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Research Article

# A Novel Algorithm for Optimizing the Covering of a Bounded Planar Domain with Simple Geometric Figures

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**Abstract.** In this paper, we address the problem of covering a given bounded domain in the plane using simple geometric figures. The proposed approach is based on a discretization of the domain, which leads to a corresponding discrete optimization problem. To solve this problem, we introduce a novel iterative algorithm that minimizes a given objective function by generating successive neighboring nodal points. As the covering elements, circular sectors with centers located outside the domain are considered. The objective is to determine the locations of the sector centers and their radii in such a way that the entire domain is completely covered, while the ratio of the total area of the covering sectors to the area of the domain is minimized. Finally, the algorithm is demonstrated on a representative example, and the resulting coverings are illustrated.

**Keywords.** Optimal coverage, Discretization of the domain, Discrete optimization.

**MSC.** 81T80; 93A30; 97M10.

**How to Cite**

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## 1 Introduction and literature context

The problem of covering a planar domain with simple geometric figures is a fundamental topic that arises in a wide range of economic, technical, and geographical applications. Problems of a military nature can also be associated with this class of optimization tasks. It should be noted that the external (covering) and internal (packing) approximation of sets by simple geometric elements has been studied since the time of Lagrange and Gauss [4, 17, 22]. For example, in [12, 14, 27, 28], the problem of covering a given region in various planes using circles of small diameter was investigated. In [18], Lebedev proposed an iterative method for the optimal covering of a nonconvex simply connected set by circles. Problems of this type also arise in the design of transportation networks or in determining optimal server locations in computer networks [9, 16].

An analysis of the existing literature shows that, in most studies, the covering elements are circular regions whose centers lie inside the domain to be covered. However, situations in which the centers of the circles are located outside the target domain are also of practical interest. In such cases, covering the entire circle is unnecessary, since a portion of it lies outside the domain. Therefore, it is more appropriate to perform the covering using circular sectors with prescribed central angles.

It is worth noting that in [14, 28], the problem of optimal coverage is addressed in terms of minimizing the radius of the covering circles. In contrast, the optimal covering of a bounded, possibly nonconvex domain by circular sectors with fixed radii and centers located outside the domain can be formulated as the problem of ensuring complete coverage of the domain while maximizing the total area of intersection between the sectors and the region of interest [18, 27].

A practical application of this problem arises in the optimal placement of land plots located outside an irrigated region for sector-based irrigation systems. Similar formulations also appear in military applications, such as the optimal deployment of radar systems for reconnaissance over enemy territory [9, 16].

Since the problem considered in this work differs from classical covering problems based on minimal radii, our objective is to develop a new procedure for the optimal coverage of a domain using circular sectors. In this setting, it is necessary to determine not only the locations of the sector centers, but also the orientations of the radius lines defining the sectors. To this end, we propose to study the problem in both continuous and discrete frameworks. In the discrete case, the domain to be covered and the circular sectors are discretized and represented in the form of a computational grid.

The remainder of the paper is organized as follows. The formal problem statement is presented in Section 2. Section 3 introduces the required preliminaries. In Section 4, the discretization of the domain is discussed, and a new algorithm for determining an optimal covering is proposed and analyzed. Numerical implementation and illustrative examples are also provided in this section. Finally, concluding remarks are given in Section 5.

## 2 Problem Statement

Let $D$ be a bounded, generally nonconvex domain in the plane (see Figure 1). Consider a set of $n$ points $\{O_i\}_{i=1}^n$ located in the plane such that

$$O_i \notin D, \qquad i = 1, 2, \ldots, n,$$

that is, all points $O_i$ lie outside the domain $D$. For each point $O_i$, consider a circle with center $O_i$ and radius $r_i$. From each such circle, we define a circular sector with fixed central angle $\alpha$, which is denoted by $P(O_i, r_i, \alpha)$. Some of these sectors may overlap. Let $k$ denote the number of distinct radii among the sectors; clearly, $k \leq n$.

**Definition 1.** If

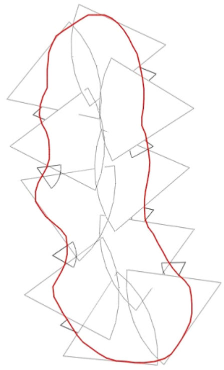

**Figure 1:** Covering a non-convex domain with circular sectors.

$$D \subset \bigcup_{i=1}^{n} P(O_i, r_i, \alpha),$$

then the family of sectors $\{P(O_i, r_i, \alpha)\}_{i=1}^{n}$ is said to cover the domain $D$.

**Definition 2.** A covering of the domain $D$ by the family of sectors $\{P(O_i, r_i, \alpha)\}_{i=1}^{n}$ is called *optimal* if the total area of the parts of the sectors lying outside the domain $D$ is minimal, that is,

$$J = \min \quad \sum_{i=1}^{n} S\big(P(O_i, r_i, \alpha) \setminus D\big), \tag{1}$$

where $S(K)$ denotes the area of a planar set $K$.

According to Definition 1, the covering of the domain $D$ by the family of sectors $P(O_i, r_i, \alpha)$ implies that for every point $(x, y) \in D$ there exists at least one sector $P(O_i, r_i, \alpha)$ such that $(x, y) \in P(O_i, r_i, \alpha)$. In general, such a sector may not be unique.

The algorithm for determining an optimal covering is constructed as follows. The positions of the sector centers $\{O_i\}$, and consequently the orientations of the sectors $P(O_i, r_i, \alpha)$, are varied so that the family of sectors covers the domain $D$ while minimizing the objective functional (1).

It should be noted that the sector $P(O_i, r_i, \alpha)$ is not uniquely defined by the parameters $O_i$, $r_i$, and $\alpha$ alone. Indeed, for a given circle, infinitely many sectors with the same central angle $\alpha$ may be constructed. Therefore, to uniquely specify a sector, it is necessary to define the directions of the radii forming the angle $\alpha$.

Let $\alpha_1^i$ and $\alpha_2^i$ ($\alpha_1^i < \alpha_2^i$) denote the azimuthal angles of the bounding radii of the sector (see Figure 2).

As shown in Figure 2, the relation

$$\alpha = \alpha_2^i - \alpha_1^i$$

holds. Consequently, if the parameters $r_i$, $\alpha_1^i$, and $\alpha_2^i$ are specified, the sector is uniquely determined. In this case, it is more convenient to denote the sector by

$$P(O_i, r_i, \alpha_1^i, \alpha_2^i).$$

Thus, the problem reduces to determining the locations of the centers $O_i$ and the corresponding azimuthal angles $\alpha_1^i$ and $\alpha_2^i$ such that the covering condition is satisfied and the objective functional (1) is minimized, subject to the constraint $\alpha = \alpha_2^i - \alpha_1^i$. If the central angle $\alpha$ is fixed, then it suffices to determine only one of the angles $\alpha_1^i$ or $\alpha_2^i$.

Finally, we note that the radii $r_i$ are not arbitrary. In this work, we assume that the covering consists of sectors with only two possible radii, denoted by $r$ and $R$. Let $m$ be the number of sectors with radius $r$ and $l$ the number of sectors with radius $R$. Then the total number of sectors is given by

$$n = m + l.$$

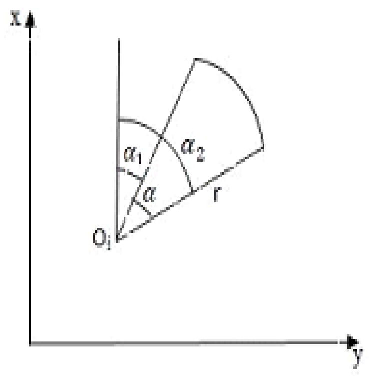


**Figure 2:** A circular sector with center $O_i$, central angle $\alpha$ and azimuths $\alpha_1$ and $\alpha_2$.

## 3 Algorithm for Finding the Optimal Cover of the Domain

We now consider the construction of an algorithm for determining an optimal covering in the sense of Definition 2. Recall that the objective functional is given by (1). Since the center points $O_i$ are located in the plane, they can be represented as $O_i = (x_i, y_i)$. Accordingly, the sector $P(O_i, r_i, \alpha)$ can be written as $P(x_i, y_i, r_i, \alpha)$.

Let

$$S\big(P(x_i, y_i, r_i, \alpha) \setminus D\big) = F_i(x_i, y_i, r_i, \alpha),$$

where $F_i$ denotes the area of the part of the $i$th sector lying outside the domain $D$. Since each radius satisfies $r_i \in \{r, R\}$, the objective functional can be decomposed into two groups. Specifically, for $i = 1, \dots, m$ we have

$$F_i(x_i, y_i, r_i, \alpha) = F_r(x_i, y_i, \alpha),$$

and for $i = m+1, \dots, m+l$,

$$F_i(x_i, y_i, r_i, \alpha) = F_R(x_i, y_i, \alpha),$$

where $m + l = n$.

Therefore, the objective functional (1) can be written as

$$F(x_1, y_1, \dots, x_n, y_n) = \sum_{i=1}^{m} F_r(x_i, y_i, \alpha) + \sum_{i=m+1}^{n} F_R(x_i, y_i, \alpha). \quad (2)$$

Thus, the problem of finding an optimal covering reduces to determining the sets of points

$$\{(x_1, y_1), \dots, (x_m, y_m)\} \quad \text{and} \quad \{(x_{m+1}, y_{m+1}), \dots, (x_n, y_n)\}$$

such that the functional (2) attains its minimum.

In the problem formulation, it is required that the center points $O_i$ lie outside the domain $D$, i.e.,

$$(x_i, y_i) \notin D, \qquad i = 1, \dots, n.$$

To define a suitable search region, let

$$x_{\text{max}} = \max\{x \mid (x, y) \in D\},$$
$$x_{\text{min}} = \min\{x \mid (x, y) \in D\},$$
$$y_{\text{max}} = \max\{y \mid (x, y) \in D\},$$
$$y_{\text{min}} = \min\{y \mid (x, y) \in D\}.$$

Denote by $Q$ the rectangle with vertices

$$(x_{\min}, y_{\min}),\ (x_{\max}, y_{\min}),\ (x_{\max}, y_{\max}),\ (x_{\min}, y_{\max}).$$

Clearly, $D \subset Q$ (see Figure 3).

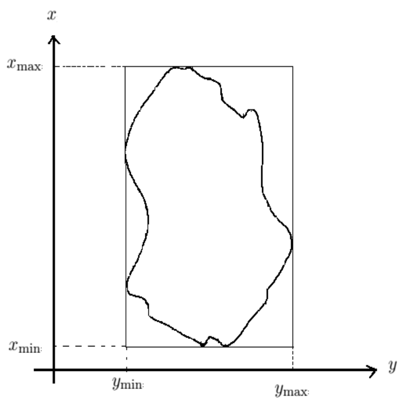


**Figure 3:** Rectangle $Q$ containing domain $D$.

It is therefore natural to restrict the search for the center points $(x_i, y_i)$ to the set $Q \setminus D$.

Introduce the vector Introduce the vector

$$z = (z_1, z_2, \ldots, z_{2n}) = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n),$$

and define the admissible set

$$K = (Q \setminus D)^n \subset \mathbb{R}^{2n}.$$

Then the constrained optimization problem can be written as

$$\min_{z \in K} \ F(z). \tag{3}$$

To solve the constrained problem (3), we employ a penalty method. Define the penalty function

$$\varphi_m(z) = \begin{cases} 0, & z \in K, \\ m\big(|z|^2 + 1\big)^2, & z \notin K, \end{cases} \tag{4}$$

where $m > 0$ is a penalty parameter.

The constrained problem (3) is thus approximated by the unconstrained optimization problem

$$\min f_m(z) = F(z) + \varphi_m(z). \tag{5}$$

Let $z_0$ denote the solution of the constrained problem (3), and let $z_0^{(m)}$ be the minimizer of (5). Then, under standard assumptions on $F$, it follows that

$$\lim_{m \to \infty} z_0^{(m)} = z_0. \tag{6}$$

Hence, for sufficiently large values of $m$, the solution of the unconstrained problem (5) provides an accurate approximation of the solution to (3).

To solve the unconstrained problem (5), a gradient-based iterative scheme can be applied, for example,

$$z^{k+1} = z^k - t \nabla f_m(z^k), \tag{7}$$

where $t > 0$ is a step size parameter. Under appropriate regularity and step size conditions, the sequence $\{z^k\}$ generated by (7) converges to a minimizer of (5).

## 4 Discretization of the Domain and Solution of the Discrete Optimization Problem

In Section 3, the problem was formulated and analyzed in the continuous setting, where it was reduced to the unconstrained optimization problem (5). Although a variety of numerical methods exist for solving such problems, in practice it is often difficult to verify the required regularity conditions for the objective function $f_m(z)$ or to guarantee convergence to a global minimum. For this reason, it is reasonable to consider alternative approaches. One effective technique is the discretization method.

To this end, we subdivide the rectangle $Q$ containing the domain $D$ into a uniform grid, as illustrated in Figure 4.

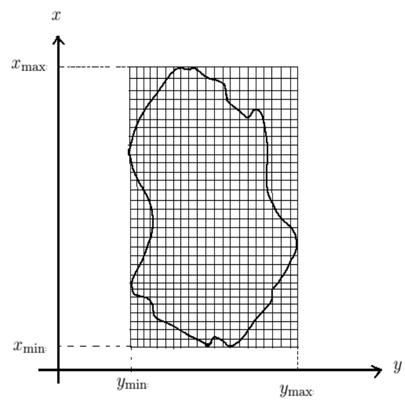


**Figure 4:** Discretization of the domain $D$ and the rectangle $Q$.

Let $n_x$ and $n_y$ be positive integers, and define the grid steps

$$h_x = \frac{x_{\max} - x_{\min}}{n_x},$$
$$h_y = \frac{y_{\max} - y_{\min}}{n_y}.$$

The grid nodes are then given by

$$x_i = x_{\min} + ih_x, \quad i = 0, \ldots, n_x,$$
$$y_j = y_{\min} + jh_y, \quad j = 0, \ldots, n_y. \tag{8}$$

Clearly, $x_0 = x_{\min}$, $x_{n_x} = x_{\max}$, $y_0 = y_{\min}$, and $y_{n_y} = y_{\max}$.

Let $z_{ij} = (x_i, y_j)$. The set of all grid nodes approximating the rectangle $Q$ is defined as

$$Q^* = \{z_{ij} \mid z_{ij} \in Q\}, \tag{9}$$

and the discrete approximation of the domain $D$ is given by

$$D^* = \{z_{ij} \mid z_{ij} \in D\}.$$

Clearly, $D^* \subset Q^*$.

To discretize the objective functional (1), we first approximate each sector $P(O_i, r_i, \alpha)$ by the set

$$P_i^* = \{z_{ij} \mid z_{ij} \in P(O_i, r_i, \alpha)\}. \tag{10}$$

For any finite set $A$, let $N(A)$ denote the number of its elements. Then $N(P_i^* \setminus D^*)$ represents the number of grid nodes belonging to the $i$th sector that lie outside the domain $D$. Summing over all sectors yields the discrete approximation of the objective functional:

$$J \approx \overline{J} = \min \sum_{i=1}^{n} N\big(P_i^* \setminus D^*\big). \tag{11}$$

Thus, the discrete optimization problem consists in determining the sector centers

$$z_{ij} = (x_i, y_j) \in Q^* \setminus D^*, \tag{12}$$

such that the functional (11) attains its minimum. This problem is nonlinear and combinatorial in nature. The discrete optimization problem (11) with (12) is generally nonlinear.

Discrete optimization problems of this type can be addressed using a variety of methods, including dynamic programming, branch-and-bound techniques, heuristic and approximate algorithms, cutting-plane methods, and integer programming approaches [1, 2, 3, 5, 6, 7, 8, 10, 11, 21, 23, 24, 29, 30]. As the grid resolution increases (i.e., as $n_x$ and $n_y$ grow), the solution of the discrete problem (11) converges to that of the continuous optimization problem (3).

We now describe the specific discrete algorithm employed in this study. Following [13, 15, 19, 20, 25, 26, 31], let $Q$ be a rectangle with vertices $(x_0, y_0)$, $(x_0, y_p)$, $(x_l, y_p)$, and $(x_l, y_0)$ such that $D \subset Q$, where $x_0 < x_{\min}$, $x_l > x_{\max}$, $y_0 < y_{\min}$, and $y_p > y_{\max}$. Define the grid steps

$$\begin{aligned} h_x &= \frac{x_l - x_0}{l}, \\ h_y &= \frac{y_p - y_0}{p}, \end{aligned} \tag{13}$$

where $l, p \in \mathbb{N}$, and the grid nodes

$$\begin{aligned} x_i &= x_0 + i h_x, \quad i = 0, \dots, l, \\ y_j &= y_0 + j h_y, \quad j = 0, \dots, p. \end{aligned}$$

Denote these nodes by $z_{ij} = (x_i, y_j)$, and define

$$Q^* = \{z_{ij} \mid z_{ij} \in Q\}, \qquad D^* = \{z_{ij} \mid z_{ij} \in D\}. \tag{14}$$

Assume that $m$ sectors have radius $r_1$ and $n$ sectors have radius $r_2$, with all sector centers located in $Q^* \setminus D^*$. Each sector is defined by

$$P\big((x^0, y^0), \alpha_1, \alpha_2, r\big) = \big\{(x, y) \,\big|\, \|(x, y) - (x^0, y^0)\| \le r,\ \alpha_1 \le \arg(x - x^0, y - y^0) \le \alpha_2\big\}. \tag{15}$$

The covering sectors are denoted by

$$\begin{aligned} P_k &= P\big((x_k^0, y_k^0), \alpha_1^k, \alpha_2^k, r_1\big), \quad k = 1, \dots, m, \\ P_k &= P\big((x_k^0, y_k^0), \alpha_1^k, \alpha_2^k, r_2\big), \quad k = m+1, \dots, m+n. \end{aligned}$$

Let

$$D_k^* = D^* \cap P_k \tag{16}$$

be the set of grid nodes of $D^*$ covered by the sector $P_k$. Define

$$D^l = \bigcup_{k=1}^{m+n} D_k^*. \tag{17}$$

Clearly, $D^l \subset D^*$. If $D^l = D^*$, then the discrete domain is completely covered. Otherwise, the set $D^* \setminus D^l$ contains uncovered grid nodes.

Since $D^* \setminus D^l$ is finite, let

$$n(D^* \setminus D^l) = f\left(x_1^0, y_1^0, \ldots, x_{m+n}^0, y_{m+n}^0, \alpha_1^1, \ldots, \alpha_1^{m+n}\right) \tag{18}$$

denote the number of uncovered nodes. The objective is to determine the sector centers $(x_k^0, y_k^0)$ and orientations $\alpha_1^k$ (with $\alpha_2^k = \alpha_1^k + 60°$) such that

$$f = 0.$$

If this condition cannot be achieved, the function

$$f\left(x_1^0, y_1^0, \ldots, x_{m+n}^0, y_{m+n}^0, \alpha_1^1, \ldots, \alpha_1^{m+n}\right) \tag{19}$$

is minimized. If the resulting minimum is unsatisfactory, further improvement can be obtained by increasing the number of sectors, i.e., the number of barrier stations.

Before analyzing the algorithm for solving the discrete optimization problem, we first describe the procedure for constructing the set $D^*$.

Let $x_i$, $i = 0, \ldots, l$, be the discretized coordinates along the $x$-axis. For each fixed $x_i$, we consider the straight line parallel to the $y$-axis that intersects the domain $D$. Clearly, this line intersects $D^*$ at least at two points. Denote the lowest and highest such points by $(x_i, y_{ib})$ and $(x_i, y_{is})$, respectively. By determining and connecting these points for all indices satisfying $0 < ib \leq i \leq is < l$, we obtain the boundary of the discrete domain $D^*$, which is denoted by $\partial D^*$. More precisely,

$$\partial D^* = \\ \left\{(x_i, y_t) \mid ib \leq t \leq is,\ (x_i, y_{ib}), (x_i, y_{is}) \in D^*,\ (x_i, y_{ib-1}), (x_i, y_{is+1}) \notin D^*,\ a \leq i \leq \mu\right\}.$$

Accordingly, the set $D^*$ can be represented as

$$D^* = \left\{(x_i, y_j) \subset Q^* \mid a < i < \mu,\ ib < j < is\right\}. \tag{20}$$

Next, we determine the set $D^p$. From relation (17), a point belongs to $D^p$ if and only if it is contained in at least one of the sets $D_k^*$. Therefore, to identify the points of $D^p$, we consider all points of $D^*$ defined by (20) and verify whether they are contained in any sector $P_k$. If a point does not belong to any sector $P_k$, it is excluded from $D^p$. The number of uncovered points is given by

$$n\left(D^* \setminus D^p\right) = n(D^*) - n(D^p),$$

and complete coverage of $D^*$ is achieved if and only if this quantity is equal to zero.

We now describe the conditions under which a node $z_{ij} = (x_i, y_j) \in D^*$ belongs to a sector $P_k$. Let $z_k^0 = (x_k^0, y_k^0)$ be the center of the circle corresponding to $P_k$, and let $\alpha_1^k$ and $\alpha_2^k = \alpha_1^k + 60°$ denote the azimuth angles defining the sector. First, we verify the distance condition

$$\|z_{ij} - z_k^0\| \leq r,$$

where $r = r_1$ for $k = 1, \ldots, m$ and $r = r_2$ for $k = m+1, \ldots, m+n$. If this condition holds, the point lies within the corresponding circle.

To determine whether $z_{ij}$ lies inside the sector $P_k$, we compute the azimuth angle of the ray originating from $z_k^0$ and passing through $z_{ij}$. First, we evaluate

$$\arctan\left(\frac{y_j - y_k^0}{x_i - x_k^0}\right),$$

and convert it to degrees using

$$\overline{\alpha} = \frac{180°}{\pi} \arctan\left(\frac{y_j - y_k^0}{x_i - x_k^0}\right). \tag{21}$$

The azimuth angle $\alpha(\overrightarrow{z_k^0 z_{ij}})$ is then defined as

$$\alpha(\overrightarrow{z_k^0 z_{ij}}) = \begin{cases} \overline{\alpha}, & x_i > x_k^0,\ y_j > y_k^0, \\ 180^\circ + \overline{\alpha}, & x_i < x_k^0,\ y_j > y_k^0, \\ 180^\circ + \overline{\alpha}, & x_i < x_k^0,\ y_j < y_k^0, \\ 360^\circ + \overline{\alpha}, & x_i > x_k^0,\ y_j < y_k^0. \end{cases}$$

The point $z_{ij}$ belongs to the sector $P_k$ if

$$\alpha_1^k \leq \alpha(\overrightarrow{z_k^0 z_{ij}}) \leq \alpha_2^k.$$

In the special case where $\alpha_1^k < 360^\circ$ and $\alpha_2^k > 360^\circ$, the above condition is replaced by

$$\alpha_1^k \leq \alpha(\overrightarrow{z_k^0 z_{ij}}) \leq 360^\circ \quad \text{or} \quad 0^\circ \leq \alpha(\overrightarrow{z_k^0 z_{ij}}) \leq \alpha_2^k - 360^\circ.$$

If either condition holds, the point $z_{ij}$ is considered an interior point of the sector.

Consequently, for given centers $(x_k^0, y_k^0)$ and azimuths $\alpha_1^k$, $k = 1, \dots, m+n$, the set $D^p$ can be constructed and the value $n(D^* \setminus D^p)$ can be evaluated. If this value is zero, then the sectors $P_k$ with $\alpha_2^k = \alpha_1^k + 60^\circ$ provide a complete covering of $D^*$, that is,

$$D^* \subset \bigcup_{k=1}^{m+n} P_k.$$

Otherwise, the parameters $(x_k^0, y_k^0)$ and $\alpha_1^k$ must be adjusted so that $n(D^* \setminus D^p)$ is minimized, which corresponds to solving the discrete optimization problem (5).

The centers $(x_k^0, y_k^0)$ are required to satisfy

$$\begin{aligned} &2 \leq d\big((x_k^0, y_k^0), D^*\big) \leq 3, \quad k = 1, \dots, m, \\ &5 \leq d\big((x_k^0, y_k^0), D^*\big) \leq 7, \quad k = m+1, \dots, m+n. \end{aligned} \tag{22}$$

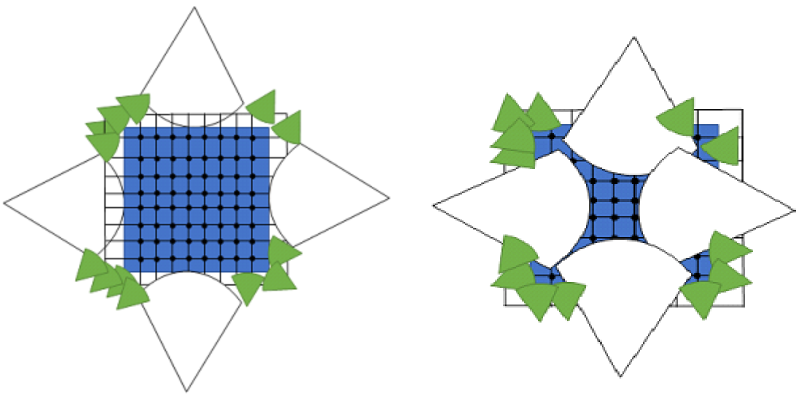

**Figure 5:** Initial configuration (left) and intermediate configuration (right).

To minimize the objective function (19), we initially select centers satisfying (22). If a selected center violates these constraints, it is replaced by a neighboring node until the conditions are met. Afterward, the set $D^p$ and the value $n(D^* \setminus D^p)$ are computed, yielding the value of the objective function $f$ defined in (18). This procedure naturally leads to Algorithm 1, which is proposed for solving the discrete optimization problem (19).

First, choose the points $(x_k^0, y_k^0)$ that satisfy the conditions (22). If condition (22) is not satisfied for a given $k$, then a neighboring nodal point $(x_k^1, y_k^1)$ of $(x_k^0, y_k^0)$ is selected such that condition (22) holds. By moving to

**Algorithm 5** Discrete optimization algorithm for optimal sector covering

**Input.** Set $i = 0$. Choose initial nodal points $z_k^0 = (x_k^0, y_k^0)$ and azimuth angles $\alpha_1^k$, $k = 1, \dots, m+n$, such that $z_k^0 \in Q^* \setminus D^*$.

**Step 1.** If for some $k$ the point $z_k^0$ does not satisfy condition (22), replace it with a neighboring nodal point that does not belong to $D^*$ and satisfies (22).

**Step 2.** Evaluate

$$f_0 = f(x_1^0, y_1^0, \dots, x_{m+n}^0, y_{m+n}^0, \alpha_1^1, \dots, \alpha_1^{m+n})$$

using (18).

**Step 3.** For each $k = 1, \dots, m+n$, generate the neighboring candidates

$$(x_k^0 \pm h_x, y_k^0), \quad (x_k^0, y_k^0 \pm h_y), \quad \alpha_1^k \pm \Delta\alpha,$$

and evaluate the corresponding objective function values

$$f_k^{x+},\ f_k^{x-},\ f_k^{y+},\ f_k^{y-},\ f_k^{\alpha+},\ f_k^{\alpha-}.$$

**Step 4.** Compute

$$f_{\text{min}} = \min_{k=1,\dots,m+n} \left\{ f_k^{x+}, f_k^{x-}, f_k^{y+}, f_k^{y-}, f_k^{\alpha+}, f_k^{\alpha-} \right\}.$$

**Step 5.** Update only the coordinate or azimuth corresponding to $f_{\text{min}}$, obtaining a new nodal configuration $(x_k^1, y_k^1, \alpha_1^k)$.

**Step 6.** If $f_{\text{min}} = 0$, terminate the algorithm.

**Step 7.** If $f_{\text{min}} \le f_0$, set $f_0 = f_{\text{min}}$, increment $i = i + 1$, and return to Step 1.

**Output.** The value $f_{\text{min}}$ and the corresponding nodal configuration, which represents a local minimum of (19).

neighboring points, the distances $d\big((x_k^1, y_k^1), D^*\big)$ appearing in (9) can either increase or decrease. Repeating this procedure a finite number of times guarantees that condition (22) is satisfied.

After this adjustment, the set $D^p$ is constructed and the value $n(D^* \setminus D^p)$ is computed. This value coincides with the objective function $f$ defined in (18). Note that $n(D^* \setminus D^p)$ can also be computed by checking each point $z_{ij} \in D^*$. If a point $z_{ij}$ does not belong to any of the sectors $P_k$, then it is counted as an element of $D^* \setminus D^p$. By examining all points in $D^*$ in this manner, the total number of uncovered points is obtained.

Based on the above discussion, we propose Algorithm 5 for solving the discrete optimization problem (19).

The proposed algorithm minimizes the objective function (19) by successively moving to neighboring nodal points until a local minimum is reached.

**Example 1.** Consider a square region $D$ (Figure 5, left) with two types of devices: four devices with radius $R = 40\,\text{km}$ and sector angle $\alpha = 60°$, and thirteen devices with radius $R = 10\,\text{km}$ and sector angle $\alpha = 60°$. In this example, we investigate the maximum coverage of the domain $D$ using Algorithm 5.

By applying Algorithm 5 and iterating through its steps, the devices gradually move toward the coverage area and their penetration angles are adjusted so as to cover increasing portions of $D$ (Figure 5, right). By perform-

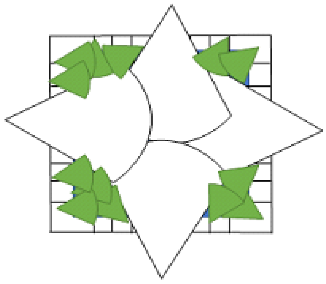

**Figure 6:** Final configuration corresponding to complete coverage of the domain $D$.

ing additional iterations of the algorithm, the devices eventually achieve complete coverage of the desired region (Figure 6).

More precisely, in the initial configuration there are $64$ nodal points within the square domain $D$ to be covered (Figure 5, left). After several iterations, an intermediate configuration is obtained (Figure 5, right), in which $46$ nodal points are covered by the devices. Upon full and accurate execution of the algorithm, the remaining $18$ uncovered nodal points are also covered, resulting in complete coverage of the square domain $D$ (Figure 6).

## 5 Conclusion

This paper proposes a new algorithm for solving discrete optimization problems arising in covering a planar domain by sectors. The proposed algorithm enables maximal coverage of a given domain $D$ using a minimum number of devices with radii $40\,\text{km}$ and $10\,\text{km}$ and a fixed sector angle of $60°$. Moreover, the algorithm minimizes the associated objective function by sequentially moving to neighboring nodal points. The developed approach is applicable to various practical problems, including military applications such as multilayer target coverage during missile launch operations. From a theoretical perspective, the algorithm can be extended to higher-dimensional covering problems, which will be investigated in future work.

## Declarations

### Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

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### Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

### Author Contributions

**Ali Shokri**: Conceptualization, Methodology, Software, Formal analysis, Investigation, Corresponding Author, Project Administration, Supervision, Original Draft, Writing, Review and Editing. **R.R. Maharramov**: Methodology, Software, Formal analysis, Investigation. **M.M. Mutallimov**: Conceptualization, Methodology, Software, Formal analysis, Investigation. **E.G. Hasimov**: Software, Formal analysis, Investigation. **I.A. Maharramov**: Software, Formal analysis, Investigation, Writing, Original Draft, Writing, Review and Editing.

**Artificial Intelligence Statement**

Artificial intelligence (AI) tools, including large language models, were used solely for language editing and improving readability. AI tools were not used for generating ideas, performing analyses, interpreting results, or writing the scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

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