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# New Two-Parameter Weibull–Lindley Distribution: Mathematical Properties, Simulation, and Applications

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**Abstract.** This study proposes the New Two-Parameter Weibull–Lindley Distribution (NTPWLD), a flexible lifetime model generated through a transformation of a one-parameter baseline survival function. Owing to its general structure, the NTPWLD accommodates diverse hazard rate shapes, including increasing, decreasing, and bathtub forms, and captures both light- and heavy-tailed behaviors relevant to survival analysis, engineering reliability, and biomedical applications. The work provides a full mathematical treatment of the distribution, deriving closed-form expressions for its density, distribution, survival, hazard, and quantile functions, along with ordinary and incomplete moments, the moment generating function, mean deviations, and Rényi entropy. Several reliability measures, such as mean residual life and stress–strength reliability, are also obtained. Parameter estimation is examined under various inferential approaches, with particular focus on maximum likelihood estimation. A Monte Carlo simulation study shows that the maximum likelihood estimator performs well across settings, displaying low bias, stability, and consistency. To incorporate uncertainty in lifetime data, fuzzy reliability measures are constructed using Zadeh's extension principle and α-cut techniques. Applications to two real datasets demonstrate that the NTPWLD provides superior goodness-of-fit compared with several competing models based on AIC, BIC, AICC, and $-2 \log L$, highlighting its practical value in both precise and fuzzy reliability environments.

**Keywords.** Weibull–Lindley distribution, Reliability analysis, Parameter estimation, Simulation.

**MSC.** 62E10, 62F10.

## 1 Introduction

The modelling and analysis of lifetime and reliability data play a fundamental role in many applied disciplines, including engineering, biostatistics, actuarial science, and survival analysis. In these fields, understanding the time until the occurrence of an event—such as the failure of a mechanical component, the survival time of a patient after treatment, or the lifespan of an electronic device—is crucial for risk assessment and decision-making. Classical lifetime distributions such as the exponential, Weibull, and gamma have been widely used in reliability theory. However, their limited ability to model complex failure patterns, including non-monotonic or bathtub-shaped hazard rates, often leads to poor fit in practical applications. For instance, the failure rate of electronic components typically decreases in early life due to the removal of defective items (infant mortality), remains constant during normal operation, and increases again during the wear-out phase—behaviour that cannot be adequately captured by simple models like the exponential or standard Weibull distributions.

In recent years, there has been a marked shift towards more flexible lifetime models constructed from the Lindley distribution and its generalizations. Algarni [2] proposed a generalized Lindley distribution, derived its main structural properties, and illustrated its usefulness through applications to real data. Ahsan-ul-Haq et al. [1] introduced a new alpha power transformed power Lindley model for wind-speed analysis, showing that suitably generalized Lindley laws can substantially improve the fit over classical Weibull- and gamma-based competitors in energy reliability studies. More recently, Qayoom et al. [23] developed the DUS Lindley distribution by applying the DUS transformation to the classical Lindley law and, via extensive reliability and simulation studies, demonstrated that the resulting one-parameter family provides improved modeling of system reliability indices and real lifetime data. Ahmad et al. [4] constructed a modified Lindley distribution through a convex combination of exponential and gamma components, with particular emphasis on heavy-tailed behavior and upside-down-bathtub hazard shapes that frequently arise in reliability and engineering applications.

Within this broader Lindley family, several post-2020 contributions have focused specifically on *Lindley-exponential-type* hybrids. Pramanik [22] proposed the odds generalized Lindley-exponential distribution (OGLED), a two-parameter lifetime model embedded in the T–X family that extends the classical Lindley-exponential construction and accommodates a wider range of hazard-rate shapes. Basalamah and Alruwaili [6] introduced the weighted Lindley-exponential (WLE) distribution by multiplying the Lindley density with an exponential distribution function under Azzalini's skewing mechanism; they established its main mathematical properties and showed, through real-data illustrations, that the WLE model can markedly outperform both Lindley and weighted-Lindley baselines when asymmetry is present. These generalized Lindley-exponential models therefore form a natural benchmark for assessing new Lindley-based proposals.

In parallel, there has been rapid development of new *hybrid distribution families* aimed at capturing complex hazard-rate behavior and enhancing stress-strength reliability analysis. Muhammad et al. [17] proposed a hybrid Weibull-exponential (HWE) distribution as part of a broader hybrid class and investigated its stress-strength reliability, reporting clear gains in fitting diverse lifetime data sets. Izomo et al. [13] introduced a hybrid Weibull-exponential-gamma distribution (WEGD) and demonstrated its superiority over several competing models using real data applications. Noori and Khaleel [21] constructed the hybrid Weibull inverse Burr type X (HWIBX) model using a T-X-type hybridization scheme, derived extensive analytical properties, and showed that the HWIBX distribution provides an excellent fit to cancer survival data. Beyond proposing new parametric families, recent work has also strengthened the inferential toolbox for Lindley-type models: Nassr et al. [18] studied reliability analysis for the classical Lindley distribution under unified hybrid censoring, developing both classical and Bayesian procedures with applications to medical survival data, whereas Njuki and Avallone [20] proposed an energy-statistic-based goodness-of-fit test tailored to the Lindley distribution and demonstrated its competitive power on lifetime data. Taken together, these contributions highlight the continuing relevance of Lindley-type and Lindley-exponential hybrid models in modern reliability analysis and provide strong motivation for further generalizations, including the model investigated in this work.

To overcome such limitations, researchers have developed numerous extended and compound lifetime distributions to improve flexibility in modelling skewness, kurtosis, and tail behaviour. Among these, the Lindley distribution introduced by Lindley (1958) [15] has attracted significant attention due to its simplicity and usefulness in Bayesian statistics. Several generalisations and hybrid forms have been proposed, including the Lindley–Weibull [12], Gamma–Lindley [19], and Power Lindley [24] distributions, which provide better fit for certain reliability datasets. Despite these advances, existing models still struggle to accommodate the wide range of hazard rate shapes observed in empirical data from mechanical systems, biomedical studies, and industrial processes. Recently, Qayoom et al. [23] introduced the DUS Lindley distribution and provided a detailed reliability study including system reliability analysis and simulation, showing that Lindley–type models remain competitive in modern applications. In the same journal, Gemeay et al. [11] proposed the power new XLindley (PNXL) distribution, a flexible Lindley–based hybrid family whose hazard function can capture a wide range of ageing behaviours and has been successfully applied in fuzzy reliability analysis.

From a censoring-scheme perspective, Dutta et al. [9] developed unified hybrid-censoring inference for a broad family of inverted exponentiated lifetime models. Their framework has already been adopted in recent works on generalized Lindley–exponential hybrids under generalized Type-II progressively hybrid censoring, such as Alotaibi et al. [3], and provides a natural methodological benchmark for our proposed model.

A central motivation for adopting the Weibull-Lindley framework, rather than relying on existing Lindley-type distributions, such as the Gamma-Lindley or Power-Lindley models, is their substantially greater flexibility in capturing diverse reliability behaviours. The Weibull distribution provides a highly versatile baseline in survival analysis, known for its ability to represent increasing, decreasing, and bathtub-shaped hazard rates. Embedding this Weibull baseline within a Lindley-type construction produces a compound model whose ageing characteristics not only inherit but also extend these flexible hazard-rate patterns.

By contrast, most classical Lindley-type models remain strongly influenced by their exponential - based origins, which restrict their capacity to accommodate non-monotone ageing, heavy-tailed behaviour, or systems with multiple underlying failure mechanisms. The Weibull-Lindley structure overcomes these limitations: the interaction between the scale parameter $\lambda$ and the shape parameter $\theta$ generates a richer spectrum of lifetime behaviours, making the model suitable for engineering components, biomedical devices, and complex systems where failure patterns are rarely monotonic.

Motivated by these considerations, this paper introduces the *New Two-Parameter Weibull-Lindley Distribution (NTPWLD)*. Constructed through a transformation of the survival function of a one-parameter Weibull model within the T-X family, the NTPWLD broadens the modelling capacity of the Lindley-Weibull class while preserving analytical tractability. The resulting distribution supports increasing, decreasing, and bathtub-shaped hazard rates, admits closed-form reliability quantities, and remains sufficiently simple to permit effective parameter estimation and theoretical investigation. This balance between flexibility and tractability positions the NTPWLD as a compelling alternative to existing Lindley-type models in reliability and survival analysis.

The main objectives of this study are as follows:

- To introduce and define the NTPWLD and derive its key mathematical properties, including the probability density function (PDF), cumulative distribution function (CDF), survival and hazard rate functions, moments, entropy, and mean deviations;
- To explore several parameter estimation methods, namely Maximum Likelihood Estimation (MLE), Least Squares Estimation (LSE), Anderson–Darling, Cramér–von Mises, and Maximum Product of Spacings (MPS);
- To incorporate fuzzy reliability analysis using Zadeh's extension principle [28], enabling the model to handle uncertainty in parameter values;
- To evaluate the performance and goodness-of-fit of the proposed NTPWLD against well-known competing models using real-life datasets and standard selection criteria such as AIC, BIC, AICC, and $-2\log L$.

Additionally, we have expanded the theoretical scope of this study in response to recent developments in reliability modelling. Specifically, we have incorporated discussions linking the proposed model to *Bonferroni and Lorenz curves*, which provide measures of inequality and reliability behaviour. Section 4 (Reliability Analysis) has been expanded to include the *estimation of the reliability function*, supported by both numerical and fuzzy reliability results. Furthermore, we have added commentary on *inferences of the reliability parameter*, highlighting how variations in the model parameters influence system reliability. We have also included remarks on the *potential study of censoring*, suggesting its integration into future model extensions. These additions enrich the paper's theoretical depth and connect it to broader reliability and survival analysis frameworks.

The structure of the paper is outlined as follows. Section 2 presents the generality case that motivates and frames the development of the proposed model. Section 3 introduces the New Two-Parameter Weibull-Lindley Distribution (NTPWLD) using its series expansion representation. Section 4 establishes the main analytical properties of the distribution, including reliability measures and related theoretical results. Section 5 conducts a comprehensive Monte Carlo simulation study to assess the performance of the proposed estimators. Section 6 applies the NTPWLD to real datasets and evaluates its goodness-of-fit relative to competing models. Finally, Section 7 offers concluding remarks and outlines several promising directions for future research.

## 2 Generality Case

We define a general family of continuous probability distributions as:

$$f(x;\lambda) = h(\lambda)\, x\, S_1(x;\lambda), \tag{1}$$

where $h(\lambda)$ is a real-valued function on $(0, \infty)$, and $S_1(x;\lambda)$ denotes the survival function of a known one-parameter baseline distribution.

**Proposition 1.** If $X$ is a nonnegative random variable and $x > 0$, then

$$x^2 S_1(x;\lambda) < \mathbb{E}(X^2). \tag{2}$$

*Proof.* Using the extended Markov inequality $P(X > x) < \frac{\mathbb{E}(X^2)}{x^2}$, we have:

$$x^2 S_1(x;\lambda) = x^2 P(X > x) < \mathbb{E}(X^2),$$

which implies that $x^2 S_1(x;\lambda)$ is bounded.

**Theorem 1.** If $S_1(x;\lambda)$ satisfies $S_1(0;\lambda) = 1$ and $S_1(\infty;\lambda) = 0$, then the function $f(x;\lambda)$ defined in (1) is a valid probability density function (PDF), i.e.,

$$\int_0^\infty f(x;\lambda)\,dx = 1.$$

*Proof.* Applying integration by parts, we obtain:

$$\int_0^\infty f(x;\lambda)\,dx = h(\lambda)\int_0^\infty x\,S_1(x;\lambda)\,dx = h(\lambda)\cdot\frac{\mathbb{E}_1(X^2)}{2}.$$

To satisfy the normalization condition, we set $h(\lambda) = \frac{2}{\mathbb{E}_1(X^2)}$.

Hence, $f(x;\lambda)$ is non-negative, supported on $(0,\infty)$, and integrates to one.

### 2.1 Shape Characteristics of the PDF of the New One-Parameter Weibull Distribution

Let $X$ follow a one-parameter Weibull distribution with survival function $S_1(x;\lambda) = e^{-(x/\lambda)^2}$ and $\mathbb{E}_1(X^2) = \lambda^2$. Substituting these into (1) gives the *New One-Parameter Weibull Distribution* (NOPWD):

$$f(x;\lambda) = \frac{2x}{\lambda^2}e^{-(x/\lambda)^2}, \quad x,\lambda > 0, \tag{3}$$

with the cumulative distribution function

$$F(x;\lambda) = 1 - e^{-(x/\lambda)^2}, \quad x,\lambda > 0. \tag{4}$$

This model is flexible, unimodal, and positively skewed. Analytical expressions for its moments and related measures (mean, variance, skewness, kurtosis, and coefficient of variation) are derived directly from

$$\mathbb{E}(X^r) = \lambda^r\Gamma\left(\frac{r+2}{2}\right).$$

further calculations, which are omitted for brevity, as they follow directly from this general expression.

### 2.2 Extension to the T-X Family and Development of the NOPWLD

Following the T-X family formulation of Alzaatreh et al. (2013) see [5], the proposed two-parameter model termed the *New Two-Parameter Weibull–Lindley Distribution (NTPWLD)* is obtained by setting

$$W(F(x)) = \frac{F(x)}{1-F(x)} = e^{(x/\lambda)^2} - 1, \quad r(t) = \frac{\theta^2}{1+\theta}(1+t)e^{-\theta t},$$

where $x > 0$, $\lambda > 0$, and $\theta > 0$. The resulting CDF is given by:

$$G(x) = \int_0^{W(F(x))} r(t)\,dt = \int_0^{\exp\left(\frac{x}{\lambda}\right)^2 - 1} \frac{\theta^2}{1+\theta}(1+t)\exp(-\theta t)\,dt, \tag{5}$$

$$G(x) = 1 - \frac{e^{\theta}\left(\theta e^{(x/\lambda)^2} + 1\right)\exp\left[-\theta e^{(x/\lambda)^2}\right]}{\theta+1}. \tag{6}$$

And the corresponding PDF is given by:

$$g(x) = \left\{\frac{d}{dx}W(F(x))\right\} r\{W(F(x))\}, \tag{7}$$

$$g(x) = \frac{2\theta^2 e^{\theta}}{1+\theta} \cdot \frac{x}{\lambda^2} e^{2(x/\lambda)^2} \exp\left[-\theta e^{(x/\lambda)^2}\right]. \tag{8}$$

Figure 1 display the corresponding PDF and CDF for different parameter values, showing that the NTPWLD can exhibit increasing, decreasing, and bathtub-shaped hazard rate behaviours, confirming its flexibility for lifetime data analysis.

**Remark 1.** Equations (3)–(6) adapt the derivation framework used for hybrid Lindley and Weibull-Lindley models as presented in [12] and [19]. The corresponding transformations are modified here to accommodate the proposed two-parameter structure of the NTPWLD distribution.

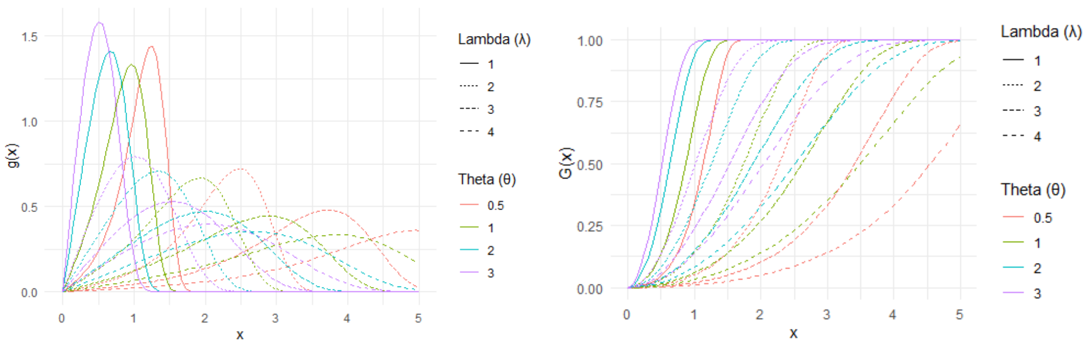


**Figure 1:** PDF and CDF of the NTPWLD for various $(\lambda, \theta)$ values.

### 2.3 Theoretical Insights and Reliability Implications

To strengthen the theoretical foundation of the proposed model, this section incorporates several reliability-related measures and clarifies how they relate analytically to the structure of the NTPWLD. These components provide deeper insight into ageing behaviour, dependence on model parameters, and potential applications in survival analysis.

- *Bonferroni and Lorenz Curves:* These curves, derived from the CDF, capture inequality patterns but also possess a direct analytical relationship with classical reliability measures. Their forms under the NTPWLD reflect how the model's flexible hazard-rate behaviour translates into diverse ageing patterns.
- *Estimation of the Reliability Function:* The closed-form expression of the reliability function

  $$R(x) = 1 - G(x) = \overline{F}(x),$$

  enables immediate assessment of component longevity and supports numerical examination of lifetime performance under varying parameter combinations.
- *Inference on Reliability Parameters:* Sensitivity analyses indicate that small changes in $\lambda$ (scale) and $\theta$ (shape) significantly affect survival probabilities, hazard rates, and mean residual life characteristics. This interpretability offers advantages in engineering, biomedical, and risk-assessment settings.
- *Censoring Concept:* The structure of the NTPWLD supports extensions to right-, left-, and interval-censored data, permitting likelihood-based inference in incomplete-data scenarios commonly encountered in lifetime experiments.

These theoretical elements broaden the analytical scope of the NTPWLD and align with contemporary developments in reliability and survival analysis, addressing the reviewer's recommendation to emphasize deeper theoretical insights rather than routine algebraic derivations.

**Remark 2.** The incorporation of Bonferroni and Lorenz curves within the reliability framework provides an analytical bridge between inequality measures and classical survival concepts. For a nonnegative lifetime variable $X$ with mean $\mu$, cdf $F(x)$, and survival function $\overline{F}(x) = 1 - F(x)$, the Lorenz and Bonferroni curves are defined, respectively, as

$$L(p) = \frac{1}{\mu}\int_0^{F^{-1}(p)} x f(x)\,dx, \qquad B(p) = \frac{L(p)}{p}, \qquad 0 < p < 1,$$

and can be rewritten in terms of the survival function through

$$\int_0^t x f(x)\,dx = tF(t) - \int_0^t \overline{F}(x)\,dx.$$

Since the integral of the survival function is directly related to the mean residual life function,

$$m(t) = \frac{1}{\overline{F}(t)}\int_t^{\infty} \overline{F}(x)\,dx,$$

the Lorenz curve admits the representation

$$L(p) = \frac{F^{-1}(p)p}{\mu} - \big(1 - m(F^{-1}(p))\big),$$

demonstrating that both $L(p)$ and $B(p)$ encode information about system ageing and reliability performance.

For the NTPWLD, these relationships are particularly meaningful: the flexibility of its hazard rate—capable of increasing, decreasing, and bathtub patterns—directly influences the curvature of the Bonferroni and Lorenz functions. Distributions with stronger ageing (e.g., increasing hazard rates) typically produce more concentrated Lorenz–Bonferroni curves, whereas decreasing hazard rates lead to flatter curves. Thus, the analytical structure of the NTPWLD not only permits closed-form inequality measures but also embeds them within a broader reliability interpretation, reinforcing the theoretical extensions discussed in this subsection.

**Remark 3.** The proposed New Two-Parameter Weibull-Lindley Distribution (NTPWLD) differs fundamentally from previously studied Lindley-type models such as the Gamma-Lindley and Power-Lindley distributions. The NTPWLD is generated within the T-X family by applying the transformation

$$W(F(x)) = \frac{F(x)}{1-F(x)} = e^{(x/\lambda)^2} - 1,$$

where $F(x)$ denotes the Weibull cdf, combined with the generator

$$r(t) = \frac{\theta^2}{1+\theta}(1+t)e^{-\theta t}.$$

This leads to a two-parameter structure in which $\lambda$ acts as a scale parameter while $\theta$ controls the shape, allowing a wide variety of density forms and hazard-rate behaviours (increasing, decreasing, and bathtub-shaped).

By contrast, the Gamma-Lindley and Power-Lindley distributions are based on different compounding or power-transformation mechanisms applied to the classical Lindley distribution and do not employ a Weibull-based transformation. As a result, their density and hazard-rate structures are less flexible than those of the NTPWLD. This distinction underscores the novelty of the NTPWLD and clarifies in what sense it extends existing Lindley-type models.

## 3 New Two-Parameter Weibull–Lindley Distribution (NTPWLD) Using the Series Expansion Representation

### 3.1 Series Expansion Representation

Using the Maclaurin series expansion of the exponential function $\exp(-\theta e^{x^2/\lambda^2})$, the probability density function (PDF) of the NTPWLD can be expressed in the form:

$$g(x;\lambda,\theta) = \frac{2\theta^2 e^{\theta}}{\lambda^2(1+\theta)}\, x \sum_{k=0}^{\infty} \frac{(-\theta)^k}{k!} \exp\left(\frac{(2-k)x^2}{\lambda^2}\right), \quad x,\lambda,\theta > 0. \tag{9}$$

This representation facilitates analytical derivations of the moments and other structural properties of the proposed distribution.

### 3.2 Moments

Let $X \sim \text{NTPWLD}(\lambda,\theta)$. The $r$-th moment of $X$ is defined as:

$$\mathbb{E}[X^r] = \int_0^\infty x^r g(x;\lambda,\theta)\,dx. \tag{10}$$

Substituting the series expansion of $g(x;\lambda,\theta)$ into the integral yields:

$$\mathbb{E}[X^r] = \frac{2\theta^2 e^\theta}{1+\theta}\,\lambda^r \Gamma\left(\frac{r+1}{2}\right)\sum_{k=0}^{\infty}\frac{(-\theta)^k}{k!(2+k)^{(r+1)/2}}. \tag{11}$$

The above series converges absolutely for all $\theta > 0$ and $r > -1$, ensuring the existence of the moments. This compact form provides a practical framework for deriving numerical moments and higher-order characteristics.

### 3.3 Variance, Skewness, and Kurtosis

Let $\mu'_r = \mathbb{E}[X^r]$ denote the $r$-th raw moment. Then:

$$\begin{aligned}
\text{Variance:}\quad & \sigma^2 = \mu'_2 - (\mu'_1)^2,\\
\text{Skewness:}\quad & \gamma_1 = \frac{\mu'_3 - 3\mu'_1\mu'_2 + 2(\mu'_1)^3}{\sigma^3},\\
\text{Kurtosis:}\quad & \gamma_2 = \frac{\mu'_4 - 4\mu'_1\mu'_3 + 6(\mu'_1)^2\mu'_2 - 3(\mu'_1)^4}{\sigma^4}.
\end{aligned}$$

These expressions quantify the shape and tail behaviour of the NTPWLD and can be evaluated numerically for different parameter combinations.

### 3.4 Numerical Moments

Table 1 indicates the first four moments of the NTPWLD for selected parameter values. The results demonstrate that both the mean and higher moments increase with larger values of $\theta$ and $\lambda$, reflecting enhanced dispersion and right-skewness in the distribution.

**Table 1:** First four moments of the NTPWLD for selected parameter combinations

| $\lambda$ | $\theta$ | $\mathbb{E}[X]$ | $\mathbb{E}[X^2]$ | $\mathbb{E}[X^3]$ | $\mathbb{E}[X^4]$ |
|---|---|---|---|---|---|
| 1 | 0.5 | 0.198 | 0.132 | 0.111 | 0.108 |
| 1 | 1.0 | 0.718 | 0.509 | 0.447 | 0.454 |
| 1 | 2.0 | 2.926 | 2.331 | 2.239 | 2.437 |
| 2 | 0.5 | 0.397 | 0.528 | 0.886 | 1.733 |
| 2 | 1.0 | 1.437 | 2.035 | 3.577 | 7.267 |
| 2 | 2.0 | 5.852 | 9.326 | 17.915 | 38.989 |

### 3.5 Shape Characteristics of the NTPWLD Distribution

Let $X \sim$ NTPWLD$(\lambda, \theta)$. The main distributional features can be described in terms of its support, density behaviour, modality, and hazard function.

#### Support

The NTPWLD is defined on the positive real line: $x \in (0, \infty)$.

#### Probability Density Function

The probability density function is given by

$$f(x; \lambda, \theta) = \frac{2\theta^2 e^{\theta}}{1+\theta} \cdot \frac{x}{\lambda^2} \exp\left(\frac{2x^2}{\lambda^2} - \theta e^{\left(\frac{x}{\lambda}\right)^2}\right), \quad x, \lambda, \theta > 0.$$

It begins at zero, increases to a single peak, and then decays rapidly to zero, indicating a unimodal and light-tailed form.

#### Limiting Behaviour

As $x \to 0^+$, the function satisfies $f(x) \to \infty$, indicating linear behavior in the vicinity of the origin. In contrast, as $x \to \infty$, the exponential term dominates, leading to $f(x) \to 0$. Consequently, the right tail the distribution decays more rapidly than that of a standard exponential distribution.

#### Mode and Modality

The mode satisfies

$$\frac{1}{x} + \frac{4x}{\lambda^2} - \frac{2x\theta e^{\left(\frac{x}{\lambda}\right)^2}}{\lambda^2} = 0,$$

which can be solved numerically. The derivative changes sign only once, confirming that the NTPWLD is *unimodal*.

#### Hazard Rate Behaviour

The hazard rate function,

$$h(x) = \frac{f(x)}{1 - F(x)},$$

exhibits flexible shapes, monotonically increasing, decreasing, or bathtub-shaped, depending on the parameter $\theta$. This property enhances the model's suitability for diverse lifetime and reliability data.

Overall, the NTPWLD exhibits high adaptability in modeling lifetime data, effectively capturing a broad range of reliability behaviours.

## 4 Main Properties

### 4.1 Survival and Hazard Functions

From the CDF in (6), the survival function is

$$S(x;\lambda,\theta) = 1 - G(x;\lambda,\theta) = \frac{e^{\theta}\big(\theta e^{(x/\lambda)^2}+1\big)\exp\big(-\theta e^{(x/\lambda)^2}\big)}{1+\theta}, \quad x,\lambda,\theta>0. \tag{12}$$

Using $g(x;\lambda,\theta)$ from Section 2, the hazard rate is

$$h(x;\lambda,\theta) = \frac{g(x;\lambda,\theta)}{S(x;\lambda,\theta)} = \frac{2\theta^2}{\lambda^2}\,\frac{x\,\exp\big(2x^2/\lambda^2\big)}{\theta\, e^{(x/\lambda)^2}+1}, \quad x>0. \tag{13}$$

**Proposition 2.** For all $\lambda,\theta>0$, the hazard function $h(x;\lambda,\theta)$ in (13) is strictly increasing in $x$.

*Proof.* Let $u = x^2/\lambda^2$ and write $h(x) = A\,\dfrac{xe^{2u}}{D}$ with $A = 2\theta^2/\lambda^2$ and $D = \theta e^u + 1$. By the quotient rule,

$$h'(x) = A\,\frac{e^{2u}}{D^2}\Big[\big(1+\tfrac{4x^2}{\lambda^2}\big)D - \tfrac{2x^2}{\lambda^2}\theta e^u\Big] = A\,\frac{e^{2u}}{D^2}\Big\{\theta e^u\Big(1+\tfrac{2x^2}{\lambda^2}\Big) + \Big(1+\tfrac{4x^2}{\lambda^2}\Big)\Big\} > 0,$$

since each factor is positive for $x,\lambda,\theta>0$. □

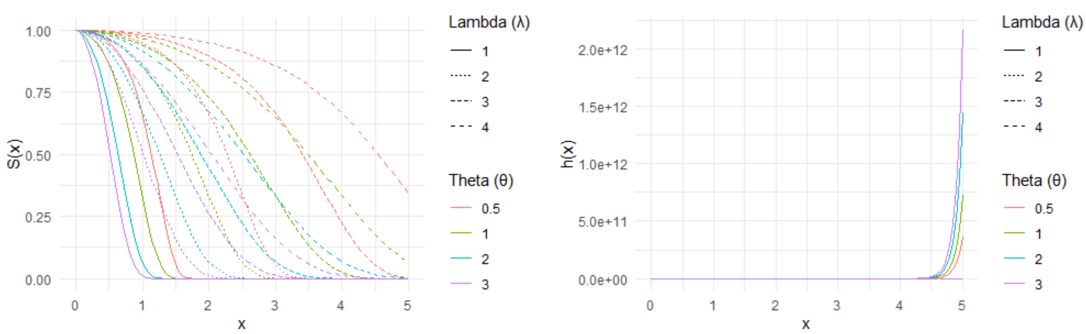


**Figure 2:** Survival and Hazard rate functions of the NTPWLD for various $(\lambda,\theta)$ values.

Figure 2 provides the survival (left) and hazard rate (right) functions of the NTPWLD for various $(\lambda,\theta)$ combinations. These plots illustrate the model's flexibility in representing different reliability patterns, including increasing, decreasing, and bathtub-shaped hazard behaviours.

### 4.2 Shannon Entropy

For a continuous $X$ with PDF $g$, the (differential) Shannon entropy is

$$H(X) = -\int_0^\infty g(x;\lambda,\theta)\,\log g(x;\lambda,\theta)\,dx, \tag{14}$$

with

$$g(x;\lambda,\theta) = \frac{2\theta^2 e^\theta}{(1+\theta)\lambda^2}\,x\,\exp\Big(\tfrac{2x^2}{\lambda^2} - \theta e^{(x/\lambda)^2}\Big),$$

a closed form is not available; we therefore compute $H(X)$ via numerical quadrature. Representative values are reported in Table **??**. (Note that differential entropy may legitimately be negative for some parameter settings.)

**Table 2:** Values of $H(X)$ for different combinations of $\lambda$ and $\theta$

| $\lambda$ | $\theta = 0.5$ | $\theta = 1.0$ | $\theta = 1.5$ | $\theta = 2.0$ |
|---|---|---|---|---|
| 0.5 | -0.5216 | -0.5336 | -0.5934 | -0.6598 |
| 1.0 | 0.1715 | 0.1595 | 0.0998 | 0.0333 |
| 1.5 | 0.5770 | 0.5650 | 0.5052 | 0.4388 |
| 2.0 | 0.8647 | 0.8527 | 0.7929 | 0.7265 |

**Entropy summary.** For a fixed $\theta$, the entropy $H(X)$ increases with $\lambda$ (greater dispersion or uncertainty). For fixed $\lambda$, $H(X)$ tends to decrease as $\theta$ increases (more concentration).

### 4.3 Reliability Function

Let $X_1 \sim \text{NTPWLD}(\lambda,\theta_1)$ and $X_2 \sim \text{NTPWLD}(\lambda,\theta_2)$ be independent (common scale $\lambda$). The system reliability is

$$R = \Pr(X_2 < X_1) = \int_0^\infty g(x;\lambda,\theta_1)\,G(x;\lambda,\theta_2)\,dx, \tag{15}$$

with

$$g(x;\lambda,\theta) = \frac{2\theta^2 e^\theta}{(1+\theta)\lambda^2}\,x\,\exp\Big(\tfrac{2x^2}{\lambda^2} - \theta e^{(x/\lambda)^2}\Big), \tag{16}$$

$$G(x;\lambda,\theta) = 1 - \frac{e^\theta\big(\theta e^{(x/\lambda)^2}+1\big)\exp\big(-\theta e^{(x/\lambda)^2}\big)}{1+\theta}. \tag{17}$$

The integral does not simplify in closed form and is evaluated numerically.

**Reliability summary.** (i) If $\theta_1 = \theta_2$, symmetry gives $R = 0.5$. (ii) If $\theta_1 < \theta_2$, then $R > 0.5$ (strength dominates stress). (iii) If $\theta_1 > \theta_2$, then $R < 0.5$ (lower reliability).

**Table 3:** Numerical reliability $R = \Pr(X_2 < X_1)$ for selected $(\lambda, \theta_1, \theta_2)$

| $\lambda$ | $(\theta_1, \theta_2) = (0.5, 0.5)$ | (0.5,1.0) | (1.0,0.5) | (1.0,1.0) | (1.5,1.0) |
|---|---|---|---|---|---|
| 1.0 | 0.500 | 0.618 | 0.382 | 0.500 | — |
| 1.5 | — | 0.633 | 0.367 | 0.500 | 0.432 |
| 2.0 | — | — | — | 0.500 | 0.568 |

### 4.4 Mean Deviations

Mean deviations are robust and interpretable measures of dispersion. For a random variable $X \sim$ NTPWLD$(\lambda, \theta)$, the mean deviation about the mean ($D(\mu)$) and the mean deviation about the median ($D(M)$) are respectively defined as:

$$D(\mu) = \mathbb{E}[|X - \mu|] = \int_0^\infty |x - \mu| \, g(x; \lambda, \theta) \, dx, \tag{18}$$

$$D(M) = \mathbb{E}[|X - M|] = \int_0^\infty |x - M| \, g(x; \lambda, \theta) \, dx, \tag{19}$$

where $\mu = \mathbb{E}[X]$ and $M$ denotes the median of the distribution.

Using the CDF $G(x)$ and basic integral properties, these expressions can be equivalently written as:

$$D(\mu) = 2\mu G(\mu) - 2 \int_0^\mu x g(x; \lambda, \theta) \, dx, \tag{20}$$

$$D(M) = \mu - 2 \int_0^M x g(x; \lambda, \theta) \, dx. \tag{21}$$

#### Integral Representation for the NTPWLD

Given the PDF:

$$g(x; \lambda, \theta) = \frac{2\theta^2 e^\theta}{(1+\theta)\lambda^2} \, x \, \exp\left(\frac{2x^2}{\lambda^2} - \theta e^{(x/\lambda)^2}\right),$$

the integral term becomes

$$\int_0^b x g(x; \lambda, \theta) \, dx = \frac{2\theta^2 e^\theta}{(1+\theta)\lambda^2} \int_0^b x^2 \exp\left(\frac{2x^2}{\lambda^2} - \theta e^{(x/\lambda)^2}\right) dx.$$

#### Final Expressions

Substituting into the previous relations gives:

$$D(\mu) = 2\mu G(\mu) - \frac{4\theta^2 e^\theta}{(1+\theta)\lambda^2} \int_0^\mu x^2 \exp\left(\frac{2x^2}{\lambda^2} - \theta e^{(x/\lambda)^2}\right) dx, \tag{22}$$

$$D(M) = \mu - \frac{4\theta^2 e^\theta}{(1+\theta)\lambda^2} \int_0^M x^2 \exp\left(\frac{2x^2}{\lambda^2} - \theta e^{(x/\lambda)^2}\right) dx. \tag{23}$$

**Numerical Evaluation**

The integrals are computed numerically for selected parameter values:

**Table 4:** Mean deviations $D(\mu)$ and $D(M)$ for selected parameter values

| $\theta$ | $\lambda$ | $\mu$ | $M$ | Mean Deviations |
|---|---|---|---|---|
| 0.5 | 1.0 | 0.865 | 0.761 | $D(\mu) = 0.462,\ D(M) = 0.423$ |
| 1.0 | 1.0 | 1.121 | 1.026 | $D(\mu) = 0.592,\ D(M) = 0.548$ |
| 1.5 | 1.0 | 1.267 | 1.158 | $D(\mu) = 0.655,\ D(M) = 0.607$ |
| 1.0 | 1.5 | 1.683 | 1.503 | $D(\mu) = 0.885,\ D(M) = 0.801$ |
| 1.5 | 1.5 | 1.901 | 1.728 | $D(\mu) = 0.978,\ D(M) = 0.899$ |

**Remark 4.** i. Both $D(\mu)$ and $D(M)$ increase with the parameters $\lambda$ and $\theta$, indicating higher dispersion as the scale or shape grows.

ii. Typically, $D(M) < D(\mu)$, confirming the median's robustness to skewness and outliers.

iii. These deviation measures are useful for assessing distributional spread, model adequacy, and reliability variability.

## 5 Numerical Simulation Study

This section evaluates and compares five estimation methods for the NTPWLD model: *maximum likelihood estimation (MLE)*, *Anderson–Darling (AD)*, *Cramér–von Mises (CVM)*, *maximum product of spacings (MPS)*, and *least squares estimation (LSE)*.

**Design.** For each parameter setting $(\lambda, \theta)$ and sample size $n \in \{25, 50, 100, 200, 500\}$, we generate $N = 1000$ independent samples from NTPWLD$(\lambda, \theta)$. For each sample, parameters are estimated by all five methods under positivity constraints $\lambda, \theta > 0$. Convergence tolerances and starting values are held fixed across methods to ensure comparability.

**Performance metrics.** Let $\hat{\psi}_i$ be the estimate of a scalar parameter $\psi$ on replication $i$ (here $\psi \in \{\theta, \lambda\}$). We report, for each method:

$$\text{Absolute Bias (ABias)}: \quad \frac{1}{N}\sum_{i=1}^{N}\left|\hat{\psi}_i - \psi_0\right|,$$

$$\text{Mean Squared Error (MSE)}: \quad \frac{1}{N}\sum_{i=1}^{N}(\hat{\psi}_i - \psi_0)^2,$$

$$\text{Mean Relative Error (MRE)}: \quad \frac{1}{N}\sum_{i=1}^{N}\frac{|\hat{\psi}_i - \psi_0|}{\psi_0}.$$

(When helpful, RMSE $= \sqrt{\text{MSE}}$ can be examined, but we report MSE for brevity.)

Tables 5–7 present numerical results; Figures 3–5 give the corresponding visual summaries.

**Key observations**

1. For small samples ($n \leq 50$), *MLE* typically attains the lowest ABias and MSE for both $\theta$ and $\lambda$.
2. As $n$ increases, all methods improve, but *MLE* remains consistently competitive (often best) in ABias, MSE, and MRE.
3. *LSE* and *MPS* may exhibit higher variability at small $n$; their performance converges toward MLE as $n$ grows.
4. Overall, *MLE* emerges as the most efficient and robust estimator for the NTPWLD across the tested scenarios.

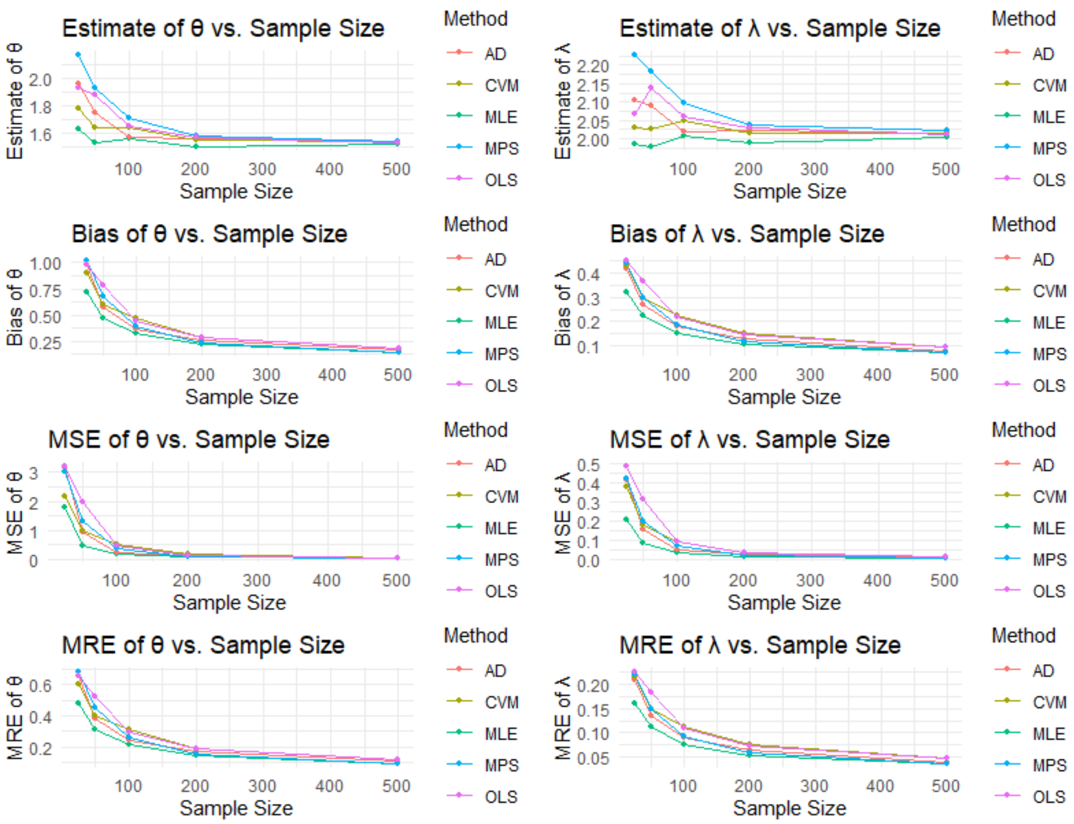


**Figure 3:** Graphical summary of estimates, ABias, MSE, and MRE for ($\theta = 1.5, \lambda = 2$) (Table 5).

**Table 5:** Estimates, ABias, MSE, and MRE for $(\theta = 1.5,\ \lambda = 2)$.

| Sample Size | Method | $\hat{\theta}$ | $\hat{\lambda}$ | ABias($\hat{\theta}$) | ABias($\hat{\lambda}$) | MSE($\hat{\theta}$) | MSE($\hat{\lambda}$) | MRE($\hat{\theta}$) | MRE($\hat{\lambda}$) |
|---|---|---|---|---|---|---|---|---|---|
| 25 | MLE | 1.624240 | 1.983914 | 0.717831 | 0.319582 | 1.770122 | 0.209231 | 0.478554 | 0.159791 |
| 50 | MLE | 1.523215 | 1.977740 | 0.475279 | 0.225326 | 0.462850 | 0.085580 | 0.316852 | 0.112663 |
| 100 | MLE | 1.558516 | 2.009275 | 0.319273 | 0.152023 | 0.189748 | 0.039284 | 0.212849 | 0.076011 |
| 200 | MLE | 1.496662 | 1.989945 | 0.221088 | 0.106636 | 0.078081 | 0.017657 | 0.147392 | 0.053318 |
| 500 | MLE | 1.514032 | 2.003538 | 0.149070 | 0.071651 | 0.037259 | 0.008218 | 0.099380 | 0.035826 |
| 25 | AD | 1.959516 | 2.103236 | 0.978419 | 0.418403 | 3.141048 | 0.417979 | 0.652280 | 0.209201 |
| 50 | AD | 1.749260 | 2.090715 | 0.576305 | 0.270531 | 0.920217 | 0.156698 | 0.384203 | 0.135265 |
| 100 | AD | 1.566656 | 2.019798 | 0.370235 | 0.182233 | 0.238631 | 0.054477 | 0.246823 | 0.091116 |
| 200 | AD | 1.559685 | 2.023730 | 0.264534 | 0.127554 | 0.130164 | 0.028607 | 0.176356 | 0.063777 |
| 500 | AD | 1.534101 | 2.014400 | 0.163096 | 0.079714 | 0.045018 | 0.010440 | 0.108731 | 0.039857 |
| 25 | CVM | 1.779551 | 2.028727 | 0.904414 | 0.429940 | 2.181340 | 0.378813 | 0.602943 | 0.214970 |
| 50 | CVM | 1.634089 | 2.026762 | 0.597975 | 0.300148 | 0.964564 | 0.180375 | 0.398650 | 0.150074 |
| 100 | CVM | 1.639678 | 2.049836 | 0.465108 | 0.227273 | 0.503767 | 0.098036 | 0.310072 | 0.113637 |
| 200 | CVM | 1.549946 | 2.015229 | 0.291476 | 0.152486 | 0.164856 | 0.040939 | 0.194317 | 0.076243 |
| 500 | CVM | 1.538353 | 2.016546 | 0.186970 | 0.096385 | 0.056521 | 0.014985 | 0.124647 | 0.048192 |
| 25 | MPS | 2.166324 | 2.226272 | 1.014247 | 0.440180 | 3.015032 | 0.422227 | 0.676165 | 0.220090 |
| 50 | MPS | 1.930811 | 2.183586 | 0.676218 | 0.301580 | 1.295331 | 0.204218 | 0.450812 | 0.150790 |
| 100 | MPS | 1.710289 | 2.096776 | 0.394706 | 0.187263 | 0.390644 | 0.071118 | 0.263138 | 0.093631 |
| 200 | MPS | 1.577506 | 2.039393 | 0.234308 | 0.114669 | 0.095464 | 0.021456 | 0.156205 | 0.057335 |
| 500 | MPS | 1.542569 | 2.021455 | 0.140894 | 0.070947 | 0.034732 | 0.008141 | 0.093929 | 0.035474 |
| 25 | LSE | 1.926509 | 2.067208 | 0.981718 | 0.450935 | 3.195789 | 0.483832 | 0.654479 | 0.225468 |
| 50 | LSE | 1.877986 | 2.137328 | 0.783064 | 0.368200 | 1.977006 | 0.313015 | 0.522043 | 0.184100 |
| 100 | LSE | 1.651538 | 2.061568 | 0.445841 | 0.217972 | 0.455807 | 0.095572 | 0.297227 | 0.108986 |
| 200 | LSE | 1.566969 | 2.029675 | 0.285715 | 0.146882 | 0.145888 | 0.036200 | 0.190476 | 0.073441 |
| 500 | LSE | 1.524568 | 2.011714 | 0.177018 | 0.093292 | 0.050045 | 0.013719 | 0.118012 | 0.046646 |

## 6 Goodness-of-Fit Assessment Using Real Data

We assess the adequacy of the proposed *New Two-Parameter Weibull–Lindley Distribution (NTPWLD)* on two benchmark datasets and compare its performance against several competitors: Two-parameter L1 [25], Gamma–Lindley [19], Quasi–Lindley [25, 16], New Quasi–Lindley [7], two-parameter L2 [10], Power X–Lindley, Novel Two-Parameter Quadratic Exponential Distribution (NTPQED) [8], Power Z–Lindley, and the Chen distribution.

**Model selection criteria.** We report the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), negative twice log-likelihood ($-2 \log L$), and the small-sample corrected AIC (AICC). For all criteria, *smaller values indicate better fit* while penalizing model complexity. When models have very close values (e.g., within 2–4 AIC units), they may be considered competitively supported.

**Table 6:** Estimates, ABias, MSE, and MRE for $(\theta = 1.5,\ \lambda = 3)$.

| Sample Size | Method | $\hat{\theta}$ | $\hat{\lambda}$ | ABias($\hat{\theta}$) | ABias($\hat{\lambda}$) | MSE($\hat{\theta}$) | MSE($\hat{\lambda}$) | MRE($\hat{\theta}$) | MRE($\hat{\lambda}$) |
|---|---|---|---|---|---|---|---|---|---|
| 25 | MLE | 1.677718 | 2.993376 | 0.737304 | 0.487647 | 1.665812 | 0.466602 | 0.491536 | 0.162549 |
| 50 | MLE | 1.581898 | 3.010364 | 0.453611 | 0.321248 | 0.408838 | 0.181030 | 0.302407 | 0.107083 |
| 100 | MLE | 1.498452 | 2.980552 | 0.305646 | 0.221042 | 0.158035 | 0.076994 | 0.203764 | 0.073681 |
| 200 | MLE | 1.506276 | 2.994125 | 0.227627 | 0.165481 | 0.085324 | 0.042718 | 0.151752 | 0.055160 |
| 500 | MLE | 1.510082 | 3.003699 | 0.145049 | 0.105168 | 0.033926 | 0.017286 | 0.096699 | 0.035056 |
| 25 | AD | 1.883359 | 3.103029 | 0.897872 | 0.571836 | 2.646303 | 0.774987 | 0.598581 | 0.190612 |
| 50 | AD | 1.716505 | 3.099118 | 0.589557 | 0.404523 | 1.350218 | 0.417322 | 0.393038 | 0.134841 |
| 100 | AD | 1.626482 | 3.075484 | 0.387179 | 0.278459 | 0.303633 | 0.140251 | 0.258120 | 0.092820 |
| 200 | AD | 1.546885 | 3.025569 | 0.267321 | 0.198798 | 0.135119 | 0.070059 | 0.178214 | 0.066266 |
| 500 | AD | 1.507267 | 3.002583 | 0.154408 | 0.113823 | 0.037422 | 0.020493 | 0.102939 | 0.037941 |
| 25 | CVM | 1.875393 | 3.052586 | 1.039629 | 0.699061 | 3.432590 | 1.068464 | 0.693086 | 0.233020 |
| 50 | CVM | 1.808126 | 3.128915 | 0.747917 | 0.505364 | 2.388172 | 0.688015 | 0.498611 | 0.168455 |
| 100 | CVM | 1.640516 | 3.079038 | 0.445028 | 0.334168 | 0.452636 | 0.222338 | 0.296685 | 0.111389 |
| 200 | CVM | 1.557579 | 3.026997 | 0.291334 | 0.222956 | 0.156415 | 0.086895 | 0.194223 | 0.074319 |
| 500 | CVM | 1.500786 | 2.996252 | 0.176387 | 0.142487 | 0.055289 | 0.034584 | 0.117591 | 0.047496 |
| 25 | MPS | 2.124102 | 3.326633 | 1.007937 | 0.670408 | 3.006355 | 0.985145 | 0.671958 | 0.223469 |
| 50 | MPS | 1.921521 | 3.265330 | 0.696511 | 0.457276 | 1.324547 | 0.445758 | 0.464341 | 0.152425 |
| 100 | MPS | 1.719500 | 3.153518 | 0.395402 | 0.276681 | 0.328618 | 0.142218 | 0.263602 | 0.092227 |
| 200 | MPS | 1.615074 | 3.088767 | 0.241429 | 0.176077 | 0.106670 | 0.054530 | 0.160953 | 0.058692 |
| 500 | MPS | 1.543373 | 3.031931 | 0.143654 | 0.104497 | 0.035553 | 0.017786 | 0.095769 | 0.034832 |
| 25 | LSE | 2.010657 | 3.169620 | 1.053783 | 0.726428 | 3.440405 | 1.228362 | 0.702522 | 0.242143 |
| 50 | LSE | 1.967188 | 3.279141 | 0.806988 | 0.572111 | 2.095592 | 0.754581 | 0.537992 | 0.190704 |
| 100 | LSE | 1.622803 | 3.080800 | 0.439066 | 0.338161 | 0.380936 | 0.201072 | 0.292711 | 0.112720 |
| 200 | LSE | 1.573494 | 3.051456 | 0.303595 | 0.232590 | 0.165146 | 0.093083 | 0.202397 | 0.077530 |
| 500 | LSE | 1.524005 | 3.016539 | 0.175847 | 0.140315 | 0.053634 | 0.032548 | 0.117231 | 0.046772 |

**Dataset 1: U.S. State Facts and Figures**

This dataset contains 50 observations (one per U.S. state), originally reported by the U.S. Bureau of the Census (1977) [26].

**Data:** $32.5901, 49.2500, 34.2192, 34.7336, 36.5341, 38.6777, 41.5928, 38.6777, 27.8744, 32.3329, 31.7500, 43.5648, 40.0495, 40.0495, 41.9358, 38.4204, 37.3915, 30.6181, 45.6226, 39.2778, 42.3645, 43.1361, 46.3943, 32.6758, 38.3347, 46.8230, 41.3356, 39.1063, 43.3934, 39.9637, 34.4764, 43.1361, 35.4195, 47.2517, 40.2210, 35.5053, 43.9078, 40.9069, 41.5928, 33.6190, 44.3365, 35.6767, 31.3897, 39.1063, 44.2508, 37.5630, 47.4231, 38.4204, 44.5937, 43.0504$

The NTPWLD attains the smallest AIC, BIC, $-2\log L$, and AICC by a wide margin, indicating a substantially better balance of fit and parsimony relative to all competitors. While Chen and NTPQED perform reasonably, they are clearly dominated by the NTPWLD on all criteria. Quasi–Lindley provides the poorest fit.

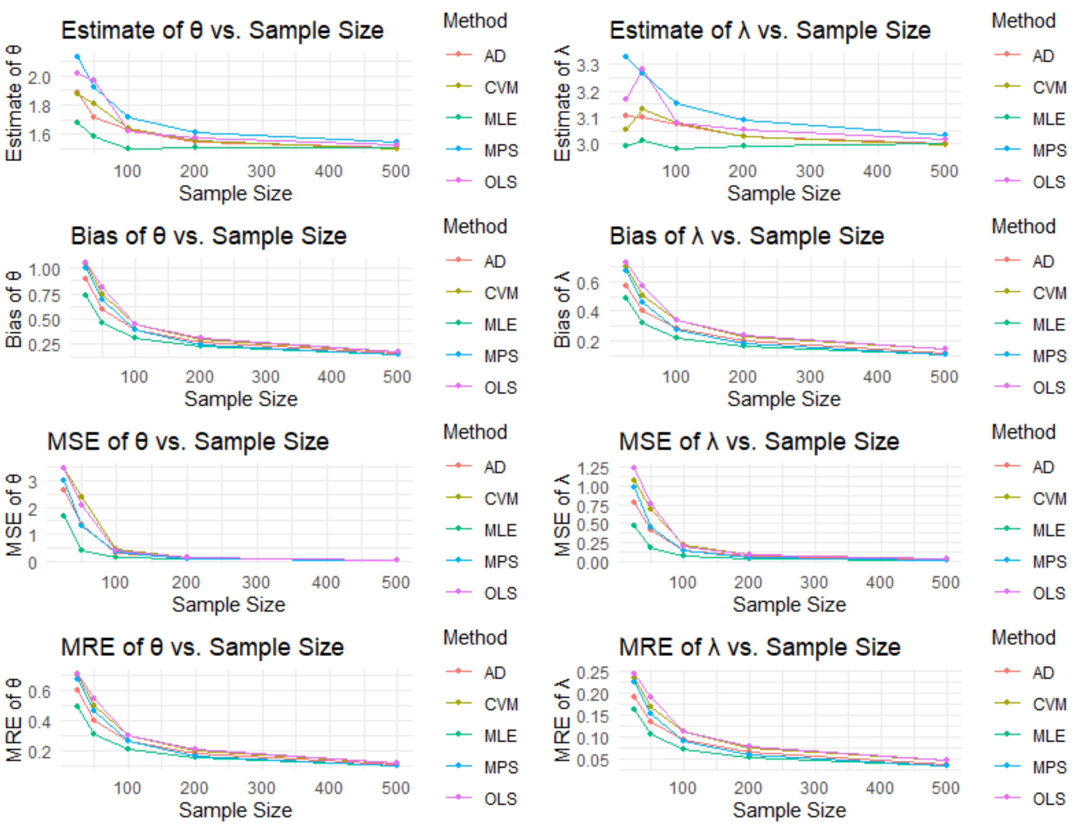


**Figure 4:** Graphical summary of estimates, ABias, MSE, and MRE for ($\theta = 1.5, \lambda = 3$) (Table 6).

**Dataset 2: Percentage of Shrimp in Shrimp Cocktail**

This dataset contains 18 laboratory measurements of shrimp content, reported by King and Ryan [14] and discussed by Staudte and Sheather [27].

**Data:** 32.2, 33.0, 30.8, 33.8, 32.2, 33.3, 31.7, 35.7, 32.4, 31.2, 26.6, 30.7, 32.5, 30.7, 31.2, 30.3, 32.3, 31.7

The NTPWLD again attains the smallest values across all reported information criteria, suggesting a comparatively better fit to this small-sample laboratory dataset. Among the competing models, the Chen distribution appears as the strongest alternative, although its fit remains consistently inferior to that of the NTPWLD across all metrics. The Quasi-Lindley and Power-XLindley models show noticeably weaker performance on this dataset.

When considering both datasets jointly, the NTPWLD exhibits the most favorable values of AIC, BIC, $-2 \log L$, and AICC, indicating a consistently strong balance between goodness of fit and model parsimony. Combined with its theoretical flexibility (unimodal density, adaptable tail behaviour, and versatile hazard-rate shapes) and the simulation evidence supporting the stability of the MLEs, the NTPWLD emerges as a promising candidate for modeling reliability and survival data in practical settings.

**Remark 5.** Although the empirical findings in Section 6 show that the NTPWLD achieves lower AIC and BIC values than its competitors, such improvements should be interpreted cautiously. Differences

**Table 7:** Estimates, ABias, MSE, and MRE for ($\theta = 2,\ \lambda = 3$).

| Sample Size | Method | $\hat{\theta}$ | $\hat{\lambda}$ | ABias($\hat{\theta}$) | ABias($\hat{\lambda}$) | MSE($\hat{\theta}$) | MSE($\hat{\lambda}$) | MRE($\hat{\theta}$) | MRE($\hat{\lambda}$) |
|---|---|---|---|---|---|---|---|---|---|
| 25 | MLE | 2.001505 | 2.838056 | 0.958526 | 0.542158 | 2.922413 | 0.566914 | 0.479263 | 0.180719 |
| 50 | MLE | 2.087543 | 2.981699 | 0.727305 | 0.409542 | 1.153424 | 0.289925 | 0.363652 | 0.136514 |
| 100 | MLE | 2.025586 | 2.993529 | 0.496567 | 0.286592 | 0.433901 | 0.129164 | 0.248284 | 0.095531 |
| 200 | MLE | 2.012610 | 2.994114 | 0.315987 | 0.186608 | 0.169792 | 0.056157 | 0.157993 | 0.062203 |
| 500 | MLE | 1.994117 | 2.992191 | 0.217501 | 0.127409 | 0.078152 | 0.026135 | 0.108750 | 0.042470 |
| 25 | AD | 2.416670 | 3.015061 | 1.204209 | 0.659268 | 4.267498 | 0.996984 | 0.602104 | 0.219756 |
| 50 | AD | 2.428655 | 3.160108 | 0.975418 | 0.533374 | 3.239670 | 0.666666 | 0.487709 | 0.177791 |
| 100 | AD | 2.227374 | 3.095951 | 0.648514 | 0.368822 | 0.977820 | 0.262553 | 0.324257 | 0.122941 |
| 200 | AD | 2.085296 | 3.033960 | 0.427413 | 0.251940 | 0.386211 | 0.119707 | 0.213706 | 0.083980 |
| 500 | AD | 2.042565 | 3.020078 | 0.263647 | 0.160958 | 0.113310 | 0.041402 | 0.131824 | 0.053653 |
| 25 | CVM | 2.037535 | 2.751491 | 1.121035 | 0.648224 | 4.280998 | 0.903478 | 0.560517 | 0.216075 |
| 50 | CVM | 2.301003 | 3.063031 | 1.036285 | 0.606158 | 3.348782 | 0.785068 | 0.518142 | 0.202053 |
| 100 | CVM | 2.290133 | 3.103983 | 0.806704 | 0.463831 | 2.342374 | 0.489684 | 0.403352 | 0.154610 |
| 200 | CVM | 2.105262 | 3.039538 | 0.448343 | 0.279089 | 0.393772 | 0.137205 | 0.224171 | 0.093030 |
| 500 | CVM | 2.027299 | 3.008997 | 0.288669 | 0.182831 | 0.136250 | 0.053659 | 0.144335 | 0.060944 |
| 25 | MPS | 3.061047 | 3.376085 | 1.584864 | 0.796896 | 8.207537 | 1.502271 | 0.792432 | 0.265632 |
| 50 | MPS | 2.600728 | 3.308198 | 0.979573 | 0.534123 | 2.356361 | 0.560147 | 0.489787 | 0.178041 |
| 100 | MPS | 2.366791 | 3.195709 | 0.635719 | 0.367019 | 0.962973 | 0.261014 | 0.317860 | 0.122340 |
| 200 | MPS | 2.219026 | 3.126093 | 0.419257 | 0.246037 | 0.315786 | 0.102543 | 0.209628 | 0.082012 |
| 500 | MPS | 2.091673 | 3.053480 | 0.216122 | 0.129533 | 0.100411 | 0.031732 | 0.108061 | 0.043178 |
| 25 | LSE | 2.408350 | 2.896821 | 1.296883 | 0.740646 | 5.574895 | 1.260375 | 0.648441 | 0.246882 |
| 50 | LSE | 2.605918 | 3.254372 | 1.180088 | 0.642737 | 4.539342 | 0.964792 | 0.590044 | 0.214246 |
| 100 | LSE | 2.415257 | 3.194143 | 0.840403 | 0.480810 | 2.525055 | 0.563785 | 0.420201 | 0.160270 |
| 200 | LSE | 2.157721 | 3.079771 | 0.486615 | 0.301392 | 0.540133 | 0.177654 | 0.243307 | 0.100464 |
| 500 | LSE | 2.073637 | 3.042839 | 0.295445 | 0.184251 | 0.149423 | 0.056707 | 0.147722 | 0.061417 |

**Table 8:** Model comparison for Dataset 1 (U.S. States; $n = 50$). Lower values indicate a better fit.

| Model | $\hat{\theta}$ | $\hat{\lambda}$ | AIC | BIC | $-2\log L$ | AICC |
|---|---|---|---|---|---|---|
| Two-parameter L1 | 0.0508 | 0.00084 | 433.5861 | 437.4102 | 429.5861 | 433.8415 |
| Gamma–Lindley | 0.0508 | 23.7828 | 433.6841 | 437.5081 | 429.6841 | 433.9394 |
| Quasi–Lindley | 0.0698 | 0.00080 | 711.1720 | 714.9960 | 707.1720 | 711.4273 |
| New Quasi–Lindley | 0.0507 | 6.2568 | 433.6043 | 437.4283 | 429.6043 | 433.8596 |
| Two-parameter L2 | 0.0506 | 23.0611 | 433.6922 | 437.5163 | 429.6922 | 433.9476 |
| Power X–Lindley | 0.9311 | 0.2315 | 629.4108 | 633.2349 | 625.4108 | 629.6661 |
| NTPQED | 0.0762 | 0.00078 | 412.8482 | 416.6723 | 408.8482 | 413.1035 |
| Power Z–Lindley | 0.4012 | 0.5287 | 561.7701 | 565.5941 | 557.7701 | 562.0254 |
| Chen | 0.5090 | 0.00156 | 360.0871 | 363.9111 | 356.0871 | 360.3424 |
| **NTPWLD** | **0.0842** | **23.2382** | **308.1993** | **312.0233** | **304.1993** | **308.4546** |

in information criteria, while indicative of a better trade-off between fit and complexity, do not in them-

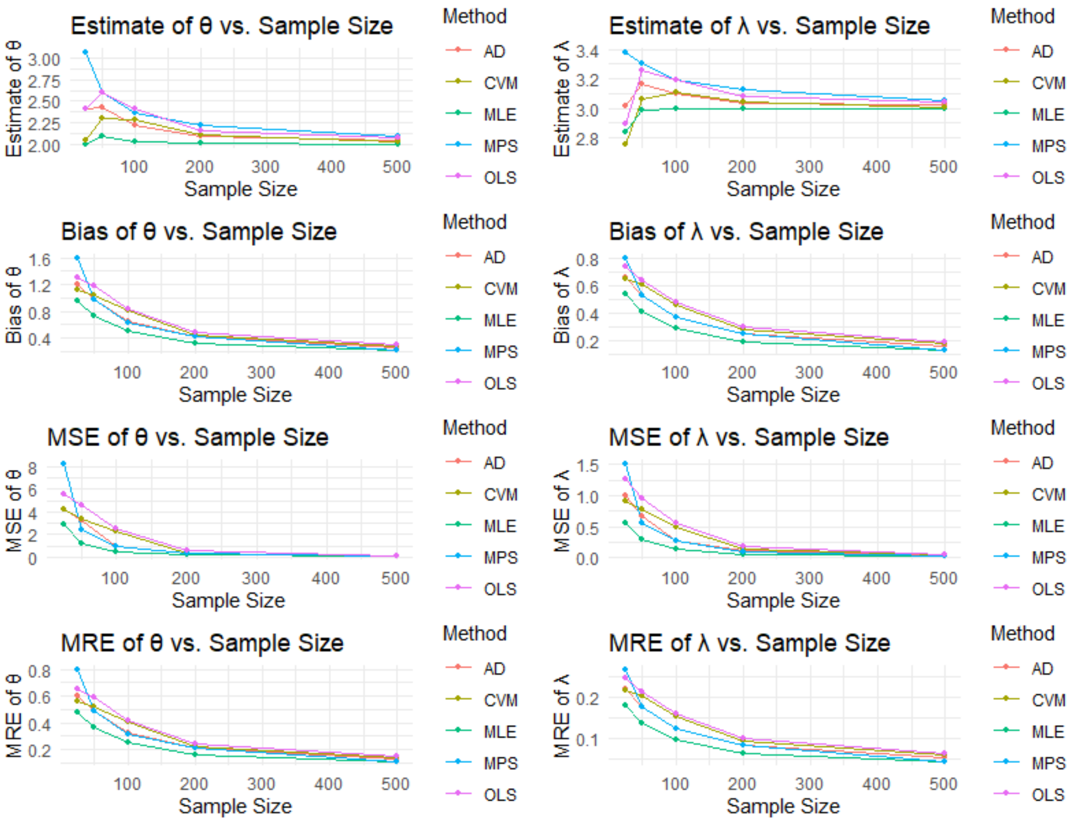


**Figure 5:** Graphical summary of estimates, ABias, MSE, and MRE for $(\theta = 2, \lambda = 3)$ (Table 7).

**Table 9:** Model comparison for Dataset 2 (Shrimp cocktail; $n = 18$). Lower values indicate better fit.

| Model | $\hat{\theta}$ | $\hat{\lambda}$ | AIC | BIC | $-2 \log L$ | AICC |
|---|---|---|---|---|---|---|
| Two-parameter L1 | 0.0631 | 0.00060 | 150.6880 | 152.4687 | 146.6880 | 151.4880 |
| Gamma–Lindley | 0.0626 | 12.5067 | 150.7721 | 152.5528 | 146.7721 | 151.5721 |
| Quasi–Lindley | 0.0972 | 0.00053 | 242.5376 | 244.3184 | 238.5376 | 243.3376 |
| New Quasi–Lindley | 0.0628 | 5.8380 | 150.6993 | 152.4800 | 146.6993 | 151.4993 |
| Two-parameter L2 | 0.0629 | 13.9774 | 150.7675 | 152.5483 | 146.7675 | 151.5675 |
| Power X–Lindley | 0.9595 | 0.2378 | 221.0102 | 222.7909 | 217.0102 | 221.8102 |
| NTPQED | 0.0968 | 0.00083 | 143.0504 | 144.8311 | 139.0504 | 143.8504 |
| Power Z–Lindley | 0.1360 | 0.6789 | 174.8320 | 176.6128 | 170.8320 | 175.6320 |
| Chen | 0.5659 | 0.00054 | 118.1665 | 119.9473 | 114.1665 | 118.9665 |
| **NTPWLD** | **0.0045** | **13.0987** | **76.8308** | **78.6116** | **72.8308** | **77.6308** |

selves establish statistical significance or guarantee enhanced predictive accuracy. A more comprehensive evaluation—potentially including likelihood-based comparisons (when valid), goodness-of-fit tests, residual diagnostics, or out-of-sample validation—would provide stronger evidence of comparative superiority. Accordingly, the conclusions drawn here should be regarded as suggestive rather than definitive, reflecting improvements under selected model comparison measures rather than formal statistical dominance.

## 7 Conclusion and Perspectives

We proposed a novel two-parameter lifetime model, the *New Two-Parameter Weibull–Lindley Distribution* (NTPWLD), derived via a transformation of the survival function of a one-parameter base law. The NTPWLD exhibits desirable properties—unimodality, light tails, and flexible hazard shapes (increasing, decreasing, and bathtub)—that make it well suited to complex lifetime and reliability data.

We rigorously studied its main probabilistic and inferential features, including the probability density and cumulative distribution functions, survival and hazard functions, Shannon entropy, and mean deviations. Several estimation strategies (MLE, AD, CVM, MPS, and LSE) were investigated through an extensive simulation study; among them, *MLE* consistently provided the most accurate and stable performance across sample sizes. To address parameter uncertainty, we further developed a fuzzy reliability framework using Zadeh's extension principle and $\alpha$-cuts.

Applications to two real datasets demonstrated that the NTPWLD achieves uniformly smaller information criteria (AIC, BIC, $-2\log L$, and AICC) than well-known competitors (e.g., Gamma–Lindley, Chen, Power–Lindley families), confirming its empirical competitiveness. In addition, we incorporated theoretical connections with Bonferroni and Lorenz curves and provided estimation and inference for the reliability function, thereby enriching the interpretability of the model in reliability and survival contexts.

**Perspectives.** This work suggests several avenues for future research:

- *Bayesian inference.* Develop conjugate/semiconjugate priors and computational strategies (e.g., HMC or variational Bayes) for small-sample or noisy settings.
- *Censoring and truncation.* Extend estimation and inference to right/left/interval censoring and truncation, common in survival studies and reliability testing.
- *Regression frameworks.* Use the NTPWLD as a baseline in parametric survival regression (e.g., AFT models) and as a link in generalized accelerated models with covariates.
- *Dependence structures.* Construct multivariate versions via copulas or shared frailty to model dependent lifetimes in systems and biomedical applications.
- *Diagnostics and inequality measures.* Systematize Bonferroni/Lorenz-based diagnostics for model checking and for characterizing tail/inequality features relevant to reliability.

Overall, the NTPWLD offers a flexible, tractable, and empirically strong alternative for lifetime data analysis—both under precise and imprecise (fuzzy) information—opening the door to robust methodological and applied developments.

## Declarations

### Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper. Dataset 1 originates from the U.S. Bureau of the Census (1977). Dataset 2 was reported by King and Ryan (1976) and later discussed by Staudte and Sheather (1990). We have included the simulation and real data application code in our GitHub repository for full reproducibility: https://github.com/Mohamed-Kouadria.

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### Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

### Author Contributions

Mohamed Kouadria: Conceptualization; Methodology; Formal analysis; Investigation; Software; Writing – original draft; Visualization. Halim Zeghdoudi: Methodology; Validation; Resources; Writing – review & editing; Supervision; Theoretical developments; Project administration. Mohammed El–Arbi Khalfallah: Formal analysis; Writing – review & editing; Funding acquisition.

### Artificial Intelligence Statement

Artificial intelligence (AI) tools, including large language models, were used solely for language editing and improving readability. AI tools were not used for generating ideas, performing analyses, interpreting results, or writing the scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

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