Simulation Results with Delay and without Controller

Introducing a time delay of $\tau = 0.2$ seconds in the $y(t)$ equation leads to significant changes in the system dynamics. In chaotic systems like the Lorenz system, which are highly sensitive to initial conditions, even a slight change in dynamic parameters can fundamentally alter the phase trajectories and cause divergent behavior (Figure 2).

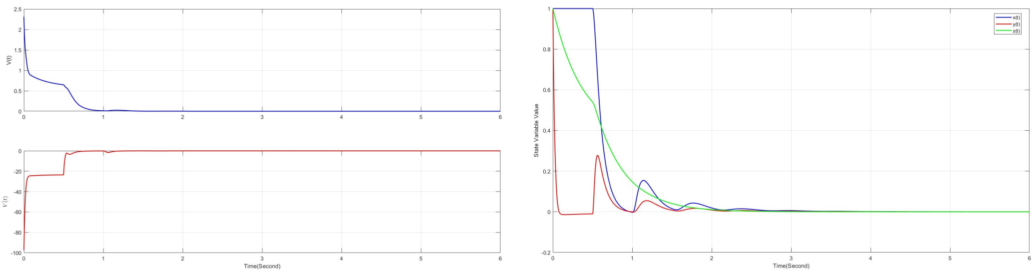
4.4 Simulation Results with Delay and with Controller

In this section, the simulation results of the controller based on the enhanced Lyapunov function under different time delay conditions (0.5, 1, 10, and 20 seconds) are presented.



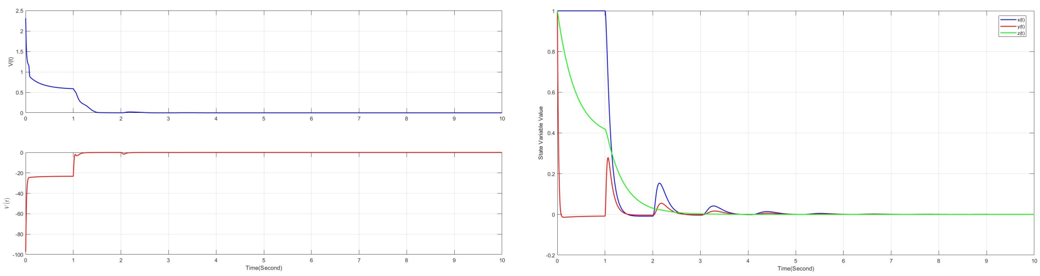
(a) State variable value **(b)** Phase space diagram

Figure 2: Lorenz system state without control (with delay $\tau = 0.2$ seconds).



(a) Lyapunov function $(V(t), \dot{V}(t))$ **(b)** State variable value

Figure 3: Lorenz system state with control (with delay $\tau = 0.5$ seconds).

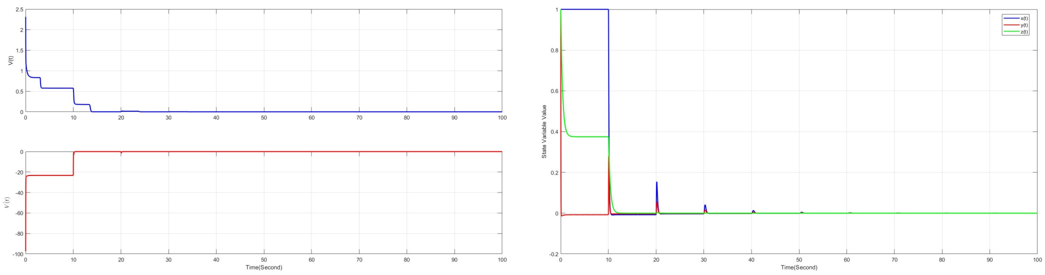


(a) Lyapunov function $(V(t), \dot{V}(t))$ **(b)** State variable value

Figure 4: Lorenz system state with control (with delay $\tau = 1.0$ seconds).

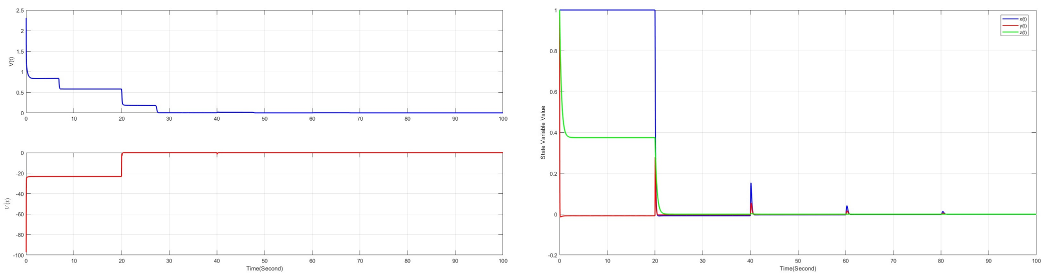
The simulation results for delays $\tau = 0.5$ (Figure 3), $\tau = 1$ (Figure 4), $\tau = 10$ (Figure 5), and $\tau = 20$ (Figure 6) indicate the effective performance of the control law in managing the system dynamics and ensuring practical stability. From these simulations, the following observations can be made:

1. In the phase diagrams, it is observed that the state variables $x(t)$, $y(t)$, and $z(t)$ converge to the equilibrium point after a certain period. This convergence indicates the success of the controller in reducing oscillations and stabilizing the system dynamics. Furthermore, the Lyapunov function $V(t)$ uniformly decreases and approaches zero over time, signifying a reduction in the system's energy and fulfilling the expectations of an effectively controlled system.



(a) Lyapunov function ($V(t)$, $\dot{V}(t)$) **(b)** State variable value

Figure 5: Lorenz system state with Control (with delay $\tau = 10.0$ seconds).



(a) Lyapunov function ($V(t)$, $\dot{V}(t)$) **(b)** State variable value

Figure 6: Lorenz system state with control (with delay $\tau = 20.0$ seconds).

2. The time derivative of the Lyapunov function, $\dot{V}(t)$, is observed to be nonpositive at all times. This characteristic, in line with the principles of the Lyapunov–Krasovskii theorem for practical stability, confirms that the system under the hybrid control is Practical stable.
3. For a small delay such as $\tau = 0.5$ seconds, $\dot{V}(t)$ quickly tends to stable negative values.
4. For larger delays such as $\tau = 10$ and $\tau = 20$ seconds, despite initial oscillations, $\dot{V}(t)$ eventually converges to stable negative values. This behavior demonstrates the significant capability of the controller in managing severe delays.
5. When the delay is small ($\tau = 0.5$ seconds), the system rapidly reaches equilibrium, and the oscillations in the state variables decay exponentially.
6. As the delay τ increases, the system enters an oscillatory phase; however, due to the effective combination of the integral and trigonometric terms in the control law, the oscillations are suppressed and the stability of the system is maintained.
7. Under large delays ($\tau = 10$ and $\tau = 20$ seconds), although the system initially exhibits severe oscillations, the controller is able to neutralize the effect of the delay through the effective combination of the integral and trigonometric terms, leading to a gradual decrease in the Lyapunov function $V(t)$. This indicates the high robustness of the proposed method against large delays.

To summarize, the examination of the simulation results authorizes that the new control law not only guarantees the convergence of the state variables, but also, through the uniform decrease of the Lyapunov function and the nonpositivity of its derivative, fully establishes the practical stability of the system in accordance with the second condition of the Lyapunov–Krasovskii theorem. This scientific achievement establishes the proposed method as an efficient and robust solution against severe time delays in the control of nonlinear dynamical systems.

Sensitivity Analysis of Controller Gains

To evaluate the controller's robustness to variations in its key gains, a sensitivity analysis was conducted on the parameters k, λ, and μ. The results reveal that the linear damping gain, k, primarily affects the convergence speed; lower values result in slower or more oscillatory responses, whereas excessively high values can produce an overly aggressive control signal. The delay compensation gain, λ, plays a critical role in ensuring system stability, particularly for large delays τ. The system shows significant sensitivity to this parameter, with stability being compromised if λ falls below a delay-dependent threshold. The nonlinear damping gain, μ, presents a key trade-off: while increasing μ effectively suppresses large-amplitude chaotic oscillations during the transient phase, it simultaneously enlarges the ultimate bound region, potentially increasing the steady-state error. This analysis underscores the complex interaction between the gains and confirms that the parameter set derived via the genetic algorithm provides a well-balanced compromise between stability, convergence speed, and final tracking accuracy.

Comparative Discussion

To contextualize the contributions of this work, a comparative analysis against established control methodologies is offered in Table 1. The proposed LKF-based controller carves out a compelling niche by balancing analytical rigor, performance, and application feasibility. Unlike Sliding Mode Control (SMC), our method generates a smooth, chattering-free control signal, which is critical for preventing mechanical wear in physical actuators [7, 34]. While Model Predictive Control (MPC) also produces smooth signals, it suffers from a very high online computational burden, especially for systems with large delays, making it impractical for many real-time applications [22, 50]. Our controller, in contrast, requires only the evaluation of a simple algebraic expression, ensuring computational efficiency. Furthermore, while Adaptive Intelligent Control (AIC) using neural or fuzzy systems offers excellent robustness to unmodeled dynamics [20, 56], it often involves complex tuning of network architectures and learning rates, and its stability guarantees are contingent on the boundedness of approximation errors. The method developed herein provides a strong, formal guarantee of Uniform Ultimate Boundedness (UUB) based on a precise analytical model, representing a powerful and practical substitute for systems where such a model is available.

5 Conclusion and Future Work

This work proposes a robust and adaptive control framework for chaotic systems subject to large, constant time delays. The approach is built upon a specially constructed Lyapunov–Krasovskii functional

Table 1: Analytical comparison of control methodologies for chaotic systems with time delay.

Feature / Method	Proposed Method (LKF-based)	Sliding Mode Control (SMC)	Model Predictive Control (MPC)	Adaptive Intelligent Control (NN/Fuzzy)
Model Dependency	**High.** Needs accurate model for LKF design (this study).	**Medium.** Needs model for surface; robust to uncertainty [12, 15, 58].	**Very High.** Critically needs accurate model; mismatch hurts [43, 50, 54].	**Low-Medium.** Needs general structure; learns unknown dynamics [2, 20, 56].
Large Delay Management	**Excellent.** Designed for large delays via LKF; tested for $\tau = 20s$ (this study).	**Good (with mods).** Standard SMC must be augmented with LKF for delays [18, 47, 51].	**Medium-Poor.** Large delays make problem computationally intractable [22, 42].	**Good.** Often integrated with LKF to handle unknown time-delays [19, 56, 40].
Handling of Strong Nonlinearity	**Excellent.** Employs direct cancellation for Lorenz system (this study).	**Excellent.** Overcomes nonlinearities via high-gain discontinuous control [15, 58].	**Good (high cost).** Handles nonlinearities but requires expensive NLP [1, 22, 35].	**Excellent.** Core strength is universal approximation property [2, 24, 59].
Chattering Suppression	**Excellent.** Continuous, smooth control law prevents chattering (this study).	**Poor (Inherent).** Discontinuous control is primary cause of chattering [7, 34, 36, 55].	**Excellent.** Smooth optimization ensures a chattering-free output [43].	**Excellent.** Continuous output; fuzzy logic often used to stop chattering [13, 28].
Formal Stability Guarantee	**Very Strong.** Rigorous mathematical proof of UUB via LKF (this study).	**Very Strong.** Guarantees finite-time convergence to stable surface [58].	**Strong (with assumptions).** Not inherent; needs terminal cost/constraints [43, 50].	**Strong (UUB).** Lyapunov proof, but contingent on bounded approx. error [20, 40, 56].
Computational Cost (Online)	**Very Low.** Requires only evaluation of a simple algebraic expression (this study).	**Low.** Involves calculating surface and a simple switching function [58].	**Very High.** Requires solving a constrained optimization problem each step [50, 54].	**Medium.** Network forward pass and adaptive updates; >SMC, <MPC [20, 33].
Implementation Simplicity	**Medium.** Complex law, simple implementation; challenge is gain tuning (this study).	**High.** Simple concept; primary challenge is chattering management [12, 58].	**Low.** Needs sophisticated solvers and complex tuning of many parameters [50, 54].	**Low-Medium.** Challenged by network/fuzzy architecture design and tuning [3, 33].

(LKF) that combines an integral term to compensate for delayed state effects with a trigonometric term that moderates nonlinear oscillations. Stability analysis demonstrates that the resulting controller guarantees Uniform Ultimate Boundedness (UUB) of the closed-loop system. Extensive simulations on the delayed Lorenz system confirm rapid convergence to a small neighborhood of the equilibrium, even for delays up to 20 seconds, and empirically support the theoretical findings through the non-positive time derivative of the LKF. Overall, the method provides a computationally efficient and low-complexity alternative to existing delay-compensation techniques such as Model Predictive Control and predictive feedback, while offering strong stability guarantees. Several avenues for further research arise from this study:

1. *Extension to time-varying delays:* Developing an LKF-based framework capable of handling unknown or time-varying delays to broaden applicability.
2. *Adaptive delay estimation:* Incorporating real-time delay estimation mechanisms to enhance performance under uncertain or drifting delays.
3. *Broader nonlinear applications:* Applying the method to higher-dimensional, networked, or more complex nonlinear chaotic systems.
4. *Experimental validation:* Implementing the controller on physical systems to further demonstrate practicality and robustness.
5. *Integration with learning-based strategies:* Exploring hybrid approaches that combine the proposed framework with data-driven or learning-augmented techniques for improved adaptability under modeling uncertainties.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

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Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Author Contributions

All authors contributed equally to the design of the study, data analysis, and writing of the manuscript, and share equal responsibility for the content of the paper.

Artificial Intelligence Statement

Artificial intelligence (AI) tools, including large language models, were used solely for language editing and improving readability. AI tools were not used for generating ideas, performing analyses, interpreting results, or writing the scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

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