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Generalized (m,n)-Fuzzy BL-Subalgebras: Algebraic Foundations, Power-Implication Structures

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Abstract. This paper offers the idea of (anti) (m,n)-fuzzy BL-subalgebras as a novel extension of classical BL-algebras within the fuzzy mathematical framework. The proposed structures generalize various types of fuzzy subalgebras, including (anti) intuitionistic, (anti) Pythagorean, (anti) Fermatean, and (anti) q-rung orthopair fuzzy BL-subalgebras for $q \geq 1$. Fundamental algebraic properties and equivalent characterizations of (m,n)-fuzzy BL-subalgebras are established through the notion of value-cuts. Furthermore, the concept of power-implication preserving (PIP) BL-algebras is introduced, and it is shown that a PIP BL-algebra exists for every prime number. Several closure properties of (m,n)-fuzzy BL-subalgebras under combination operations are also derived within this framework. From an applied perspective, the developed theoretical results can serve as a mathematical foundation for modeling and reasoning in fuzzy control systems and optimization processes, particularly in decision-making environments characterized by uncertainty and graded information.

Keywords. Fuzzy optimization, Fuzzy logic, PIP BL-algebra, (m,n)-fuzzy BL-subalgebra, (m,n)-fuzzy nil radical.

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1 Introduction

Fuzzy logic is a mathematical approach developed to model reasoning in contexts where information is imprecise, uncertain, or inherently vague, with its origins tracing back to the 1960s. In contrast to classical Boolean logic, which operates on strict true/false evaluations, fuzzy logic accommodates intermediate truth values ranging continuously between 0 and 1. This framework serves as an effective link between human intuitive reasoning and computational precision, enabling systems to operate reliably even when faced with ambiguity.

A fuzzy subset (FS), introduced by Zadeh in 1965 [15], extends the notion of a classical (crisp) subset by assigning to each element a degree of membership, rather than restricting membership to a binary "in or out" classification.

A fuzzy subset extends classical set theory by allowing partial membership, enabling systems to handle ambiguity and gradation. It underpins fuzzy logic, which powers applications, proving essential in scenarios where precision is less critical than adaptability. A t-norm is a mathematical function used in fuzzy logic to extend the principles of classical logic AND operation. It operates on values within the interval $[0, 1]$ and is fundamental in combining degrees of truth in systems dealing with uncertainty or partial information. BL-algebras were introduced by Czech mathematician Petr Hájek in the scope of fuzzy Logic in 1998. Hájek developed BL-algebras (where BL stands for Basic Logic) as the algebraic counterpart to his axiomatic system for fuzzy logic, unifying earlier structures like MV-algebras, G$\ddot{o}$del algebras, and product algebras under a single framework. This work formalized the connection between t-norm-based fuzzy logics and their algebraic semantics, marking a foundational milestone in non-classical logic research. Fuzzy subsets are the semantic basis for fuzzy logic, which uses degrees of truth to reason under uncertainty. The updated works in the scope of BL-algebras is cited to [1, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 16].

Classical sets, despite their simplicity and mathematical beauty, are inadequate for modeling real phenomena. Labeled sets, and especially fuzzy sets, remove these shortcomings and provide powerful tools for modeling real complex systems. Based on the fact that most real-world problems are imprecise, the application of fuzzy theory is an important tool in solving mathematical problems based on this modeling. Therefore, the combination of the two theories of regular sets and fuzzy theory is a fundamental tool in solving current mathematical problems, and in this study, we will discuss the combination of BL-subalgebras as regular sets and fuzzy theories. Regarding these points, we orient the opinion of (m, n)-fuzzy BL-subalgebras and investigate their properties. The (m, n)-fuzzy BL-subalgebras provide a parameterized framework to generalize fuzzy BL-algebraic structures, enhance their applicability to complex problems, and deepen theoretical understanding of algebraic robustness under fuzziness. This innovation aligns with broader goals in fuzzy mathematics to refine tools for handling uncertainty in computational and logical systems. One of the advantages and motivations of this work is to BL-algebras model the semantics of fuzzy logic. By adjusting m and n, one can explore variations in logical axioms (e.g., implications or t-norms), supporting the study of alternative fuzzy logic or their algebraic counterparts. Studying how m and n affect subalgebra properties reveals insights into the sensitivity and stability of BL-algebras under fuzzy operations. This analysis is critical for ensuring reliability in systems where inputs are approximate or imprecise. In applications like decision-making, automated reasoning, or fuzzy control, uncertainties often vary in nature. The parameters m and n allow tailoring the algebraic structure to specific contexts, accommodating noise, partial truth, or multi-source vagueness. By incorporating parameters m and n, the definition of fuzzy BL-subalgebras becomes more versatile. These parameters adjust the strictness or leniency of membership conditions, allowing the framework to generalize standard fuzzy BL-subalgebras. Indeed, the motivation for introducing (m, n)-fuzzy BL-subalgebras stems from the need to enhance the flexibility and applicability of fuzzy algebraic structures in modeling real-world problems with varying degrees of uncertainty.

In the aim of this study, we need to present the notion of power-implication preserving BL-algebras (PIP BL-algebras) as a popularization of BL-algebras. This idea makes a novel notation as K-divisible implicative) BL-algebra for any given $k \in \mathbb{N}$. The power-implication preserving BL-algebras extends traditional fuzzy logic by formalizing how scaled implications retain algebraic coherence. They address both theoretical questions (e.g., stability of BL-axioms under parameterized operations) and practical needs (e.g., adaptive systems requiring tunable logical rules). By integrating power operations into the algebraic framework, PIP BL-algebras enhance the versatility of fuzzy mathematics in modeling complex, dynamic real-world phenomena. In this regard, we extend the (m, n)-fuzzy BL-subalgebras by some binary operations and binary relations especially the combination of (m, n)-fuzzy BL-subalgebras. In final, this study introduces the nil radical (m, n)-fuzzy BL-subalgebras and considers the equal conditions for the homomorphic image of nil radical (m, n)-fuzzy BL-subalgebras and nil radical homomorphism image of (m, n)-fuzzy BL-subalgebras, via the algebraic tools such as homomorphisms.

Our investigation yields several key original contributions that distinguish this work from existing literature:

1. Unified fuzzy framework: The proposed (anti) (m, n)-fuzzy BL-subalgebra offers a versatile and unified definition, consolidating various fuzzy subalgebra types such as (anti) intuitionistic fuzzy, Pythagorean fuzzy, Fermatean fuzzy, and q-rung ortho-pair fuzzy BL-subalgebras (with $q \geq 1$) into one cohesive algebraic structure.

2. PIP BL-algebra innovation: This work introduces the concept of power-implication preserving (PIP) BL-algebras, marking a notable advancement in the domain. By formalizing the preservation of scaled implications ($x^m \to y^n$), it ensures algebraic consistency. A particularly noteworthy achievement is the demonstration that a corresponding PIP

BL-algebra exists for every prime number, offering a fascinating intersection between number theory and fuzzy algebra.

3. Operational closure under combination: A critical discovery lies in the closure property of union and intersection combinations within (anti) (m, n)-fuzzy BL-subalgebras in the context of PIP BL-algebras. Unlike traditional BL-algebras, this framework ensures the structural integrity of combined subalgebras, highlighting its robustness and resilience.

4. Characterization through level cuts: By establishing equivalencies between algebraic properties and corresponding level subsets ((a, b)-cuts), the study bridges the gap between fuzzy theory and crisp algebraic structures. This integration provides deeper conceptual clarity and strengthens the connection between set-theoretic and algebraic approaches.

5. Introduction of nil radical fuzzy subalgebras: The theoretical framework is further expanded with the inclusion of nil radical (m, n)-fuzzy BL-subalgebras. Detailed analysis clarifies their behavior under homomorphisms, with precise conditions ensuring that the homomorphic image of a nil radical fuzzy subalgebra aligns with the nil radical of its original homomorphic image.

This research significantly enriches the mathematical foundation of fuzzy algebra while offering practical methodologies for modeling intricate systems involving flexible logical operations and multi-valued uncertainties. By addressing essential questions about the stability of BL-axioms under parameterized functions, it opens new avenues for designing adaptive systems grounded in fuzzy logic.

2 Notations and Preliminaries

This section develops the preliminary concepts required for our research.

Definition 1. [15] Let X be a nonempty set. A fuzzy set (FS) on X is a set of ordered pairs

$$A = \{(x, \mu(x)) \mid x \in X\},$$

where $\mu : X \to [0, 1]$ is the membership function assigning to each $x \in X$ a degree of membership $\mu(x)$ in $[0, 1]$.

Definition 2. [2] Let $m, n \in \mathbb{N}$. An (m, n)-fuzzy set, denoted E, over the universal set X is given by

$$E = \{(x, \mu_E(x), \nu_E(x)) \mid x \in X\},$$

where $\mu_E, \nu_E : X \to [0,1]$ assign to each $x \in X$ its degree of membership and non-membership, respectively, subject to the constraint

$$0 \leq \mu_E^m(x) + \nu_E^n(x) \leq 1 \quad \text{for all } x \in X.$$

If $q \geq 1$ and $m = n = q$, then E is called a q-rung orthopair fuzzy set (or q-rung orthopair-FS).

Definition 3. [6] A $(X, \curlywedge, \curlyvee, *, \to, 0, 1)$ is called a BL-algebra, if for all $x, y, z \in X$:

$(BL1)$ $(X, \curlywedge, \curlyvee, 0, 1)$ is a bounded lattice,

$(BL2)$ $(X, *, 1)$ is a commutative monoid,

$(BL3)$ $x * y \leq z \Leftrightarrow x \leq y \to z$,

$(BL4)$ $x \curlywedge y = x * (x \to y)$,

$(BL5)$ $(x \to y) \curlyvee (y \to x) = 1$.

Theorem 1. [6] Let $(X, \curlywedge, \curlyvee, *, \to, 0, 1)$ be a BL-algebra and $x, y, z \in X$. Then

(i) $x * y \leq x \curlywedge y$.

(ii) $x \leq y \Leftrightarrow x \to y = 1$.

(iii) If $x \leq y$, then $z \to x \leq z \to y, y \to z \leq x \to z$ and $x * z \leq y * z$.

(iv) $x \to (y \to z) = (x * y) \to z = y \to (x \to z)$.

(v) $1 \to x = x, x \to x = 1$ and $x \to 1 = 1$.

(vi) $x \curlyvee y = ((x \to y) \to y) \curlywedge ((y \to x) \to x)$.

(vii) $x * (y \curlyvee z) = (x * y) \curlyvee (x * z)$.

3 On (m, n)-Fuzzy BL-Subalgebras

In this section, we orient the notation for (m, n)-fuzzy BL-subalgebras and their anti-counterparts, and we investigate their properties. We extend the notions of (anti) (m, n)-fuzzy BL-subalgebras by building on fuzzy BL-subalgebras and on the complements of fuzzy BL-subalgebras. In particular, the complement plays a key role in generating (anti) (m, n)-fuzzy BL-subalgebras. We also generalize (m, n)-fuzzy BL-subalgebras to intersections, unions, and nested radicals of the form (m, n)-fuzzy BL-subalgebras.

From here on, we view $(X, \curlywedge, \curlyvee, *, \to, 0, 1)$ as a BL-algebra, with X denoting the underlying set. We then derive the following consequences. Let $A = \{(x, \mu(x)) \mid x \in X\}$ denote a FS on X. For all $k \in \mathbb{N}$, $x \in X$, and any map $\mu : X \to [0, 1]$, define

$$\mu^k(x) = (\mu(x))^k, \qquad E^{(k,l)} = \{(x, \mu_E^k(x), \nu_E^l(x)) \mid x \in X\}.$$

Clearly, $E^{(1,1)} = E$. For all $k \geq l \geq 2$, we have $\mu^k \subseteq \mu^l$. Moreover, if $k \leq k'$ and $l \leq l'$, and $E^{(l,k)}$ is an (m, n)-FS of X, then $E^{(k',l')}$ is also an (m, n)-FS of X.

Definition 4. Let $E = \{(x, \mu_E(x), \nu_E(x)) \mid x \in X\}$ be an (m, n)-FS on X. Then

(i) E is an (m, n)-fuzzy BL-subalgebra of X if, for all $x, y \in X$,

$$\mu_E^m(x * y) \geq \min\{\mu_E^m(x), \mu_E^m(y)\}, \quad \mu_E^m(x \to y) \geq \min\{\mu_E^m(x), \mu_E^m(y)\},$$

$$\nu_E^n(x * y) \leq \max\{\nu_E^n(x), \nu_E^n(y)\}, \quad \nu_E^n(x \to y) \leq \max\{\nu_E^n(x), \nu_E^n(y)\}.$$

(ii) E is an anti (m, n)-fuzzy BL-subalgebra of X if, for all $x, y \in X$,

$$\mu_E^m(x * y) \leq \max\{\mu_E^m(x), \mu_E^m(y)\}, \quad \mu_E^m(x \to y) \leq \max\{\mu_E^m(x), \mu_E^m(y)\},$$

$$\nu_E^n(x * y) \geq \min\{\nu_E^n(x), \nu_E^n(y)\}, \quad \nu_E^n(x \to y) \geq \min\{\nu_E^n(x), \nu_E^n(y)\}.$$

An FS case: If $E = \{(x, \mu_E(x)) \mid x \in X\}$ is a fuzzy set on X, then

$$E \text{ is a fuzzy BL-subalgebra of } X \iff \mu_E(x * y) \geq \min\{\mu_E(x), \mu_E(y)\}, \quad \mu_E(x \to y) \geq \min\{\mu_E(x), \mu_E(y)\},$$

and E is an anti fuzzy BL-subalgebra of X if

$$\mu_E(x * y) \leq \max\{\mu_E(x), \mu_E(y)\}, \quad \mu_E(x \to y) \leq \max\{\mu_E(x), \mu_E(y)\}.$$

Let X be a BL-algebra and $\emptyset \neq Y \subseteq X$. We discuss that $(Y, \curlywedge, \curlyvee, *, \to, 0, 1)$ is a subalgebra of X, if for all $x, y \in Y$, $x * y \in Y$ and $x \to y \in Y$ and will denote it by $Y \leq_{sub} X$.

Example 1. Let X be a BL-algebra and $Y \leq_{\text{sub}} X$. Define the characteristic function

$$\chi_Y(t) = \begin{cases} 1, & t \in Y, \\ 0, & t \notin Y. \end{cases}$$

For every $m \in \mathbb{N}$ and $t \in X$, set $\chi_Y^m(t) = (\chi_Y(t))^m$.

Since Y is a subalgebra, for all $x, y \in Y$, we have $x * y \in Y$. Hence

$$\chi_Y^m(x * y) = \chi_Y(x * y) = 1 \geq \chi_Y(x) \curlywedge \chi_Y(y) = \chi_Y^m(x) \curlywedge \chi_Y^m(y).$$

If $x * y \notin Y$, then either $x \notin Y$ or $y \notin Y$, which gives

$$\chi_Y^m(x*y) = \chi_Y(x*y) = 0 \leq \chi_Y(x) \curlywedge \chi_Y(y) = \chi_Y^m(x) \curlywedge \chi_Y^m(y).$$

Thus, for all $m, n \in \mathbb{N}$,

$$\chi_Y^m(x*y) \geq \chi_Y^m(x) \curlywedge \chi_Y^m(y),$$

and

$$(\chi_Y^c(x*y))^n = 1 - \chi_Y^n(x*y) \leq \chi_Y^c(x) \curlyvee \chi_Y^c(y) = (\chi_Y^c(x))^n \curlyvee (\chi_Y^c(y))^n,$$

where $\chi_Y^c = 1 - \chi_Y$.

Consequently,

$$\chi_Y^m(x*y) \geq \chi_Y^m(x) \curlywedge \chi_Y^m(y), \qquad (\chi_Y^c(x*y))^n \leq (\chi_Y^c(x))^n \curlyvee (\chi_Y^c(y))^n.$$

Hence, the set

$$E = \{(x, \chi_Y(x), \chi_Y^c(x)) \mid x \in X\},$$

is an (m, n)-fuzzy BL-subalgebra of X.

In what follows, we present a (2,2)-fuzzy BL-subalgebra of an infinite BL-algebra.

Example 2. Let $X = ([0,1], \curlywedge, \curlyvee, *, \rightarrow, 0, 1)$ (the Gödel algebra), which is uncountably infinite. Define

$$E = \{(x, \mu_E(x), \nu_E(x)) \mid x \in X\},$$

with

$$\mu_E(x) = \begin{cases} 0.9, & x = 1, \\ 0.7, & x \in [0.6, 1), \\ 0.5, & x \in [0.3, 0.6), \\ 0.2, & x \in [0, 0.3), \end{cases} \qquad \nu_E(x) = \begin{cases} 0.1, & x = 1, \\ 0.2, & x \in [0.6, 1), \\ 0.3, & x \in [0.3, 0.6), \\ 0.6, & x \in [0, 0.3). \end{cases}$$

Clearly, $\mu_E^2(x)$, $\nu_E^2(x) \leq 1$ for all $x \in X$, so E is a (2,2)-FS.

Since $x * y = \min(x, y)$, we have $\mu_E(\min(x, y))$ equal to $\mu_E(x)$ or $\mu_E(y)$. Hence

$$\mu_E^2(x*y) = \mu_E^2(\min(x,y)) \geq \min\{\mu_E^2(x), \mu_E^2(y)\}.$$

Similarly,

$$\nu_E^2(x*y) \leq \max\{\nu_E^2(x), \nu_E^2(y)\}.$$

If $x \leq y$, then $x \rightarrow y = 1$ and $\mu_E(1) = 0.9$ is the maximum, so

$$\mu_E^2(x \rightarrow y) = 0.9^2 = 0.81 \geq \min\{\mu_E^2(x), \mu_E^2(y)\}.$$

If $x > y$, then $x \to y = y$ and thus

$$\mu_E^2(x \to y) = \mu_E^2(y) \geq \min\{\mu_E^2(x), \mu_E^2(y)\}.$$

For the non-membership part, if $x \leq y$ then $x \to y = 1$ and $\nu_E(1) = 0.1$ is the minimum, so

$$\nu_E^2(x \to y) = 0.1^2 = 0.01 \leq \max\{\nu_E^2(x), \nu_E^2(y)\}.$$

If $x > y$ then $x \to y = y$ and

$$\nu_E^2(x \to y) = \nu_E^2(y) \leq \max\{\nu_E^2(x), \nu_E^2(y)\}.$$

Thus, the required inequalities hold in all cases, and E is a (2,2)-fuzzy BL-subalgebra of X.

Remark 1. An (m, n)-FS on X specializes to several well-known FS frameworks:

(i) Intuitionistic-FS on X when $m = n = 1$.

(ii) Pythagorean-FS on X when $m = n = 2$.

(iii) Fermatean-FS on X when $m = n = 3$.

(iv) q-rung orthopair-FS on X when $m = n = q \geq 1$.

Remark 2. An anti (m, n)-FS on X specializes to several well-known anti-FS frameworks:

(i) Anti-Intuitionistic-FS on X when $m = n = 1$.

(ii) Anti-Pythagorean-FS on X when $m = n = 2$.

(iii) Anti-Fermatean-FS on X when $m = n = 3$.

(iv) Anti-q-rung orthopair-FS on X when $m = n = q \geq 1$.

From now on, we denote $E = \{(x, \mu_E(x), \nu_E(x)) \mid x \in X\}$ simply by E for brevity.

Proposition 1. Let E be an (m, n)-fuzzy BL-subalgebra of X and let $x, y \in X$. Then

(i) $\mu_E(1) \geq \mu_E^m(1) \geq \mu_E^m(x)$ and $\nu_E^n(1) \leq \nu_E^n(x) \leq \nu_E(x)$.

(ii) $\mu_E(x \curlywedge y) \geq \mu_E^m(x \curlywedge y) \geq \mu_E^m(x) \curlywedge \mu_E^m(y)$ and

$$\nu_E^n(x \curlywedge y) \leq \nu_E^n(x) \curlyvee \nu_E^n(y) \leq \nu_E(x) \curlyvee \nu_E(y).$$

(iii) $\mu_E(x \curlyvee y) \geq \mu_E^m(x \curlyvee y) \geq \mu_E^m(x) \curlywedge \mu_E^m(y)$ and

$$\nu_E^n(x \curlyvee y) \leq \nu_E^n(x) \curlyvee \nu_E^n(y) \leq \nu_E(x) \curlyvee \nu_E(y).$$

Proof. Let $x, y \in X$. By Theorem 1, we obtain

$$\begin{aligned}
\mu_E(1) &\geq \mu_E^m(1) = \mu_E^m(x \to x) \geq \mu_E^m(x) \curlywedge \mu_E^m(x),\\
\nu_E^n(1) &= \nu_E^n(x \to x) \leq \nu_E^n(x) \curlyvee \nu_E^n(x) = \nu_E^n(x) \leq \nu_E(x),\\
\mu_E(x \curlywedge y) &\geq \mu_E^m(x \curlywedge y) = \mu_E^m\big(x * (x \to y)\big)\\
&\geq \mu_E^m(x) \curlywedge \mu_E^m(x \to y) \geq \mu_E^m(x) \curlywedge \mu_E^m(y),\\
\nu_E^n(x \curlywedge y) &= \nu_E^n\big(x * (x \to y)\big) \leq \nu_E^n(x) \curlyvee \nu_E^n(x \to y) \leq \nu_E^n(x) \curlyvee \nu_E^n(y),\\
\mu_E(x \curlyvee y) &\geq \mu_E^m(x \curlyvee y) = \mu_E^m\big(((x \to y) \to y) \curlywedge ((y \to x) \to x)\big)\\
&\geq \mu_E^m(x) \curlywedge \mu_E^m(y),\\
\nu_E^n(x \curlyvee y) &= \nu_E^n\big(((x \to y) \to y) \curlywedge ((y \to x) \to x)\big)\\
&\leq \nu_E^n((x \to y) \to y) \curlyvee \nu_E^n((y \to x) \to x)\\
&\leq \nu_E^n(x) \curlyvee \nu_E^n(y) \leq \nu_E(x) \curlyvee \nu_E(y).
\end{aligned}$$

□

Let E and $F = \{(x, \mu_F(x), \nu_F(x)) \mid x \in X\}$ be (m, n)-fuzzy BL-subalgebras of X. We say $E \subseteq F$ if, for all $x \in X$,

$$\mu_E(x) \leq \mu_F(x) \quad \text{and} \quad \nu_E(x) \geq \nu_F(x).$$

Moreover, for all $n \geq n'$ and all $m \leq m'$, we have

$$E^{(n,n')} \subseteq E^{(n',n)} \quad \text{and} \quad E^{(m',n)} \subseteq E^{(m,n')}.$$

Define the intersection and union of E and F by

$$E \cap F = \{(x, \min\{\mu_E(x), \mu_F(x)\}, \max\{\nu_E(x), \nu_F(x)\}) \mid x \in X\},$$

$$E \cup F = \{(x, \max\{\mu_E(x), \mu_F(x)\}, \min\{\nu_E(x), \nu_F(x)\}) \mid x \in X\}.$$

Clearly, both $E \cap F$ and $E \cup F$ are (m, n)-fuzzy BL-subalgebras of X.

Theorem 2. Let A be a fuzzy BL-subalgebra of X. Then

$$[A, \tfrac{1}{m}, \tfrac{1}{n}] \;=\; \{(x,\ \sqrt[m]{\mu(x)},\ \sqrt[n]{\mu^c(x)}) \mid x \in X\}$$

is an (m, n)-fuzzy BL-subalgebra of X.

Proof. Let $x \in X$. Then

$$(\sqrt[m]{\mu(x)})^m + (\sqrt[n]{\mu^c(x)})^n = \mu(x) + 1 - \mu(x) = 1,$$

so $[A, \frac{1}{m}, \frac{1}{n}]$ is an (m, n)-fuzzy subset of X.

Assume now that A is a fuzzy BL-subalgebra of X. We verify the two main conditions.

$$(\sqrt[m]{\mu(x \to y)})^m = \mu(x \to y) \geq \mu(x) \curlywedge \mu(y) = (\sqrt[m]{\mu(x)})^m \curlywedge (\sqrt[m]{\mu(y)})^m.$$

Also,

$$\begin{aligned}(\sqrt[n]{\mu^c(x \to y)})^n &= 1 - \mu(x \to y) \leq 1 - (\mu(x) \curlywedge \mu(y)) \leq (1 - \mu(x)) \curlyvee (1 - \mu(y)) \\ &= \mu^c(x) \curlyvee \mu^c(y) = (\sqrt[n]{\mu^c(x)})^n \curlyvee (\sqrt[n]{\mu^c(y)})^n.\end{aligned}$$

Similarly,

$$(\sqrt[m]{\mu(x * y)})^m \geq (\sqrt[m]{\mu(x)})^m \curlywedge (\sqrt[m]{\mu(y)})^m$$

and

$$(\sqrt[n]{\mu^c(x * y)})^n \leq (\sqrt[n]{\mu^c(x)})^n \curlyvee (\sqrt[n]{\mu^c(y)})^n.$$

Thus, all defining conditions hold, and $[A, \frac{1}{m}, \frac{1}{n}]$ is an (m,n)-fuzzy BL-subalgebra of X. □

Corollary 1. Let A be a fuzzy BL-subalgebra of X. Then

$$[[A, \tfrac{1}{m}, \tfrac{1}{n}]] = \{(x, \sqrt[m]{\mu^c(x)}, \sqrt[n]{\mu(x)}) \mid x \in X\},$$

is an anti-(m,n)-fuzzy BL-subalgebra of X.

Theorem 3. Let A be a fuzzy BL-subalgebra of X. Then

$$\langle A, \tfrac{1}{m}, \tfrac{1}{n} \rangle = \{(x, \sqrt[m]{\nu^c(x)}, \sqrt[n]{\nu(x)}) \mid x \in X\}$$

is an (m,n)-fuzzy BL-subalgebra of X.

Proof. Let $x \in X$. Then

$$(\sqrt[m]{\nu(x)})^m + (\sqrt[n]{\nu^c(x)})^n = \nu(x) + 1 - \nu(x) = 1,$$

so

$$\langle A, \tfrac{1}{m}, \tfrac{1}{n} \rangle = \{(x, \sqrt[m]{\nu^c(x)}, \sqrt[n]{\nu(x)}) \mid x \in X\},$$

is an (m,n)-fuzzy subset of X.

Now suppose that A is a fuzzy BL-subalgebra of X. We have

$$(\sqrt[m]{\nu^c(x \to y)})^m = \nu^c(x \to y) \geq \nu(x) \curlywedge \nu(y) = (\sqrt[m]{\nu^c(x)})^m \curlywedge (\sqrt[m]{\nu^c(y)})^m,$$

and

$$(\sqrt[n]{\nu(x \to y)})^n = \nu(x \to y) \leq \nu(x) \curlyvee \nu(y) = (\sqrt[n]{\nu^c(x)})^n \curlyvee (\sqrt[n]{\nu^c(y)})^n.$$

Similarly,

$$(\sqrt[m]{\nu^c(x*y)})^m \ \geq \ (\sqrt[m]{\nu^c(x)})^m \curlywedge (\sqrt[m]{\nu^c(y)})^m,$$

and

$$(\sqrt[n]{\nu(x*y)})^n \ \leq \ (\sqrt[n]{\nu(x)})^n \curlyvee (\sqrt[n]{\nu(y)})^n.$$

Hence,

$$\langle A, \tfrac{1}{m}, \tfrac{1}{n}\rangle \ = \ \{(x, \ \sqrt[m]{\nu^c(x)}, \ \sqrt[n]{\nu(x)}) \mid x \in X\},$$

is an (m,n)-fuzzy BL-subalgebra of X. □

Corollary 2. Let A be a fuzzy BL-subalgebra of X. Then

$$\langle\langle A, \tfrac{1}{m}, \tfrac{1}{n}\rangle\rangle = \{(x, \ \sqrt[m]{\nu(x)}, \ \sqrt[n]{\nu^c(x)}) \mid x \in X\},$$

is an anti(m,n)-fuzzy BL-subalgebra of X.

Let E be an (m,n)-FS of X and let $x \in X$. Define

$$E^c \ = \ \{(x, \ \nu_E^{\frac{n}{m}}(x), \ \mu_E^{\frac{m}{n}}(x)) \mid x \in X\}.$$

Theorem 4. Let E be an (m,n)-fuzzy BL-subalgebra of X. Then E^c is an anti (m,n)-fuzzy BL-subalgebra of X.

Proof. Let $x,y \in X$. Then $(\nu_E^{\frac{n}{m}}(x))^m + (\mu_E^{\frac{m}{n}}(x))^n = \mu_E^m(x) + \nu_E^n(x) \leq 1$ and so E^c is an (m,n)-FS of X. In addition,

$$\begin{aligned}
(\nu_E^{\frac{n}{m}}(x \to y))^m &= (\nu_E(x \to y))^n \leq \max\{\nu_E^n(x), \nu_E^n(y)\} = \max\{(\nu_E^{\frac{n}{m}}(x))^m, (\nu_E^{\frac{n}{m}}(y))^m\},\\
(\nu_E^{\frac{n}{m}}(x * y))^m &= (\nu_E(x * y))^n \leq \max\{\nu_E^n(x), \nu_E^n(y)\} = \max\{(\nu_E^{\frac{n}{m}}(x))^m, (\nu_E^{\frac{n}{m}}(y))^m\},\\
(\mu_E^{\frac{m}{n}}(x \to y))^n &= (\mu_E(x \to y))^m \geq \min\{\mu_E^m(x), \mu_E^m(y)\} = \min\{(\mu_E^{\frac{m}{n}}(x))^n, (\mu_E^{\frac{m}{n}}(y))^n\},\\
(\mu_E^{\frac{m}{n}}(x * y))^n &= (\mu_E(x * y))^n \geq \min\{\mu_E^n(x), \mu_E^n(y)\} = \min\{(\mu_E^{\frac{m}{n}}(x))^n, (\mu_E^{\frac{m}{n}}(y))^n\}.
\end{aligned}$$

Therefore, E^c is an anti (m,n)-fuzzy BL-subalgebra of X. □

E be an (m,n)-fuzzy BL-subalgebra of X. Denote $E^{(1)} = E^c, E^{(2)} = (E^c)^c$ and for all $k \geq 2, E^{(k)} = (E^{(k-1)})^c$.

Also, for all (m,n)-fuzzy BL-subalgebras E and $F = \{(x, \mu_F(x), \nu_F(x)) \mid x \in X\}$ of X, we discuss that $E = F$, if for all $x \in X, \mu_E(x) = \mu_F(x)$ and $\nu_E(x) = \nu_F(x)$.

Corollary 3. Let E be an (m,n)-fuzzy BL-subalgebra of X and $k \in \mathbb{N}$. Then

(i) $E^{(2k-1)}$ is an anti (m,n)-fuzzy BL-subalgebra of X.

(ii) $E^{(2k)} = E$.

Let E be an (m,n)-FS of X and $x \in X$. Define $\lceil E \rceil = \{(x, \nu_E^{\frac{m}{n}}(x), \mu_E^{\frac{n}{m}}(x)) \mid x \in X\}$.

Theorem 5. Let E be an (m,n)-fuzzy BL-subalgebra of X. Then $\lceil E \rceil$ is an (n,m)-fuzzy BL-subalgebra of X.

Proof. Let $x, y \in X$. Then $(\mu_E^{\frac{m}{n}}(x))^n + (\nu_E^{\frac{n}{m}}(x))^m = \mu_E^m(x) + \nu_E^n(x) \leq 1$ and so $\lceil E \rceil$ is an (m,n)-FS of X. In addition,

$$(\mu_E^{\frac{m}{n}}(x \to y))^n = (\mu_E(x \to y))^m \geq \min\{\mu_E^m(x), \mu_E^m(y)\} = \min\{(\mu_E^{\frac{m}{n}}(x))^n, (\mu_E^{\frac{m}{n}}(y))^n\},$$
$$(\mu_E^{\frac{m}{n}}(x * y))^n = (\mu_E(x * y))^n \geq \min\{\mu_E^n(x), \mu_E^n(y)\} = \min\{(\mu_E^{\frac{m}{n}}(x))^n, (\mu_E^{\frac{m}{n}}(y))^n\},$$
$$(\nu_E^{\frac{n}{m}}(x \to y))^m = (\nu_E(x \to y))^n \leq \max\{\nu_E^n(x), \nu_E^n(y)\} = \max\{(\nu_E^{\frac{n}{m}}(x))^m, (\nu_E^{\frac{n}{m}}(y))^m\},$$
$$(\nu_E^{\frac{n}{m}}(x * y))^m = (\nu_E(x * y))^n \leq \max\{\nu_E^n(x), \nu_E^n(y)\} = \max\{(\nu_E^{\frac{n}{m}}(x))^m, (\nu_E^{\frac{n}{m}}(y))^m\}.$$

Therefore, $\lceil E \rceil$ is an (n,m)-fuzzy BL-subalgebra of X. □

Let E be an (m,n)-fuzzy BL-subalgebra of X. Denote $E^{(0)} = E, E^{\{1\}} = \lceil E \rceil, E^{\{2\}} = \lceil \lceil E \rceil \rceil$ and for all $k \geq 2$, $E^{\{k\}} = \lceil E^{\{k-1\}} \rceil$.

Corollary 4. Let E be an (m,n)-fuzzy BL-subalgebra of X and $k \in \mathbb{N}$. Then

(i) $E^{\{2k-1\}}$ is an (n,m)-fuzzy BL-subalgebra of X.

(ii) $E^{\{2k\}}$ is an (m,n)-fuzzy BL-subalgebra of X.

(iii) $E^{\{2k\}} = E$.

Definition 5. Let E be an (m,n)-FS of X, and let $\alpha, \beta \in [0,1]$. The (m,n)-fuzzy level set of E at (α, β) is

$$E^(\alpha, \beta) \;=\; \{\, x \in X \mid \mu_E^{(m)}(x) \geq \alpha,\ \nu_E^{(n)}(x) \leq \beta \,\},$$

where $\mu_E^{(m)}$ and $\nu_E^{(n)}$ denote the m- and n-level membership functions of E, respectively.

Example 3. Let E be the $(2,3)$-FS of an arbitrary set $X = \{a, b, c\}$, given by

$$E \;=\; \{(a, 0.8, 0.3),\ (b, 0.7, 0.5),\ (c, 0.9, 0.4)\}.$$

Hence

$$E^{(0.6,0.1)} \;=\; \{x \in \{a,b,c\} \mid \mu_E^{(2)}(x) \geq 0.6,\ \nu_E^{(3)}(x) \leq 0.1\} \;=\; \{a, c\},$$

since $0.8^2 \geq 0.6$ and $0.3^3 \leq 0.1$ for a, and $0.9^2 \geq 0.6$ and $0.4^3 \leq 0.1$ for c, while $0.7^2 \leq 0.6$ for b yields $b \notin E^{(0.6,0.1)}$.

Theorem 6. Let E be an (m,n)-FS of X and let $\alpha, \beta \in [0,1]$. Then $E^{(\alpha,\beta)}$ is a BL-subalgebra of X whenever E is an (m,n)-fuzzy BL-subalgebra of X.

Proof. Let $x, y \in E^{(\alpha,\beta)}$. Then $\mu_E^{(m)}(x) \geq \alpha$ and $\nu_E^{(n)}(y) \leq \beta$. We have

$$\mu_E^{(m)}(x * y) \geq \mu_E^{(m)}(x) \curlywedge \mu_E^{(m)}(y) \geq \alpha \curlywedge \alpha = \alpha,$$
$$\nu_E^{(n)}(x * y) \leq \nu_E^{(n)}(x) \curlyvee \nu_E^{(n)}(y) \leq \beta \curlyvee \beta = \beta,$$
$$\mu_E^{(m)}(x \to y) \geq \mu_E^{(m)}(x) \curlywedge \mu_E^{(m)}(y) \geq \alpha \curlywedge \alpha = \alpha,$$
$$\nu_E^{(n)}(x \to y) \leq \nu_E^{(n)}(x) \curlyvee \nu_E^{(n)}(y) \leq \beta \curlyvee \beta = \beta.$$

Thus, $x * y \in E^{(\alpha,\beta)}$ and $x \to y \in E^{(\alpha,\beta)}$.

Conversely, suppose $\mu_E^{(m)}(x) \curlywedge \mu_E^{(m)}(y) = \alpha$ and $\nu_E^{(n)}(x) \curlyvee \nu_E^{(n)}(y) = \beta$. Then $\mu_E^{(m)}(x) \geq \alpha$, $\mu_E^{(m)}(y) \geq \alpha$ and $\nu_E^{(n)}(x) \leq \beta$, $\nu_E^{(n)}(y) \leq \beta$, so, $x, y \in E^{(\alpha,\beta)}$. Since $E^{(\alpha,\beta)}$ is a BL-subalgebra, we also have $x * y \in E^{(\alpha,\beta)}$ and $x \to y \in E^{(\alpha,\beta)}$. Therefore,

$$\mu_E^{(m)}(x * y) \geq \alpha = \mu_E^{(m)}(x) \curlywedge \mu_E^{(m)}(y), \qquad \nu_E^{(n)}(x * y) \leq \beta = \nu_E^{(n)}(x) \curlyvee \nu_E^{(n)}(y),$$
$$\mu_E^{(m)}(x \to y) \geq \alpha = \mu_E^{(m)}(x) \curlywedge \mu_E^{(m)}(y), \qquad \nu_E^{(n)}(x \to y) \leq \beta = \nu_E^{(n)}(x) \curlyvee \nu_E^{(n)}(y).$$

Hence, E is an (m, n)-fuzzy BL-subalgebra of X. □

4 PIP BL-Algebra and (m, n)-Fuzzy BL-Subalgebras

In this section we introduce the notation of power-implication preserving PIP BL-algebras and study their key properties. We show that Boolean algebras and Product BL-algebras are PIP BL-algebras, whereas Gödel BL-algebras and Łukasiewicz BL-algebras are not PIP BL-algebras.

Our aim is to extend the theory of (m, n)-fuzzy BL-subalgebras and to develop some strong structural results for BL-algebras. This motivates the study of (m, n)-fuzzy nilpotent radicals in BL-algebras, which depend on the k-divisible implicative conditions.

Let $n \in \mathbb{N}$ and $(X, *, 0)$ be a BL-algebra. For any $x \in X$, define

$$x^2 = x * x, \quad x^3 = x^2 * x, \quad x^n = x^{n-1} * x.$$

Since $(X, *, 1)$ is a commutative monoid, for all $k \geq 2$ we have

$$(x * y)^k = x^k * y^k.$$

If E is an (m, n)-fuzzy BL-subalgebra of X and $k \geq 2$, then for all $x \in X$,

$$\mu_E^{(m)}(x^k) \geq \mu_E^{(m)}(x), \qquad \nu_E^{(n)}(x^k) \leq \nu_E^{(n)}(x).$$

Definition 6. Let X be a BL-algebra and $k \in \mathbb{N}$. We say that X is a k-divisible implicative BL-algebra (i.e., a PIP BL-algebra) if for all x,y $\in X$, $\quad (x \to y)^k = x^k \to y^k$.

Example 4. (i) Let $2 \leq k \in \mathbb{N}$ and $X = [0, 1]$ be the Product BL-algebra, with

$$x * y = xy \quad \text{and} \quad x \to y = \begin{cases} \frac{y}{x}, & x > y, \\ 1, & \text{otherwise.} \end{cases}$$

For any $t \in X$, $t^n = t^{n-1} * t = \cdots = \underbrace{tt \cdots t}_{n \text{ times}}$. Since the map $f(t) = t^k$ is strictly increasing on $[0, 1]$, we have $x > y \iff x^k > y^k$.

Hence,

$$(x \to y)^k = \begin{cases} \left(\frac{y}{x}\right)^k, & x > y, \\ 1^k, & \text{otherwise,} \end{cases} \quad \text{which is} \begin{cases} \frac{y^k}{x^k}, & x > y, \\ 1, & \text{otherwise.} \end{cases}$$

For any $t, s \in X$,

$$\left(\frac{t}{s}\right)^k = \left(\frac{t}{s}\right)^{k-1} * \frac{t}{s} = \frac{t^k}{s^k}.$$

Therefore, $(x \to y)^k = x^k \to y^k$, so X is a k-divisible implicative (i.e., PIP) BL-algebra.

(ii) Let $2 \leq k \in \mathbb{N}$ and $X = [0, 1]$ be the Gödel BL-algebra, with

$$x * y = \min(x, y), \qquad x \to y = \begin{cases} 1, & x \leq y, \\ y, & x > y. \end{cases}$$

Clearly, $t^k = \underbrace{t * t * \cdots * t}_{k \text{ times}} = \min(\underbrace{t, \ldots, t}_{k \text{ times}}) = t$, so for any $y \in X$, $(x \to y)^k = x \to y$. Moreover $x^k \to y^k = x \to y$. Hence $(x \to y)^k = x^k \to y^k$, and X is a k-divisible implicative BL-algebra.

Example 5. Let $2 \leq k \in \mathbb{N}$ and $X = [0, 1]$ be the Lukasiewicz BL-algebra, which $x * y = \max(0, x + y - 1)$ and $x \to y = \min(1, 1 - x + y)$. For any $k \in \mathbb{N}$, by induction, obviously $x^k = \max(0, kx - (k - 1))$, and so

$$(x \to y)^k = \max(0, k \times (x \to y) - (k - 1)) = \max(0, k \times \min(1, 1 - x + y) - (k - 1))$$

and

$$\begin{aligned} x^k \to y^k &= \min(1, 1 - x^k + y^k) \\ &= \min(1, 1 - \max(0, kx - (k - 1)) + \max(0, ky - (k - 1))). \end{aligned}$$

Now, X is a k-divisible implicative BL-algebra iffy for all $x, y \in X$, $(x \to y)^k = x^k \to y^k$, which do not hold, necessarily. For instance, if $k = 2$, $x = 0.8$, and $y = 0.3$, then

$$x \to y = \min(1, 1 - x + y) = \min(1, 1 - 0.8 + 0.3) = \min(1, 0.5) = 0.5 \Rightarrow$$
$$(x \to y)^k = (x \to y)^2 = (0.5)^2 = \max(0, 2 \cdot 0.5 - 1) = \max(0, 1 - 1) = 0,$$
$$x^k = x^2 = \max(0, 2 \cdot x - 1) = \max(0, 2 \cdot 0.8 - 1) = \max(0, 1.6 - 1) = 0.6$$
$$y^k = y^2 = \max(0, 2 \cdot y - 1) = \max(0, 2 \cdot 0.3 - 1) = \max(0, 0.6 - 1) = 0$$
$$x^k \to y^k = 0.6 \to 0 = \min(1, 1 - 0.6 + 0) = \min(1, 0.4) = 0.4$$

Thus $(x \to y)^k = 0 \neq 0.4 = x^k \to y^k$, and the Lukasiewicz BL-algebra is not a k-divisible implicative BL-algebra.

In what follows, it is tried to construct a new class of power-implication preserving (PIP) BL-algebras with finite fields as underlying set.

Theorem 7. Let p be a prime. Then there's at least a power-implication preserving (PIP) BL-algebra X to $|X| = p$.

Proof. Let $(X, +, \cdot)$ be a field. Assume $(X, +, \cdot)$ is finite, there's a prime p to $(X, +, \cdot)$ is isomorphic to $(\mathbb{Z}_p, \oplus, \odot)$. For any $\overline{x}, \overline{y} \in \mathbb{Z}_p$, define

$$\overline{x} \curlywedge \overline{y} = \begin{cases} \overline{y} & \text{if} \quad \overline{x} = \overline{1}, \\ \overline{x} & \text{if} \quad \overline{y} = 1, \ \textit{and} \\ \overline{\max\{x, y\}} & \text{o.w.,} \end{cases} \qquad \overline{x} \curlyvee \overline{y} = \begin{cases} \overline{1} & \text{if} \quad \overline{x} = \overline{1} \text{ or } \overline{y} = \overline{1}, \\ \overline{\min\{x, y\}} & \text{o.w.,} \end{cases}.$$

Also, define $* : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$ by $\overline{x} * \overline{y} = \overline{xy}$ and $\to : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$ by

$$\overline{x} \to \overline{y} = \begin{cases} \overline{yx^{-1}} & x < y, \\ \overline{y} & x = 1, \\ \overline{1} & \text{o.w,} \end{cases}$$

. Now, prove that $(\mathbb{Z}_p, \curlywedge, \curlyvee, *, \to, \overline{0}, \overline{1})$ is a BL-algebra. Obviously, $(\mathbb{Z}_p, \curlywedge, \curlyvee, \overline{0}, \overline{1})$ is a bounded lattice relative to the order $\leq_{\mathbb{Z}_p}$, which $\overline{x} \leq_{\mathbb{Z}_p} \overline{y}$ iffy $y \leq x$ and $(\mathbb{Z}_p, *, \overline{1})$ is a commutative monoid. Thus BL1 and BL2 are valid.

$(BL3)$: Let $\overline{x}, \overline{y}, \overline{z} \in \mathbb{Z}_p$ and $\overline{x} * \overline{y} \leq_{\mathbb{Z}_p} \overline{z}$. Then $\overline{x} * \overline{y} \leq_{\mathbb{Z}_p} \overline{z}$ iff $\overline{xy} \leq_{\mathbb{Z}_p} \overline{z}$ if and only $z \leq xy$ iff $zy^{-1} \leq x$ iff $\overline{x} \leq_{\mathbb{Z}_p} \overline{zy^{-1}}$ iff $\overline{x} \leq_{\mathbb{Z}_p} (\overline{y} \to \overline{z})$.

$(BL4)$: Let $\overline{x}, \overline{y} \in \mathbb{Z}_p$. If $x < y$, then $\overline{x} \curlywedge \overline{y} = \overline{y} = \overline{xyx^{-1}} = \overline{x} * \overline{yx^{-1}} = \overline{x} * (\overline{x} \to \overline{y})$. If $x = 1$, then $\overline{x} \curlywedge \overline{y} = \overline{y} = \overline{1} * \overline{y} = \overline{1} * (\overline{1} \to \overline{y}) = \overline{x} * (\overline{x} \to \overline{y})$. If $x = y$, then $\overline{x} \curlywedge \overline{y} = \overline{x} = \overline{x} * \overline{1} = \overline{x} * (\overline{x} \to \overline{x}) = \overline{x} * (\overline{x} \to \overline{y})$. If $x \geq y$, then $\overline{x} \curlywedge \overline{y} = \overline{x} = \overline{x} \curlywedge \overline{1} = \overline{x} * (\overline{x} \to \overline{y}) = \overline{x} * (\overline{x} \to \overline{y})$.

$(BL5)$: Let $\overline{x}, \overline{y} \in \mathbb{Z}_p$. If $x \leq y$, then $\overline{y} \to \overline{x} = \overline{1}$. If $y \leq x$, then $\overline{x} \to \overline{y} = \overline{1}$. Then in any cases, $(\overline{x} \to \overline{y}) \curlyvee (\overline{y} \to \overline{x}) = \overline{1}$.

Now, for all $k \in \mathbb{N}$ and any $\overline{y} \in \mathbb{Z}_p, \overline{x}^n = \overline{x^n}$ and

$$(\overline{x} \to \overline{y})^k = \begin{cases} (\overline{yx^{-1}})^k, & x < y \\ (\overline{y})^k, & x = 1 \\ \overline{1}, & \text{o.w.} \end{cases} = \begin{cases} \overline{y^k x^{-k}}, & x < y \\ (\overline{y^k}), & x = 1 \\ \overline{1}, & \text{o.w.} \end{cases}.$$

Computations yield, $\overline{y^k} = (\overline{1} \to \overline{y})^k = \overline{1}^k \to \overline{y}^k = \overline{y^k}$ and for all $x < y$,

$$(\overline{x} \to \overline{y})^k = \overline{y^k x^{-k}} = \overline{x^{-k}} \to \overline{y^k},$$

since $x^k < y^k$. Therefore, X is a k-divisible implicative BL-algebra. □

Example 6. Consider the field $(\mathbb{Z}_5, \oplus, \odot)$ and a lattice $(\mathbb{Z}_5, \curlywedge, \curlyvee)$ in Figure 1. Then $(\mathbb{Z}_5, \curlywedge, \curlyvee, *, \to, \overline{0}, \overline{1})$ is a power-implication preserving (PIP) BL-algebra given by Tables 1 and 2.

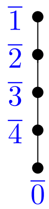

Figure 1: $(\mathbb{Z}_5, \leq)$

$*$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{1}$	$\overline{3}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{1}$	$\overline{4}$	$\overline{2}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

Table 1: $(X, *)$

$\to$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{2}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{4}$	$\overline{2}$
$\overline{3}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{3}$
$\overline{4}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$

Table 2: $(X, \to)$

Definition 7. Let E be an (m, n)-FS of a set X. We define the (m, n)-fuzzy nil radical of E as

$$\sqrt[mn]{E} = (\mu_{\sqrt[mn]{E}}, \nu_{\sqrt[mn]{E}}),$$

where

$$\mu_{\sqrt[mn]{E}}(x) = \bigvee_{k \geq 1} \mu_E(x^k), \quad \nu_{\sqrt[mn]{E}}(x) = \bigwedge_{k \geq 1} \nu_E(x^k).$$

Example 7. Let $(X, \curlywedge, \curlyvee, *, \to, 0, 1)$ be a BL-algebra, where $X = \{0, a, b, 1\}$ with $0 \leq a \leq b \leq 1$. The underlying tables are given as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

and

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Clearly, E is a $(2, 3)$-FS on X given by

$$E = \{(0, 0.2, 0.9),\ (a, 0.6, 0.7),\ (b, 0.7, 0.6),\ (1, 0.9, 0.4)\}.$$

Clearly, for any $k \geq 1$,

$$0^k = 0, \mu_E(0^k) = 0.2, \nu_E(0^k) = 0.9, \mu_{\sqrt[6]{E}}(0) = \bigvee\{0.2, 0.2, \ldots\} = 0.2, \nu_{\sqrt[6]{E}}(0)$$
$$= \bigwedge\{0.9, 0.9, \ldots\} = 0.9,$$

$$a^k = a, \mu_E(a^k) = 0.6, \nu_E(a^k) = 0.7, \mu_{\sqrt[6]{E}}(a) = \bigvee\{0.6, 0.6, \ldots\} = 0.6, \nu_{\sqrt[6]{E}}(a)$$
$$= \bigwedge\{0.7, 0.7, \ldots\} = 0.7,$$
$$b^k = b, \mu_E(b^k) = 0.7, \nu_E(b^k) = 0.6, \mu_{\sqrt[6]{E}}(b) = \bigvee\{0.7, 0.7, \ldots\} = 0.7, \nu_{\sqrt[6]{E}}(b)$$
$$= \bigwedge\{0.6, 0.6, \ldots\} = 0.6,$$
$$1^k = 1, \mu_E(1^k) = 0.9, \nu_E(1^k) = 0.4, \mu_{\sqrt[6]{E}}(1) = \bigvee\{0.9, 0.9, \ldots\} = 0.9, \nu_{\sqrt[6]{E}}(1)$$
$$= \bigwedge\{0.4, 0.4, \ldots\} = 0.4.$$

Hence $\sqrt[6]{E} = E$.

Theorem 8. Let $(X, *, 0)$ be a BL-algebra and let E be an (m, n)-fuzzy BL-subalgebra of X. Then:

(i) $\sqrt[mn]{E} = \left(\lim_{k\to\infty} \mu_E(x^k), \lim_{k\to\infty} \nu_E(x^k)\right)$.

(ii) If X has the PIP property, then $\sqrt[mn]{E}$ is an (m, n)-fuzzy BL-subalgebra of X.

Proof. Let $x \in X$ and $k \geq 1$.

(i) Assume E is an (m, n)-FS of X. For all $x \in X$,

$$0 \leq \mu_E^m(x^k) + \nu_E^n(x^k) \leq 1,$$

hence

$$\lim_{k\to\infty} \left(\mu_E^m(x^k) + \nu_E^n(x^k)\right) \leq 1.$$

Therefore,

$$1 \geq \bigvee_{k\geq 1} \mu_E^m(x^k) \wedge \bigwedge_{k\geq 1} \nu_E^n(x^k) = \mu^m_{\sqrt[mn]{E}}(x) + \nu^n_{\sqrt[mn]{E}}(x).$$

Hence $\sqrt[mn]{E}$ is an (m, n)-FS of X.

(ii) By (i), $\sqrt[mn]{E}$ is an (m, n)-FS of X, and since $(X, *, 1)$ is a commutative monoid, we have

$$\mu^m_{\sqrt[mn]{E}}(x * y) = \Big(\bigvee_{k\geq 1} \mu_E^m((x * y)^k)\Big) = \bigvee_{k\geq 1} \mu_E^m(x^k * y^k) \geq \bigvee_{k\geq 1} \left(\mu_E^m(x^k) \wedge \mu_E^m(y^k)\right)$$
$$\geq \Big(\bigvee_{k\geq 1} \mu_E^m(x^k)\Big) \wedge \Big(\bigvee_{k\geq 1} \mu_E^m(y^k)\Big) = \mu^m_{\sqrt[mn]{E}}(x) \wedge \mu^m_{\sqrt[mn]{E}}(y).$$

Similarly,

$$\nu^n_{\sqrt[mn]{E}}(x * y) = \bigwedge_{k\geq 1} \nu_E^n((x * y)^k) \leq \bigwedge_{k\geq 1} \left(\nu_E^n(x^k) \vee \nu_E^n(y^k)\right) \leq$$

$$\Big(\bigwedge_{k\geq 1}\nu_E^n(x^k)\Big)\vee\Big(\bigvee_{k\geq 1}\nu_E^n(y^k)\Big)=\nu^n_{\sqrt[mn]{E}}(x)\vee\nu^n_{\sqrt[mn]{E}}(y).$$

If X is a PIP, then for all $k\in\mathbb{N}$,

$$(x\to y)^k=x^k\to y^k.$$

Consequently,

$$\mu^m_{\sqrt[mn]{E}}(x\to y)\geq\mu^m_{\sqrt[mn]{E}}(x)\wedge\mu^m_{\sqrt[mn]{E}}(y),\qquad\nu^n_{\sqrt[mn]{E}}(x\to y)\leq\nu^n_{\sqrt[mn]{E}}(x)\vee\nu^n_{\sqrt[mn]{E}}(y).$$

Thus $\sqrt[mn]{E}$ satisfies the fuzzy BL-subalgebra axioms, and hence is an (m,n)-fuzzy BL-subalgebra of X.

□

Theorem 9. Let $(X,*,0)$ be a BL-algebra and let E,F be (m,n)-fuzzy BL-subalgebras of X. Then:

(i) $E\subseteq\sqrt[mn]{E}$.

(ii) If $E\subseteq F$, then $\sqrt[mn]{E}\subseteq\sqrt[mn]{F}$.

(iii) $\sqrt[mn]{\sqrt[mn]{E}}=\sqrt[mn]{E}$.

Proof. (i) Let $x\in X$. Then

$$\mu_{\sqrt[mn]{E}}(x)=\bigvee_{k\geq 1}\mu_E(x^k)=\mu_E(x)\vee\bigvee_{k\geq 2}\mu_E(x^k),$$

hence $\mu_E(x)\leq\mu_{\sqrt[mn]{E}}(x)$ for all $x\in X$. Thus $\mu_E\leq\mu_{\sqrt[mn]{E}}$ pointwise, and consequently $E\subseteq\sqrt[mn]{E}$.

Similarly,

$$\nu_{\sqrt[mn]{E}}(x)=\bigwedge_{k\geq 1}\nu_E(x^k)=\nu_E(x)\wedge\bigwedge_{k\geq 2}\nu_E(x^k),$$

so $\nu_{\sqrt[mn]{E}}(x)\leq\nu_E(x)$ for all $x\in X$. This yields $\nu_{\sqrt[mn]{E}}\leq\nu_E$, and hence $E\subseteq\sqrt[mn]{E}$ as desired.

(ii) Let $x\in X$. Then $\mu_E(x)\leq\mu_F(x)$ and $\nu_E(x)\geq\nu_F(x)$. Consequently,

$$\mu_{\sqrt[mn]{E}}(x)=\bigvee_{k\geq 1}\mu_E(x^k)\leq\bigvee_{k\geq 1}\mu_F(x^k)=\mu_{\sqrt[mn]{F}}(x),$$

and

$$\nu_{\sqrt[mn]{E}}(x)=\bigwedge_{k\geq 1}\nu_E(x^k)\geq\bigwedge_{k\geq 1}\nu_F(x^k)=\nu_{\sqrt[mn]{F}}(x).$$

Hence $\sqrt[mn]{E}\subseteq\sqrt[mn]{F}$.

(iii) From (i) we have $\sqrt[mn]{E} \subseteq \sqrt[mn]{\sqrt[mn]{E}}$. Let $x \in X$. Since $(X, *, 1)$ is commutative,

$$\mu_{\sqrt[mn]{\sqrt[mn]{E}}}(x) = \bigvee_{k\geq 1} \mu_{\sqrt[mn]{E}}(x^k) = \bigvee_{k\geq 1}\bigvee_{l\geq 1} \mu_E((x^k)^l)$$

$$= \bigvee_{k\geq 1}\bigvee_{l\geq 1} \mu_E(x^{kl}) = \bigvee_{r\geq 1} \mu_E(x^r) = \mu_{\sqrt[mn]{E}}(x).$$

Similarly,

$$\nu_{\sqrt[mn]{\sqrt[mn]{E}}}(x) = \bigwedge_{k\geq 1} \nu_{\sqrt[mn]{E}}(x^k) = \bigwedge_{k\geq 1}\bigwedge_{l\geq 1} \nu_E((x^k)^l) = \bigwedge_{r\geq 1} \nu_E(x^r) = \nu_{\sqrt[mn]{E}}(x).$$

Thus $\sqrt[mn]{\sqrt[mn]{E}} = \sqrt[mn]{E}$. □

Definition 8. Let E and F be (m,n)-fuzzy subsets of X. Define the following fuzzy operations on E and F:

$$E * F = \{ (x,\ \mu_{E*F}(x),\ \nu_{E*F}(x)) \mid x \in X \},$$
$$E \to F = \{ (x,\ \mu_{E\to F}(x),\ \nu_{E\to F}(x)) \mid x \in X \},$$

where

$$\mu_{E*F}(x) = \bigvee_{a*b=x} (\mu_E(a) \wedge \mu_F(b)), \qquad \nu_{E*F}(x) = \bigwedge_{a*b=x} (\nu_E(a) \vee \nu_F(b)),$$

$$\mu_{E\to F}(x) = \bigvee_{a\to b=x} (\mu_E(a) \wedge \mu_F(b)), \qquad \nu_{E\to F}(x) = \bigwedge_{a\to b=x} (\nu_E(a) \vee \nu_F(b)).$$

Example 8. Consider the BL-algebra in Example 6, $(2,3)$-fuzzy set E, and $(2,3)$-fuzzy set F as follows:

$$E = \{(\bar{0}, 0.8, 0.5), (\bar{4}, 0.7, 0.6), (\bar{3}, 0.6, 0.7), (\bar{2}, 0.5, 0.8), (\bar{1}, 0.4, 0.9)\},$$
$$F = \{(\bar{0}, 0.95, 0.3), (\bar{4}, 0.85, 0.4), (\bar{3}, 0.75, 0.5), (\bar{2}, 0.65, 0.6), (\bar{1}, 0.55, 0.7)\}\}.$$

Computations show that,

$$E * F = \{(\bar{0}, 0.8, 0.5), (\bar{1}, 0.7, 0.6), (\bar{2}, 0.7, 0.6), (\bar{3}, 0.65, 0.6), (\bar{4}, 0.6, 0.7)\}\}$$
$$E \to F = \{(\bar{0}, 0.95, 0.3), (\bar{1}, 0.85, 0.4), (\bar{2}, 0.75, 0.5), (\bar{3}, 0.65, 0.6), (\bar{4}, 0.55, 0.7)\}\}.$$

Theorem 10. Let E be an (m,n)-FS of X. The following are equivalent:

(i) E is an (m,n)-fuzzy BL-subalgebra of X.

(ii) $(E * E)^{(m,n)} \subseteq E^{(m,n)}$ and $(E \to E)^{(m,n)} \subseteq E^{(m,n)}$.

Proof. Let $x, y \in X$. There exist $c, d \in X$ such that, using the definitions of μ and ν,

$$\mu^m_{E*E}(x) + \nu^n_{E*E}(x) = \left(\bigvee_{a*b=x} (\mu_E(a) \wedge \mu_E(b))\right)^m + \left(\bigwedge_{a*b=x} (\nu_E(a) \vee \nu_E(b))\right)^n.$$

Since

$$\left(\bigvee_{a*b=x} (\mu_E(a) \wedge \mu_E(b))\right)^m \leq \bigvee_{a*b=x} \mu^m_E(a) \wedge \mu^m_E(b),$$

$$\bigwedge_{a*b=x} (\nu_E(a) \vee \nu_E(b))^n \geq \bigwedge_{a*b=x} \nu^n_E(a) \vee \nu^n_E(b),$$

we obtain

$$\mu^m_{E*E}(x) + \nu^n_{E*E}(x) \leq \mu^m_E(c) + \nu^n_E(d) \leq 1.$$

Hence $E * E$ is an (m, n)-FS of X, and thus $(E * E)^{(m,n)}$ is an (m, n)-FS of X.

Assume E is an (m, n)-fuzzy BL-subalgebra of X. Then for all $x \in X$,

$$\mu^m_{E*E}(x) = \bigvee_{a*b=x} (\mu_E(a) \wedge \mu_E(b))^m = \bigvee_{a*b=x} (\mu^m_E(a) \wedge \mu^m_E(b)) \leq \bigvee_{a*b=x} \mu^m_E(a*b) = \mu^m_E(x),$$

and

$$\nu^n_{E*E}(x) = \bigwedge_{a*b=x} (\nu_E(a) \vee \nu_E(b))^n = \bigwedge_{a*b=x} (\nu^n_E(a) \vee \nu^n_E(b)) \geq \bigwedge_{a*b=x} \nu^n_E(a*b) = \nu^n_E(x).$$

Hence $(E * E)^{(m,n)} \subseteq E^{(m,n)}$. A similar argument with the implication shows

$$(E \to E)^{(m,n)} \subseteq E^{(m,n)}.$$

Conversely, suppose $(E * E)^{(m,n)} \subseteq E^{(m,n)}$ and $(E \to E)^{(m,n)} \subseteq E^{(m,n)}$. Then for all $x, y \in X$,

$$\mu^m_E(x * y) \geq \mu^m_{E*E}(x * y) = \bigvee_{a*b=x*y} (\mu_E(a) \wedge \mu_E(b))^m \geq \mu^m_E(x) \wedge \mu^m_E(y),$$

and

$$\nu^n_E(x * y) \leq \nu^n_{E*E}(x * y) = \bigwedge_{a*b=x*y} (\nu_E(a) \vee \nu_E(b))^n \leq \nu^n_E(x) \vee \nu^n_E(y).$$

Similarly, from $(E \to E)^{(m,n)} \subseteq E^{(m,n)}$ it follows that

$$\mu^m_E(x \to y) \geq \mu^m_E(x) \wedge \mu^m_E(y), \qquad \nu^n_E(x \to y) \leq \nu^n_E(x) \vee \nu^n_E(y).$$

Hence E is an (m, n)-fuzzy BL-subalgebra of X. □

Theorem 11. Let E and F be (m,n)-fuzzy BL-subalgebras of X. If $E \subseteq F$, then

$$E^{(m,1)} \subseteq E * F.$$

Proof. Let $x \in X$. Since $x * 1 = x$, we have

$$\mu_{E*F}(x) \;=\; \bigvee_{a*b=x} \big(\mu_E(a) \wedge \mu_F(b)\big) \;\geq\; \mu_E(x) \wedge \mu_F(1) \;\geq\; \mu_E(x)^m \wedge \mu_E(1) = \mu_E^m(x),$$

and

$$\nu_{E*F}(x) \;=\; \bigwedge_{a*b=x} \big(\nu_E(a) \vee \nu_F(b)\big) \;\leq\; \nu_E(x) \vee \nu_F(1) \;\leq\; \nu_E(x) \vee \nu_E(1) = \nu_E(x).$$

Hence $\mu_E^m \leq \mu_{E*F}$ and $\nu_{E*F} \leq \nu_E$, which yields $E^{(m,1)} \subseteq E * F$. □

Let E be an (m,n)-fuzzy BL-subalgebra of X. We say that E is a two-sided (m,n)-fuzzy BL-subalgebra if μ_E forms a fuzzy BL-subalgebra of X and ν_E forms an anti-fuzzy BL-subalgebra of X.

Example 9. Let A be a fuzzy BL-subalgebra of X. Then by Theorem 2, $[A, \frac{1}{m}, \frac{1}{n}]$ is a two-sided (m,n)-fuzzy BL-subalgebra of X. Let $E = [A, \frac{1}{m}, \frac{1}{n}] = \{(x, \sqrt[m]{\mu(x)}, \sqrt[n]{\mu^c(x)})\}$, if consider $\mu_E(x) = \sqrt[m]{\mu(x)}$ and $\nu_E(x) = \sqrt[n]{\mu^c(x)})\}$, then by Theorem 2,

$$\begin{aligned}
\mu_E(x \to y) &= (\sqrt[m]{\mu(x \to y))})^m \geq (\sqrt[m]{\mu(x)})^m \curlywedge (\sqrt[m]{\mu(y)})^m = \mu_E(x) \curlywedge \mu_E(x)\\
\mu_E(x * y) &= (\sqrt[m]{\mu(x * y))})^m \geq (\sqrt[m]{\mu(x)})^m \curlywedge (\sqrt[m]{\mu(y)})^m = \mu_E(x) \curlywedge \mu_E(x)\\
\nu_E(x \to y) &= (\sqrt[n]{\mu^c(x \to y)})^n \leq \mu^c(x) \curlyvee \mu^c(y) = (\sqrt[n]{\mu^c(x)})^n \curlyvee (\sqrt[n]{\mu^c(y)})^n\\
&= \nu_E(x) \curlyvee \nu_E(x)\\
\nu_E(x * y) &= (\sqrt[n]{\mu^c(x * y)})^n \leq \mu^c(x) \curlyvee \mu^c(y) = (\sqrt[n]{\mu^c(x)})^n \curlyvee (\sqrt[n]{\mu^c(y)})^n\\
&= \nu_E(x) \curlyvee \nu_E(x).
\end{aligned}$$

It follows that μ_E is a fuzzy BL-subalgebra and ν_E is an anti fuzzy BL-subalgebra of X, so $[A, \frac{1}{m}, \frac{1}{n}]$ is a two-sided (m,n)-fuzzy BL-subalgebra of X.

Corollary 5. Let E and F be (m,n)-fuzzy BL-subalgebras of X. Then

(i) $E^{(m,1)} \subseteq E * E$.

(ii) If E is a two-sided fuzzy BL-subalgebra of X, then $E \subseteq E * E$.

(iii) If E, F are two-sided fuzzy BL-subalgebras of X and $E \subseteq F$, then $E \subseteq E * F$.

Corollary 6. Let A be a fuzzy BL-subalgebra of X. Then $[A, \frac{1}{m}, \frac{1}{n}] \subseteq [A, \frac{1}{m}, \frac{1}{n}] * [A, \frac{1}{m}, \frac{1}{n}]$.

Theorem 12. Let E and F be (m,n)-fuzzy BL-subalgebras of X. Then

(i) $\sqrt[mn]{E} \cap \sqrt[mn]{F} = \sqrt[mn]{E \cap F}$,

(ii) If $E \subseteq F$, then $(\sqrt[mn]{E})^{(m,n)} \subseteq (\sqrt[mn]{E*F})^{(m,n)}$.

Proof. (i) Let $x, y \in X$. By Theorem 9, $\sqrt[mn]{E \cap F} \subseteq \sqrt[mn]{E} \cap \sqrt[mn]{F}$.

$$\begin{aligned}(\mu_{\sqrt[mn]{E}} \cap \mu_{\sqrt[mn]{F}})(x) &= \mu_{\sqrt[mn]{E}}(x) \wedge \mu_{\sqrt[mn]{F}}(x) \\ &= \left(\bigvee_{k\geq 1} \mu_E(x^k)\right)^m \wedge \left(\bigvee_{k\geq 1} \mu_F(x^k)\right)^m \\ &\leq \bigvee_{k\geq 1} \left(\mu_F(x^k) \wedge \mu_E(x^k)\right) \\ &= \bigvee_{k\geq 1} \mu_{E\cap F}(x^k) = (\mu_{\sqrt[mn]{E\cap F}})(x).\end{aligned}$$

Similarly,

$$\begin{aligned}(\nu_{\sqrt[mn]{E}} \cup \nu_{\sqrt[mn]{F}})(x) &= \nu_{\sqrt[mn]{E}}(x) \vee \nu_{\sqrt[mn]{F}}(x) \\ &= \left(\bigwedge_{k\geq 1} \nu_E(x^k)\right) \vee \left(\bigwedge_{k\geq 1} \nu_F(x^k)\right) \\ &\geq \bigwedge_{k\geq 1} \left(\nu_F(x^k) \vee \nu_E(x^k)\right) \\ &= \bigwedge_{k\geq 1} \nu_{E\cup F}(x^k) = (\nu_{\sqrt[mn]{E\cup F}})(x).\end{aligned}$$

Hence

$$\sqrt[mn]{E} \cap \sqrt[mn]{F} = \sqrt[mn]{E \cap F}.$$

(ii) Let $x \in X$. Then

$$\begin{aligned}\mu^m_{\sqrt[mn]{E*F}}(x) &= \bigvee_{k\geq 1} \mu^m_{E*F}(x^k) = \bigvee_{k\geq 1}\left(\bigvee_{a*b=x^k} \mu^m_E(a) \wedge \mu^m_F(b)\right) \\ &\geq \bigvee_{k\geq 1} \left(\mu^m_F(x^k) \wedge \mu^m_E(1)\right) \geq \bigvee_{k\geq 1} \left(\mu^m_F(x^k) \wedge \mu^m_E(x^k)\right) \\ &= \left(\bigvee_{k\geq 1} \mu^m_F(x^k)\right) \wedge \left(\bigvee_{k\geq 1} \mu^m_E(x^k)\right) \\ &= \mu^m_{\sqrt[mn]{E}}(x) \wedge \mu^m_{\sqrt[mn]{F}}(x) = (\mu^m_{\sqrt[mn]{E}} \cap \mu^m_{\sqrt[mn]{F}})(x).\end{aligned}$$

Hence $\mu^m_{\sqrt[mn]{E}} \cap \mu^m_{\sqrt[mn]{F}} \subseteq \mu^m_{\sqrt[mn]{E*F}}$.

Similarly,

$$\begin{aligned}\nu^n_{\sqrt[mn]{E*F}}(x) &= \bigvee_{k\geq 1} \nu^n_{E*F}(x^k) = \bigvee_{k\geq 1} \bigwedge_{a*b=x^k} \left(\nu^n_E(a) \vee \nu^n_F(b)\right) \\ &\leq \bigvee_{k\geq 1} \left(\nu^n_F(x^k) \vee \nu^n_E(1)\right) \leq \bigvee_{k\geq 1} \left(\nu^n_F(x^k) \vee \nu^n_E(x^k)\right)\end{aligned}$$

$$= \Big(\bigvee_{k\geq 1} \nu^n_F(x^k)\Big) \vee \Big(\bigvee_{k\geq 1} \nu^n_E(x^k)\Big)$$
$$= \nu^n_{\sqrt[mn]{E}}(x) \vee \nu^n_{\sqrt[mn]{F}}(x) = (\nu^n_{\sqrt[mn]{E}} \cup \nu^n_{\sqrt[mn]{F}})(x).$$

Hence $\nu^n_{\sqrt[mn]{E*F}} \subseteq \nu^n_{\sqrt[mn]{E}} \cup \nu^n_{\sqrt[mn]{F}}$.
If, in addition, $E \subseteq F$, then $\mu_E \subseteq \mu_F$ and $\nu_F \subseteq \nu_E$. Consequently,

$$\mu^m_{\sqrt[mn]{E}} \cap \mu^m_{\sqrt[mn]{F}} = \mu^m_{\sqrt[mn]{E}}, \qquad \nu^n_{\sqrt[mn]{E}} \cup \nu^n_{\sqrt[mn]{F}} = \nu^n_{\sqrt[mn]{E}}.$$

Therefore,
$$(\sqrt[mn]{E})^{(m,n)} \subseteq (\sqrt[mn]{E * F})^{(m,n)}.$$

□

Theorem 13. Let E be an (m, n)-fuzzy BL-subalgebra of X. Then
$$\big(\sqrt[mn]{E}\big)^{(m,n)} = \big(\sqrt[mn]{E * E}\big)^{(m,n)}.$$

Proof. Let $x \in X$. Then

$$\mu^m_{E*E}(x) = \bigvee_{k\geq 1} \mu^m_E(a) \wedge \mu^m_E(b) \quad \text{over} \quad a * b = x^k,$$
$$\leq \mu^m_E(a * b) = \mu^m_E(x),$$

and

$$\nu^n_{E*E}(x) = \bigwedge_{k\geq 1} \nu^n_E(a) \vee \nu^n_E(b) \quad \text{over} \quad a * b = x^k,$$
$$\geq \nu^n_E(a * b) = \nu^n_E(x).$$

Hence $\mu^m_{\sqrt[mn]{E*E}} \subseteq \mu^m_{\sqrt[mn]{E}}$ and $\nu^n_{\sqrt[mn]{E}} \subseteq \nu^n_{\sqrt[mn]{E*E}}$. Therefore,
$$\big(\sqrt[mn]{E * E}\big)^{(m,n)} \subseteq \big(\sqrt[mn]{E}\big)^{(m,n)}.$$

By applying Theorem 12 (or the corresponding dual argument), the reverse inclusion holds as well, and we obtain
$$\big(\sqrt[mn]{E}\big)^{(m,n)} = \big(\sqrt[mn]{E * E}\big)^{(m,n)}.$$

□

Let $(X,*,0)$ and $(X',*',0')$ be BL-algebras. A map $\Psi : X \to X'$ is called a BL-homomorphism if

$$\Psi(x*y) = \Psi(x) *' \Psi(y) \quad \text{for all } x, y \in X.$$

If Ψ is onto, it is called an epimorphism.

For any FS A of X and B of X', define the pushforward of μ along Ψ by

$$(\mu_\Psi)(y) = \Psi(\mu)(y) = \begin{cases} \bigvee_{y=\Psi(x)} \mu(x), & \Psi^{-1}(y) \neq \emptyset, \\ 0, & \Psi^{-1}(y) = \emptyset, \end{cases}$$

and the pullback of ν along Ψ by

$$(\nu_\Psi^{-1})(x) = \Psi^{-1}(\nu)(x) = \nu(\Psi(x)).$$

Theorem 14. Let E and F be (m,n)-fuzzy BL-subalgebras of X and X', respectively. For a homomorphism $\Psi : X \to X'$, the following hold:

(i) $\Psi^{-1}(F)$ is an (m,n)-fuzzy BL-subalgebra of X.

(ii) $\sqrt[mn]{\Psi^{-1}(F)} = \Psi^{-1}\big(\sqrt[mn]{F}\big)$.

(iii) $\Psi^{-1}(F) \subseteq \Psi^{-1}\big(\sqrt[mn]{F}\big)$.

Proof. (i) Let $x, y \in X$. Since

$$\big(\Psi^{-1}(\mu_F(x))\big)^m + \big(\Psi^{-1}(\nu_F(x))\big)^n = \mu_F(\Psi(x))^m + \nu_F(\Psi(x))^n \leq 1,$$

it follows that

$$\Psi^{-1}(F) = \{(x, \Psi^{-1}(\mu_F), \Psi^{-1}(\nu_F)) \mid x \in X\}$$

is an (m,n)-fuzzy BL-subalgebra of X.

Moreover, for the operations on X, we have

$$\big(\mu_{\Psi^{-1}(F)}(x*y)\big)^m = \mu_F\big(\Psi(x*y)\big)^m \geq \mu_F\big(\Psi(x)\big)^m \curlywedge \mu_F\big(\Psi(y)\big)^m$$
$$= \big(\Psi^{-1}(\mu_F)(x)\big)^m \curlywedge \big(\Psi^{-1}(\mu_F)(y)\big)^m,$$

and similarly

$$\big(\nu_{\Psi^{-1}(F)}(x*y)\big)^n = \nu_F\big(\Psi(x*y)\big)^n \leq \nu_F\big(\Psi(x)\big)^n \curlyvee \nu_F\big(\Psi(y)\big)^n$$
$$= \big(\Psi^{-1}(\nu_F)(x)\big)^n \curlyvee \big(\Psi^{-1}(\nu_F)(y)\big)^n.$$

Hence $\Psi^{-1}(F)$ is an (m,n)-fuzzy BL-subalgebra of X.

(ii) For $x \in X$,

$$(\mu_{\sqrt[mn]{\Psi^{-1}(F)}})(x) = \bigvee_{k\geq 1} (\mu_{\Psi^{-1}(F)})(x^k) = \bigvee_{k\geq 1} \mu_F(\Psi(x^k)) = \bigvee_{k\geq 1} \mu_F(\Psi(x)^k)$$

and hence

$$(\mu_{\sqrt[mn]{\Psi^{-1}(F)}})(x) = \mu_{\sqrt[mn]{F}}(\Psi(x)) = \mu_{\Psi^{-1}(\sqrt[mn]{F})}(x).$$

Similarly,

$$(\nu_{\sqrt[mn]{\Psi^{-1}(F)}})(x) = \nu_{\sqrt[mn]{F}}(\Psi(x)) = \nu_{\Psi^{-1}(\sqrt[mn]{F})}(x).$$

Therefore,

$$\sqrt[mn]{\Psi^{-1}(F)} = \Psi^{-1}(\sqrt[mn]{F}).$$

(iii) By Theorem 9, we have $\Psi^{-1}(F) \subseteq \sqrt[mn]{\Psi^{-1}(F)}$. Using part (ii), this yields

$$\Psi^{-1}(F) \subseteq \sqrt[mn]{\Psi^{-1}(F)} = \Psi^{-1}(\sqrt[mn]{F}),$$

i.e.,

$$\Psi^{-1}(F) \subseteq \Psi^{-1}(\sqrt[mn]{F}).$$

□

Theorem 15. Let E and F be (m, n)-fuzzy BL-subalgebras of X and X', respectively, and let $\Psi : X \to X'$ be an isomorphism. Then

(i) $\Psi(E)$ is an (m, n)-fuzzy BL-subalgebra of X'.

(ii) $\sqrt[mn]{\Psi(E)} \subseteq \sqrt[mn]{\Psi(\sqrt[mn]{E})}$.

(iii) $\sqrt[mn]{\Psi(E)} = \Psi(\sqrt[mn]{E})$.

Proof. (i) Let $x', y' \in X'$. Then there exist $c, d \in X$ such that

$$\Psi^m(\mu_E)(y') + \Psi^n(\nu_E)(y') = \bigvee_{\Psi(x)=y'} \mu_E^m(x) + \bigvee_{\Psi(x)=y'} \nu_E^n(x) = \mu_E^m(c) + \nu_E^n(d) \leq 1,$$

and hence $\Psi(E)$ is an (m, n)-fuzzy BL-subset of X'. Since Ψ is surjective, there exist $x, y \in X$ with $\Psi(x) = x'$ and $\Psi(y) = y'$. Thus,

$$\begin{aligned}((\mu_\Psi^m)_E)(x' * y') &= \Psi^m(\mu_E)(x' * y') \\ &= \bigvee_{\Psi(a)=x' * y'} \mu_E^m(a) \geq \mu_E^m(x * y) \\ &\geq \mu_E^m(x) \wedge \mu_E^m(y) \\ &= \left(\bigvee_{\Psi(x)=x'} \mu_E^m(x)\right) \wedge \left(\bigvee_{\Psi(y)=y'} \mu_E^m(y)\right)\end{aligned}$$

$$= \Psi^m(\mu_E)(x') \wedge \Psi^m(\mu_E)(y').$$

Similarly, since Ψ is injective,

$$\begin{aligned} ((\nu_\Psi^n)_E)(x' * y') &= \Psi^n(\nu_E)(x' * y') \\ &= \bigvee_{\Psi(a)=x' * y'} \nu_E^n(a) \leq \nu_E^n(x * y) \\ &\leq \nu_E^n(x) \wedge \nu_E^n(y) \\ &= \Psi^n(\nu_E)(x') \wedge \Psi^n(\nu_E)(y'). \end{aligned}$$

Hence $\Psi(E)$ is an (m, n)-fuzzy BL-subalgebra of X'.

(ii) Let $y \in X'$. Since Ψ is surjective, $\Psi^{-1}(y) \neq \varnothing$. Then

$$\mu_{\Psi(\sqrt[mn]{E})}(y) = \bigvee_{k \geq 1} \mu_{\Psi(E)}(y^k) = \bigvee_{k \geq 1} \bigvee_{\Psi(x)=y^k} \mu_E(x) \geq \bigvee_{\Psi(x)=y} \mu_E(x) = \mu_{\Psi(E)}(y),$$

and

$$\nu_{\Psi(\sqrt[mn]{E})}(y) = \bigwedge_{k \geq 1} \nu_{\Psi(E)}(y^k) = \bigwedge_{k \geq 1} \bigvee_{\Psi(x)=y^k} \nu_E(x) \leq \bigvee_{\Psi(x)=y} \nu_E(x) = \nu_{\Psi(E)}(y).$$

Thus,

$$\mu_{\Psi(E)} \subseteq \mu_{\Psi(\sqrt[mn]{E})}, \qquad \nu_{\Psi(E)} \supseteq \nu_{\Psi(\sqrt[mn]{E})},$$

so that $\Psi(E) \subseteq \Psi(\sqrt[mn]{E})$.

(iii) Let $y \in X'$. Since Ψ is surjective, $\Psi^{-1}(y) \neq \varnothing$. For any $\varepsilon > 0$,

$$\begin{aligned} \mu_{\sqrt[mn]{\Psi(E)}}(y) - \varepsilon &\leq \mu_{\Psi(E)}(y^m) = \mu_{\Psi(E)}(\Psi(x^m)) \\ &= \mu_{\Psi^{-1}(\Psi(E))}(x^m) = \mu_E(x^m) \leq \mu_{\sqrt[mn]{E}}(x) \\ &\leq \bigvee_{\Psi(z)=y} \mu_{\Psi(\sqrt[mn]{E})}(y) = \mu_{\Psi(\sqrt[mn]{E})}(y), \end{aligned}$$

and similarly,

$$\begin{aligned} \nu_{\sqrt[mn]{\Psi(E)}}(y) + \varepsilon &\geq \nu_{\Psi(E)}(y^m) = \nu_{\Psi(E)}(\Psi(x^m)) \\ &= \nu_{\Psi^{-1}(\Psi(E))}(x^m) = \nu_E(x^m) \geq \nu_{\sqrt[mn]{E}}(x) \\ &\geq \bigwedge_{\Psi(z)=y} \nu_{\Psi(\sqrt[mn]{E})}(y) = \nu_{\Psi(\sqrt[mn]{E})}(y). \end{aligned}$$

Thus,

$$\mu_{\sqrt[mn]{\Psi(E)}} \subseteq \mu_{\Psi(\sqrt[mn]{E})}, \qquad \nu_{\sqrt[mn]{\Psi(E)}} \supseteq \nu_{\Psi(\sqrt[mn]{E})},$$

and therefore $\sqrt[mn]{\Psi(E)} \subseteq \Psi(\sqrt[mn]{E})$.

On the other hand, since Ψ is surjective, by the definition of supremum there exists $x \in X$ such that

$$\mu_{\Psi(\sqrt[mn]{E})}(y)-\varepsilon \le \mu_{\sqrt[mn]{E}}(x) = \bigvee_{k\ge 1}\mu_E(x^k), \qquad \nu_{\Psi(\sqrt[mn]{E})}(y)+\varepsilon \ge \nu_{\sqrt[mn]{E}}(x) = \bigwedge_{k\ge 1}\nu_E(x^k).$$

Moreover,

$$\mu_{\Psi(\sqrt[mn]{E})}(y) = \bigvee_{\Psi(x)=y}\mu_{\sqrt[mn]{E}}(x) = \bigvee_{k\ge 1}\bigvee_{\Psi(x)=y}\mu_E(x^k),$$

and

$$\nu_{\Psi(\sqrt[mn]{E})}(y) = \bigvee_{\Psi(x)=y}\nu_{\sqrt[mn]{E}}(x) = \bigwedge_{k\ge 1}\bigvee_{\Psi(x)=y}\nu_E(x^k).$$

Thus, for some $m \in \mathbb{N}$,

$$\mu_{\Psi(\sqrt[mn]{E})}(y)-\varepsilon \le \bigvee_{\Psi(z)=y^m}\mu_E(z) = \mu_{\Psi(E)}(y^m) \le \bigvee_{k\ge 1}\mu_{\Psi(E)}(y^k) = \mu_{\sqrt[mn]{\Psi(E)}}(y),$$

$$\nu_{\Psi(\sqrt[mn]{E})}(y)+\varepsilon \ge \bigwedge_{\Psi(w)=y^m}\nu_E(w) = \nu_{\Psi(E)}(y^m) \ge \bigwedge_{k\ge 1}\nu_{\Psi(E)}(y^k) = \nu_{\sqrt[mn]{\Psi(E)}}(y).$$

Consequently,

$$\mu_{\Psi(\sqrt[mn]{E})} \subseteq \mu_{\sqrt[mn]{\Psi(E)}}, \qquad \nu_{\Psi(\sqrt[mn]{E})} \supseteq \nu_{\sqrt[mn]{\Psi(E)}},$$

and thus,

$$\Psi(\sqrt[mn]{E}) = \sqrt[mn]{\Psi(E)}.$$

□

5 Conclusion

This study introduced and systematically analyzed the concept of generalized (m,n)-fuzzy BL-subalgebras as an advanced framework extending classical fuzzy algebraic structures. By parameterizing the membership and non-membership degrees through the integers m and n, the proposed model unifies and generalizes several established fuzzy subalgebras, including intuitionistic, Pythagorean, Fermatean, and q-rung orthopair fuzzy BL-subalgebras. The theoretical developments presented here contribute significantly to the algebraic foundation of fuzzy logic and its applications in uncertain reasoning.

The principal achievements of this research can be summarized as follows:

- The introduction of power-implication preserving (PIP) BL-algebras, which establish a structured algebraic mechanism for scaling implications while preserving logical consistency. It was proven that a PIP BL-algebra exists for every prime number, creating an elegant intersection between number theory and fuzzy algebra.
- A unified framework for (m, n)-fuzzy BL-subalgebras was developed, ensuring closure under essential algebraic operations such as intersection and union. This closure property enhances the robustness and internal consistency of the fuzzy algebraic system.
- Novel relationships between algebraic properties and their corresponding level-cut representations were established, bridging the gap between fuzzy semantics and crisp algebraic characterizations.
- The concept of (m, n)-fuzzy nil radical BL-subalgebras was formulated and examined. Theorems regarding their homomorphic images demonstrate algebraic stability and invariance under structural mappings.

Future investigations may focus on:

- Exploring computational algorithms for determining (m, n)-fuzzy BL-subalgebra membership functions in high-dimensional optimization problems.
- Extending the proposed framework to hybrid fuzzy–neutrosophic systems and their optimization behavior.
- Investigating real-world control and decision models where parameterized fuzziness (via m, n) offers measurable performance benefits.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

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Competing Interests

The authors declare that they have no competing interests relevant to the content of this paper.

Authors' Contributions

All authors contributed equally to the design of the study, data analysis, and writing of the manuscript, and share equal responsibility for the content of the paper.

Contribution of AI Tools

The authors used AI-based tools solely as part of their writing and editing workflow. Specifically, AI-assisted capabilities were employed to improve language quality, clarity, grammar, and stylistic consistency. The authors did not rely on AI to generate original scientific content, data, analyses, or interpretations.

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