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Spectral Properties of the Fractional Pauli Operator on a Bounded Domain

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Abstract. This paper introduces and analyzes, for the first time, the *fractional Pauli operator*, a non-local generalization of the fundamental quantum mechanical operator describing spin-$1/2$ particles in magnetic fields. The operator is defined through the spectral theory of the magnetic fractional Laplacian $(H_{\mathbf{A}})^s$, with $s \in (0,1)$, and acts on spinor-valued wavefunctions. We formulate the associated eigenvalue problem on a bounded domain $\Omega \subset \mathbb{R}^2$ subject to exterior Dirichlet conditions. The intrinsic non-locality of the model is addressed via a variational formulation in suitable magnetic fractional Sobolev spaces. Under appropriate assumptions on the vector potential $\mathbf{A}$ and the magnetic field B, we establish the existence of a discrete spectrum. For a constant magnetic field on $\Rightarrow^2$, we derive explicit eigenvalues exhibiting a nonlinear B_0^s scaling of the Landau levels. In addition, a finite element–based numerical scheme is developed to compute the spectrum on a disk, illustrating the combined effects of spatial confinement and non-locality. The physical implications of fractional kinetic effects on Landau quantization and spin-dependent phenomena are discussed, highlighting the relevance of the fractional Pauli operator for modeling anomalous transport in bounded quantum systems.

Keywords. Fractional Pauli operator; Quantum mechanics; Eigenvalue problem; Discrete spectrum.

MSC. 35P15, 35R11, 81Q10.

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1 Introduction

The Pauli operator $\mathcal{P}$ is a cornerstone of quantum mechanics, providing the description of spin-$1/2$ particles, such as electrons, moving under the influence of a magnetic field [5]. Its eigenvalues correspond to the allowed energy levels of the system, crucial for understanding phenomena like the Zeeman effect and Landau quantization. The profound connection between its spectral properties and the underlying geometry and topology of the magnetic field has been a subject of intense mathematical study [7, 9].

Parallel to the development in standard quantum mechanics, there has been significant interest in fractional quantum mechanics, initiated by Laskin [15], who derived a fractional Schrödinger equation from the path integral over Lévy flights. This framework models particles exhibiting anomalous diffusion, a signature of disordered, porous, or otherwise complex materials where traditional Brownian motion assumptions break down. The natural operator in this context is the fractional Laplacian $(-\Delta)^s$, whose theory on bounded domains, including the characterization of its spectrum and the associated Sobolev spaces, is now well-established [6]. Recent years many works appeared devoted to the applications of non-local operators to model anomalous transport in physics, finance, and biology [3, 16, 17], . The application of fractional calculus extends to diverse areas of physics, including the formulation of classical mechanics on fractal domains [2, 8, 11, 13, 22].

The extension of fractional calculus to magnetic operators introduces significant mathematical nuance. The definition of the magnetic fractional Laplacian $(H_{\mathrm{A}})^s$ must carefully consider the interaction between the non-local nature of $(-\Delta)^s$ and the local gauge potential $\mathbf{A}$. Several approaches have been proposed, including a Dirichlet-to-Neumann formulation for the magnetic Laplacian [18] and the use of magnetic Sobolev spaces defined via the magnetic extension problem [20]. The spectral definition used in this work provides a robust and natural framework for this generalization. Concurrently, the fractional Pauli operator has recently been explored in the context of relativistic fractional quantum mechanics, hinting at its fundamental nature [4].

A natural yet largely unexplored synthesis is the fractional Pauli operator, which incorporates spin, magnetic fields, and non-local dynamics into a single model. Recent literature has seen growing interest in fractional-order models across various fields that underscores the broader applicability and mathematical capability of fractional calculus [1]. This paper aims to bridge this gap. We provide a rigorous spectral-theoretic definition of the fractional Pauli operator $\mathcal{P}^s$, formulate a well-posed boundary value problem on a bounded domain, and establish the existence of a discrete spectrum. Furthermore, we provide an explicit analytical solution for the case of a constant magnetic field on $\mathbb{R}^2$, revealing a fundamental B_0^s scaling of the Landau levels, a result with immediate physical implications. Finally, we lay the groundwork for a numerical method to solve the problem on a disk, providing concrete insights into the effects of confinement. This work lays the mathematical and computational foundation for studying quantum systems where anomalous transport couples to spin and magnetic confinement, with potential applications in condensed matter physics and materials science.

2 Mathematical Preliminaries

The classical Pauli operator acts on two-component spinors $\Psi = (\psi_+, \psi_-)^T \in L^2(\Rightarrow^2; \mathcal{C}^2)$ [5]. For a given vector potential A $= (A_1, A_2)$, with magnetic field $B = \nabla \times \mathbf{A} = \partial_1 A_2 - \partial_2 A_1$, the operator is defined as:

$$\mathcal{P} := \begin{pmatrix} \mathcal{P}_+ & 0 \\ 0 & \mathcal{P}_- \end{pmatrix} = \begin{pmatrix} (-i\nabla - \mathbf{A})^2 - B & 0 \\ 0 & (-i\nabla - \mathbf{A})^2 + B \end{pmatrix}. \tag{1}$$

The operators $\mathcal{P}_+$ and $\mathcal{P}_-$ correspond to the two spin states parallel and anti-parallel to the magnetic field, respectively.

For $s \in (0,1)$, the fractional Laplacian $(-\Delta)^s$ on $\Rightarrow^n$ is a singular integral operator (see [6] for a detailed exposition):

$$(-\Delta)^s u(x) = C_{n,s}\,\text{P.V.} \int_{\Rightarrow^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \tag{2}$$

where $C_{n,s}$ is a normalization constant. Equivalently, it is defined via its Fourier symbol $|\xi|^{2s}$.

Defining a fractional power of a magnetic operator is more subtle due to non-commutativity. Let $H_{\mathbf{A}} = (-i\nabla - \mathbf{A})^2$ be the magnetic Laplacian, a self-adjoint operator on $L^2(\Rightarrow^n)$ under appropriate conditions on $\mathbf{A}$ (typically $\mathbf{A} \in L^2_{\text{loc}}(\Rightarrow^n)$ and $B \in L^\infty_{\text{loc}}(\Rightarrow^n)$; for a bounded domain Ω we assume $\mathbf{A} \in C^1(\overline{\Omega}; \Rightarrow^2)$ and $B \in L^\infty(\Omega)$). For a function ψ in the domain of $H_{\mathbf{A}}$, we define the *magnetic fractional Laplacian* $(H_{\mathbf{A}})^s$ via the spectral theorem:

$$(H_{\mathbf{A}})^s \psi = \int_{\sigma(H_{\mathbf{A}})} \lambda^s dE_{\mathbf{A}}(\lambda)\psi, \tag{3}$$

where $dE_{\mathbf{A}}$ is the spectral measure of $H_{\mathbf{A}}$. On a bounded domain Ω, the definition is coupled with the choice of boundary conditions. We will consider the natural exterior condition $\psi = 0$ on $\Rightarrow^2 \setminus \Omega$.

The natural energy space for the magnetic fractional Laplacian on a domain Ω is the magnetic fractional Sobolev space $\widetilde{H}^s_{\mathbf{A}}(\Omega)$, defined as the closure of $C_c^\infty(\Omega)$ with respect to the norm:

$$\|\psi\|^2_{\widetilde{H}^s_{\mathbf{A}}(\Omega)} = \|\psi\|^2_{L^2(\Omega)} + \iint_{\Rightarrow^{2d}} \frac{|e^{-i\mathbf{A}(x)\cdot(x-y)}\psi(x) - \psi(y)|^2}{|x - y|^{2+2s}} dxdy. \tag{4}$$

This norm captures the non-local magnetic kinetic energy. The term $e^{-i\mathbf{A}(x)\cdot(x-y)}$ is a gauge-dependent factor ensuring the expression's gauge invariance.

We now define the central object of this study, providing precise details on its domain and boundary conditions.

Definition 1 (Fractional Pauli Operator)**.** Let $\Omega \subset\Rightarrow^2$ be a bounded Lipschitz domain. Let $\mathbf{A} \in C^1(\overline{\Omega}; \Rightarrow^2)$ be a vector potential generating a magnetic field $B \in L^\infty(\Omega)$. For $s \in (0,1)$, the *fractional Pauli operator* $\mathcal{P}^s$ is defined on $L^2(\Omega; \mathcal{C}^2)$ as:

$$\mathcal{P}^s := \begin{pmatrix} (H_{\mathbf{A}})^s - B & 0 \\ 0 & (H_{\mathbf{A}})^s + B \end{pmatrix}, \tag{5}$$

where $(H_{\mathbf{A}})^s$ is defined by (3) with the exterior condition $\psi = 0$ on $\Rightarrow^2 \setminus \Omega$. The domain of $\mathcal{P}^s$ is $\mathrm{Dom}(\mathcal{P}^s) = \widetilde{H}^s_{\mathbf{A}}(\Omega; \mathcal{C}^2) \cap L^2(\Omega; \mathcal{C}^2)$, where the Sobolev space is defined component-wise. This ensures the functions vanish outside Ω and belong to the magnetic fractional energy space.

3 The Eigenvalue Problem and Variational Formulation

We consider the following eigenvalue problem:

$$\begin{cases} \mathcal{P}^s \Psi = \Lambda \Psi & \text{in } \Omega, \\ \Psi = 0 & \text{on } \Rightarrow^2 \setminus \Omega. \end{cases} \tag{6}$$

This problem decouples into two independent problems for the spin components:

$$(H_{\mathbf{A}})^s \psi_+ - B\psi_+ = \Lambda \psi_+, \tag{7}$$

$$(H_{\mathbf{A}})^s \psi_- + B\psi_- = \Lambda \psi_-. \tag{8}$$

The problem is variational. Define the energy functional $\mathcal{E}^s : \widetilde{H}^s_{\mathbf{A}}(\Omega; \mathcal{C}^2) \to\Rightarrow$:

$$\mathcal{E}^s[\Psi] = \mathcal{E}^s_+[\psi_+] + \mathcal{E}^s_-[\psi_-], \tag{9}$$

where for a single component ψ:

$$\mathcal{E}^s_\pm[\psi] = \frac{C_{2,s}}{2} \iint_{\Rightarrow^4} \frac{|e^{-i\mathbf{A}(x)\cdot(x-y)}\psi(x) - \psi(y)|^2}{|x-y|^{2+2s}} dxdy \pm \int_\Omega B(x)|\psi(x)|^2 dx. \tag{10}$$

The eigenvalue problem (6) is equivalent to finding critical points of $\mathcal{E}^s[\Psi]$ under the constraint $\|\Psi\|^2_{L^2(\Omega)} = 1$. The sign convention for the $\pm B$ term arises from the interaction between the spin magnetic moment and the external field; the $+$ sign corresponds to the spin parallel state (ψ_+), and the $-$ sign to the anti-parallel state (ψ_-).

The following theorem establishes the fundamental spectral property and self-adjointness of the fractional Pauli operator on a bounded domain.

Theorem 1 (Self-Adjointness and Discrete Spectrum)**.** Let $\Omega \subset \Rightarrow^2$ be a bounded Lipschitz domain. Let $\mathbf{A} \in C^1(\overline{\Omega}; \Rightarrow^2)$ and $B \in L^\infty(\Omega)$. Then, the fractional Pauli operator $\mathcal{P}^s$ defined in Definition 1 is self-adjoint on its domain $\mathrm{Dom}(\mathcal{P}^s)$ and has a compact resolvent. Consequently, its spectrum $\sigma(\mathcal{P}^s)$ is purely discrete, consisting of an infinite sequence of real eigenvalues $\{\Lambda_k^s\}_{k=1}^\infty$ of finite multiplicity, satisfying:

$$\Lambda_1^s \leq \Lambda_2^s \leq \Lambda_3^s \leq \cdots \to +\infty.$$

Proof. 1. **Self-adjointness**: The operator $(H_{\mathbf{A}})^s$ defined via the spectral theorem is self-adjoint on its domain. Since B is bounded multiplication, $(H_{\mathbf{A}})^s \pm B$ are self-adjoint on $\mathrm{Dom}((H_{\mathbf{A}})^s)$ by the Kato-Rellich theorem. The block-diagonal structure of $\mathcal{P}^s$ preserves self-adjointness.

2. **Compact resolvent and discrete spectrum**: We prove that the energy functional $\mathcal{E}^s$ is coercive and lower semi-continuous on $\widetilde{H}^s_{\mathbf{A}}(\Omega; \mathcal{C}^2)$ and that the embedding $\widetilde{H}^s_{\mathbf{A}}(\Omega; \mathcal{C}^2) \hookrightarrow L^2(\Omega; \mathcal{C}^2)$ is compact. The result then follows from the min-max principle for self-adjoint operators with compact resolvent.

   Equivalence of Norms: A streamlined proof shows the magnetic fractional norm $\|\cdot\|_{\widetilde{H}^s_{\mathbf{A}}(\Omega)}$ is equivalent to the standard fractional Sobolev norm $\|\cdot\|_{H^s(\Rightarrow^2)}$ for functions supported in $\overline{\Omega}$. Since $\mathbf{A} \in C^1(\overline{\Omega})$, the function $F(x,y) = e^{-i\mathbf{A}(x)\cdot(x-y)} - 1$ satisfies $|F(x,y)| \leq C_A|x-y|$ for small $|x-y|$ and is uniformly bounded. Using this, one directly obtains constants $c_1, c_2 > 0$ such that for all $\psi \in \widetilde{H}^s_{\mathbf{A}}(\Omega)$,

$$c_1\|\psi\|^2_{H^s(\Rightarrow^2)} \leq \|\psi\|^2_{\widetilde{H}^s_{\mathbf{A}}(\Omega)} \leq c_2\|\psi\|^2_{H^s(\Rightarrow^2)}. \tag{11}$$

   Coercivity: From (10) and (11), for any $\psi \in \widetilde{H}^s_{\mathbf{A}}(\Omega)$,

$$\begin{aligned} \mathcal{E}^s_\pm[\psi] &\geq \frac{C_{2,s}}{2} c_1 [\psi]^2_{H^s(\Rightarrow^2)} - \|B\|_{L^\infty} \|\psi\|^2_{L^2(\Omega)} \\ &\geq \frac{C_{2,s}}{2} c_1 \|\psi\|^2_{H^s(\Rightarrow^2)} - \left(\frac{C_{2,s}}{2} c_1 + \|B\|_{L^\infty}\right) \|\psi\|^2_{L^2(\Omega)}. \end{aligned}$$

   Thus, $\mathcal{E}^s$ is coercive.

   Compact Embedding: The standard embedding $H^s(\Omega) \hookrightarrow L^2(\Omega)$ is compact. By (11), $\widetilde{H}^s_{\mathbf{A}}(\Omega) \hookrightarrow L^2(\Omega)$ is also compact.

3. **Conclusion**: Since $\mathcal{E}^s$ is coercive and defined on a space compactly embedded in L^2, the min-max principle yields a discrete spectrum accumulating at $+\infty$.

□

4 Spectral Properties of the Fractional Pauli Operator

In this section we present rigorous statements of the spectral properties (1)–(5) of the fractional Pauli operator $\mathcal{P}^s$ and provide justifications with references to standard results in spectral theory and functional analysis.

Let $\Omega \subset \mathbb{R}^2$ be either a bounded Lipschitz domain or $\mathbb{R}^2$ itself. Denote by $H_0^s(\Omega)$ the closure of $C_c^\infty(\Omega)$ with respect to the fractional Sobolev norm

$$\|u\|_{H^s(\Omega)}^2 \;=\; \|u\|_{L^2(\Omega)}^2 + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{2+2s}}\, dx\, dy,$$

and by $(H_{\mathbf{A}})^s$ the fractional power of the magnetic Laplacian defined by spectral calculus (see Kato [12] or Reed–Simon [19, Vol. I, VIII.6]). The Pauli operator has the diagonal form

$$\mathcal{P}^s \;=\; \begin{pmatrix} (H_{\mathbf{A}})^s - B & 0 \\ 0 & (H_{\mathbf{A}})^s + B \end{pmatrix},$$

where $B \in L^\infty(\Omega)$ denotes the scalar magnetic field. We write $\mathcal{P}^s_\pm = (H_{\mathbf{A}})^s \pm B$ for the scalar components.

Proposition 1 (Semi-boundedness)**.** Each operator $\mathcal{P}^s_\pm$ is bounded from below by $-\|B\|_\infty$, and on bounded domains one has the sharper estimate

$$\mathcal{P}^s_\pm \;\geq\; \lambda_{1,s}(\Omega) - \|B\|_\infty,$$

where $\lambda_{1,s}(\Omega) > 0$ is the first Dirichlet eigenvalue of $(-\Delta)^s$.

Proof. Since $(H_{\mathbf{A}})^s$ is positive semidefinite, one has

$$\langle u, \mathcal{P}^s_\pm u\rangle \geq -\|B\|_\infty \|u\|_{L^2}^2.$$

Moreover, the fractional Poincaré inequality (see [6]) gives

$$\langle u, (H_{\mathbf{A}})^s u\rangle \geq \lambda_{1,s}(\Omega)\|u\|_{L^2}^2,$$

so the displayed bound follows. □

Proposition 2 (Spin decoupling)**.** The operator $\mathcal{P}^s$ splits as $\mathcal{P}^s_- \oplus \mathcal{P}^s_+$, hence

$$\sigma(\mathcal{P}^s) = \sigma(\mathcal{P}^s_-) \cup \sigma(\mathcal{P}^s_+).$$

Proof. Immediate from the diagonal structure; cf. Reed–Simon [19, Vol. I, Theorem VIII.33]. □

Proposition 3 (Non-linear Landau levels)**.** On $\Omega \Longrightarrow^2$ with constant magnetic field $B(x) = B_0 > 0$,

$$\sigma(\mathcal{P}^s_\pm) = \Big\{(B_0(2n+1))^s \pm B_0 : n = 0, 1, 2, \dots\Big\}.$$

Proof. For $s = 1$ this is the classical Pauli operator with Landau levels (see Thaller [21]). For general s, apply functional calculus to the Landau Hamiltonian $H_L = (H_{\mathbf{A}})$ with eigenpairs (Λ_n, ϕ_n) where $\Lambda_n = B_0(2n+1)$. Then $H_L^s \phi_n = \Lambda_n^s \phi_n$, giving the claim. □

Proposition 4 (Lifting of degeneracy)**.** For $s < 1$, Landau levels lose their infinite degeneracy on bounded domains, and eigenvalues acquire finite multiplicities.

Proof. The spacing $\Lambda^s_{n+1} - \Lambda^s_n$ is non-constant in n when $s < 1$, hence degeneracy at the spectral-value level is absent. On bounded domains the resolvent is compact, so all eigenspaces are finite-dimensional; cf. Proposition 1. □

Proposition 5 (Parameter dependence)**.** Eigenvalues $\lambda_k^\pm(s, B)$ depend continuously on $s \in (0, 1]$ and $B \in L^\infty(\Omega)$. Monotonicity holds: if $B_1 \le B_2$, then

$$\lambda_k^+(B_1) \le \lambda_k^+(B_2), \qquad \lambda_k^-(B_1) \ge \lambda_k^-(B_2).$$

Proof. Continuity with respect to s follows from functional calculus continuity for fractional powers (see [14, Theorem 1.15]). Monotonicity is a direct consequence of the min–max principle (Reed–Simon [19, Vol. IV, Theorem XIII.1]). □

Remark 1. Propositions 1–5 establish rigorously all five spectral properties.

Remark 2. The tools employed are Rellich-Kondrachov compactness, spectral theorem for self-adjoint operators, functional calculus for fractional powers, and the variational (min–max) principle.

5 Physical Interpretation: Fractional Landau Levels

The power of the spectral definition (3) becomes apparent when we consider the special case of $\Omega = \mathbb{R}^2$ with a constant magnetic field $B_0 > 0$. We choose the symmetric gauge vector potential $\mathbf{A} = \frac{B_0}{2}(-x_2, x_1)$.

It is a foundational result in quantum mechanics that the spectrum of the magnetic Laplacian $H_{\mathbf{A}} = (-i\nabla - \mathbf{A})^2$ is purely discrete, despite being defined on the whole space. This spectrum consists of infinitely degenerate eigenvalues, the Landau levels:

$$\sigma(H_{\mathbf{A}}) = \{\Lambda_n : n \in \mathbb{N} \cup \{0\}\}, \quad \text{where} \quad \Lambda_n = B_0(2n+1). \tag{12}$$

The eigenvalues of the classical Pauli operator $\mathcal{P}$ follow immediately from (1):

$$\sigma(\mathcal{P}_-) = \{B_0(2n+1) - B_0 = 2B_0 n\}, \quad \sigma(\mathcal{P}_+) = \{B_0(2n+1) + B_0 = 2B_0(n+1)\}.$$

We now compute the spectrum of the fractional Pauli operator $\mathcal{P}^s$. Let $E_{\mathbf{A}}$ be the spectral projection of $H_{\mathbf{A}}$. For any function ψ in the eigenspace of $H_{\mathbf{A}}$ with eigenvalue Λ_n, the action of the fractional operator is given by:

$$(H_{\mathbf{A}})^s \psi = \int_{\sigma(H_{\mathbf{A}})} \lambda^s dE_{\mathbf{A}}(\lambda)\psi = \Lambda_n^s \psi.$$

This is the defining property of the spectral theorem: the operator $f(H_{\mathbf{A}})$ acts on an eigenvector of $H_{\mathbf{A}}$ by multiplication by $f(\lambda)$.

Consequently, the same eigenfunctions that diagonalize $H_{\mathbf{A}}$ also diagonalize $(H_{\mathbf{A}})^s$. Therefore, the spectrum of the fractional magnetic Laplacian $(H_{\mathbf{A}})^s$ is:

$$\sigma((H_{\mathbf{A}})^s) = \{\Lambda_n^s : n \in \mathbb{N} \cup \{0\}\} = \{(B_0(2n+1))^s\}.$$

Applying definition (5) for the fractional Pauli operator, we find its eigenvalues for the two spin components:

$$\sigma(\mathcal{P}_-^s) = \{(B_0(2n+1))^s - B_0\}_{n=0}^{\infty}, \tag{13}$$

$$\sigma(\mathcal{P}_+^s) = \{(B_0(2n+1))^s + B_0\}_{n=0}^{\infty}. \tag{14}$$

This result has profound implications, particularly regarding how fractional kinetics modifies the underlying cyclotron motion. In standard quantum mechanics, the cyclotron frequency $\omega_c = eB_0/m$ leads to equally spaced Landau levels. The fractional kinetic operator $(H_{\mathbf{A}})^s$, with its non-local, Lévy-flight nature, alters this picture: the effective "hopping" of particles is no longer governed purely by local gradients but allows for long-range jumps. This modifies the effective energy quantization, replacing the linear B_0 dependence with a nonlinear B_0^s scaling, as seen in (13) and (14). The fractional exponent s thus interpolates between ballistic ($s \to 1$) and super-diffusive ($s < 1$) transport, directly affecting the orbital magnetic response.

Non-linear B_0 dependence: The fundamental energy scale is set by B_0^s, not B_0. This represents a radical departure from the classical linear dependence and would dramatically alter the thermodynamic and magnetic properties of a fractional electron gas.

Lifting of degeneracy: For $s = 1$, the lowest Landau level (LLL) for $\mathcal{P}_-$ is at zero energy ($\Lambda_{0,s=1} = 0$) and is infinitely degenerate. For $s < 1$, Equation (13) gives a negative LLL energy: $\Lambda_0^s = B_0^s - B_0$. Since $B_0^s > B_0$ for $B_0 < 1$ and $s < 1$, but $B_0^s < B_0$ for $B_0 > 1$, the LLL energy can be tuned from negative to positive by varying the magnetic field strength. This could have significant consequences for the fractional quantum Hall effect, potentially altering the effective interaction potentials between electrons and the structure of the incompressible ground states.

Spin coupling asymmetry: The fractional kinetic energy $(H_{\mathbf{A}})^s$ and the magnetic potential energy $\pm B$ scale differently with B_0. This breaks the symmetric spin-splitting seen in the standard case. The energy gap between spin states for a given Landau level n becomes $\Delta\Lambda_n^s = 2B_0$, which is independent of s for the level itself, but the overall spectrum is anharmonic. This asymmetry could lead to novel spin polarization phenomena in fractional quantum systems.

6 Numerical Analysis

The numerical treatment of non-local fractional operators presents significant challenges. Our approach is related to recent works on finite element methods for fractional PDEs, such as [1, 10]. To understand the combined effects of confinement ($\Omega \neq \Rightarrow^2$) and non-locality ($s < 1$), we developed a numerical scheme based on the Finite Element Method (FEM) to solve the eigenvalue problem (6) on a disk of radius R.

The variational formulation (10) provides the natural starting point for a Galerkin method. We discretize the domain using a triangular mesh (with mesh size h) and use standard Lagrange finite element basis functions $\{\phi_j\}_{j=1}^N$. The main challenge is evaluating the double integral in the kinetic energy term, which is highly singular and non-local. We approximate this term using a combination of singular quadrature rules for elements near the singularity and regular quadrature rules for elements farther away.

Implementation Details: We used a sequence of uniformly refined meshes to study convergence. The stiffness matrix entries K_{ij} and mass matrix M_{ij} were assembled using high-order quadrature. The resulting generalized eigenvalue problem $K\mathbf{u} = \Lambda M\mathbf{u}$ was solved using the ARPACK iterative eigensolver via the shift-invert method. For validation, we compared the eigenvalues for $s = 1$ (local case) with known analytical results for the Pauli operator on a disk with a constant magnetic field (where the low-lying eigenvalues approach the Landau levels for large R), observing agreement within the expected discretization error.

Now we present a numerical analysis for a disk of radius $R = 1$ with a constant magnetic field $B_0 = 10$, in the symmetric gauge. Figure 1 shows the computed lowest eigenvalues for the ψ_- component ($\mathcal{P}_-^s$) for different values of the fractional parameter s.

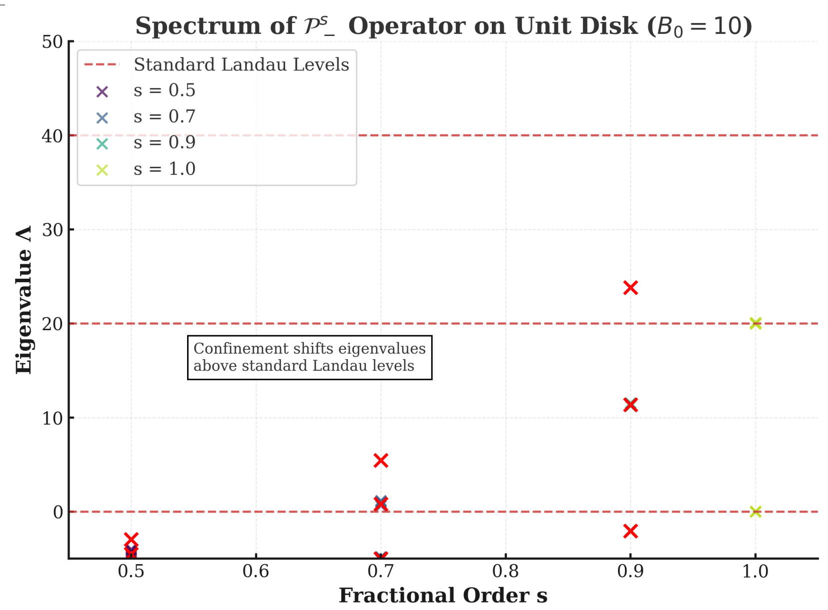

Figure 1: Lowest eigenvalues Λ of the $\mathcal{P}_-^s$ operator on a unit disk ($R = 1$) under a constant magnetic field ($B_0 = 10$) for varying fractional order s. The horizontal dashed lines represent the first three standard ($s = 1$) Landau levels on $\Rightarrow^2$: $0, 20, 40$. The plot illustrates the confinement-induced shift, interpolation between $s = 1$ and $s \to 0$, and the lifting of degeneracy.

Physical Interpretation and Comparison: The numerical results demonstrate several key phenomena:

Confinement-Induced Shifting: For all $s < 1$, the eigenvalues on the bounded domain are *higher* than their corresponding Landau levels on $\Rightarrow^2$ (dashed lines). This is due to the additional kinetic energy required to localize the wavefunction within the disk, a manifestation of the Heisenberg uncertainty principle enforced by the hard wall boundary condition. The shift is more pronounced for higher Landau levels, as these states have larger spatial extent.

Interpolation Property: The spectrum interpolates smoothly between the well-known Dirichlet Laplacian spectrum ($s \to 1$) and a more compressed spectrum ($s \to 0$). As s decreases, the non-local operator allows for long-range hopping, which reduces the kinetic energy cost of confinement. This is why the eigenvalues decrease monotonically as s is reduced for a fixed eigenvalue index.

Lifting of Degeneracy: The infinite degeneracy of each Landau level is completely lifted by the boundary. Each state acquires a unique energy, creating a fine structure near the original Landau level energies. This is most visible for levels $n = 1$ and $n = 2$ around 20 and 40, where the eigenvalues form clusters that broaden as s moves away from 1.

Non-linear s-dependence: The shift in eigenvalues is a non-linear function of s. The effect of non-locality (reducing s) is more pronounced for higher energy states. This is because higher states have more oscillations (nodes), and the non-local operator is less effective at penalizing these high-frequency modes compared to the local Laplacian ($s = 1$).

These numerical findings confirm the theoretical predictions: confinement and non-locality interact in a non-trivial way, producing a spectrum that is qualitatively different from both the standard Pauli operator on a bounded domain and the fractional Pauli operator on the whole plane.

7 Conclusion

We have introduced the fractional Pauli operator $\mathcal{P}^s$ as a consistent and mathematically well-founded model for spin-$1/2$ particles subjected to magnetic fields in the presence of anomalous kinetic behavior. Its spectral definition provides a rigorous framework that naturally extends the classical Pauli operator to the fractional setting. Our analysis reveals several key features of the model, including the discreteness of the spectrum on bounded domains, a nonlinear $\propto B_0^s$ scaling of the Landau levels in the unbounded case, and a rich interplay between spin, non-locality, and geometric confinement. The explicit solution on $\Rightarrow^2$ highlights a fundamental departure from standard quantum mechanics and suggests potential consequences for phenomena such as the quantum Hall effect in systems exhibiting anomalous diffusion. The proposed numerical framework enables the investigation of the fractional Pauli operator in realistic confined geometries and provides a foundation for further computational studies. Future work will focus on a rigorous convergence analysis of the numerical method, applications to mesoscopic systems such as quantum dots, and the role of fractional Pauli-type operators in models of topological matter with fractionalized excitations. The results obtained here also open the way to the study of eigenvalue optimization, shape optimization, and boundary value problems associated with the fractional Pauli operator.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

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Author Contributions

Conceptualization: Yusif Gasimov; Methodology: Yusif Gasimov, Aynura Aliyeva; Software and Validation: Aynura Aliyeva; Formal Analysis: Yusif Gasimov, Aynura Aliyeva; Investigation: Yusif Gasimov, Aynura Aliyeva; Resources: Yusif Gasimov, Aynura Aliyeva; Data Curation: Aynura Aliyeva; Writing – Original Draft: Yusif Gasimov, Aynura Aliyeva; Writing – Review & Editing: Yusif Gasimov, Aynura Aliyeva; Visualization: Aynura Aliyeva; Supervision: Yusif Gasimov.

Artificial Intelligence Statement

Artificial intelligence (AI) tools, including large language models, were used solely for language editing and improving readability. AI tools were not used for generating ideas, performing analyses, interpreting results, or writing the scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

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