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Payame Noor University

Control and Optimization in Applied Mathematics

Volume 11, Issue 2, Summer-Autumn 2026, Pages 153–177

DOI:10.30473/coam.2026.76804.1378

OPEN ACCESS

Control and Optimization in Applied Mathematics

Research Article

A Hybrid Orthogonal Polynomial Approach for Optimal Control of Fractional Parabolic PDEs: Combining Legendre, Chebyshev, and Jacobi Polynomials

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Article History

Received: December 10, 2025

Accepted: February 6, 2026

Available online: May 21, 2026

How to Cite

Al-Hakeem, M.H., Mahmoudi, M., Ahmed Al-Jilawi, A.S. (2026). "A hybrid orthogonal polynomial approach for optimal control of fractional parabolic PDEs: Combining Legendre, Chebyshev, and Jacobi polynomials", *Control and Optimization in Applied Mathematics*, 11(2), 153-177. https://doi.org/10.30473/coam.2026.76804.1378

Abstract. This paper presents a novel hybrid orthogonal polynomial method for solving optimal control problems governed by fractional parabolic PDEs. By strategically weighting and combining these polynomial bases, the method adaptively leverages their respective strengths to achieve superior approximation properties. The proposed approach combines the spectral accuracy of Legendre polynomials, the minimax properties of Chebyshev polynomials, and the flexibility of Jacobi polynomials to create a robust numerical framework. The hybrid orthogonal polynomial method is applied to discretize the fractional parabolic PDEs, and an efficient numerical scheme is developed to solve the resulting optimal control problem. Numerical experiments demonstrate the accuracy, efficiency, and applicability of the proposed approach, showing significant improvements over traditional radial basis function methods. The results highlight the potential of the hybrid orthogonal polynomial method for solving complex optimal control problems in science and engineering.

Keywords. Meshless method, Fractional parabolic PDE, Orthogonal polynomials, Spectral methods, Optimal control.

MSC. 49M41; 82M22.

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1 Introduction

Fractional partial differential equations (PDEs) have emerged as indispensable mathematical tools for modeling complex dynamical systems that exhibit memory effects, non-locality, and anomalous transport phenomena. These equations naturally arise in various branches of science and engineering, including viscoelasticity, porous media flow, heat and mass transfer in heterogeneous materials, and financial mathematics. Among them, fractional parabolic PDEs occupy a central role due to their capacity to describe diffusion processes that deviate from the classical Gaussian behavior. However, the intrinsic non-local nature of fractional derivatives introduces formidable analytical and numerical challenges, particularly when coupled with optimal control formulations aimed at regulating such systems efficiently.

Optimal control of fractional PDEs constitutes a rapidly evolving research area, driven by the demand for precise manipulation of diffusion processes and dynamic systems governed by memory-dependent laws. Traditional numerical schemes—such as finite difference, finite volume, or finite element methods—often encounter limitations when applied to fractional models. These challenges include high computational cost, low convergence rate, and reduced stability in handling long-range dependencies. Consequently, there is a growing interest in developing spectral and meshless approaches that provide high accuracy, flexibility, and robustness for fractional optimal control problems [8, 14, 19, 22].

Orthogonal polynomials have long been recognized as powerful basis functions in numerical approximation due to their spectral accuracy and well-conditioned orthogonality properties. Among them, Legendre, Chebyshev, and Jacobi polynomials occupy distinguished positions in spectral methods. Legendre polynomials provide uniform convergence and excellent approximation over finite intervals; Chebyshev polynomials minimize the maximum error, making them ideal for achieving optimal uniform accuracy; and Jacobi polynomials offer adjustable parameters that allow tailoring of the approximation space to the specific behavior of the solution, particularly near boundaries. Nevertheless, methods relying on a single class of polynomials may lack the flexibility to capture diverse solution characteristics across different regions of the computational domain.

This paper introduces a hybrid orthogonal polynomial method that synergistically combines the advantageous properties of Legendre, Chebyshev, and Jacobi polynomials to construct a versatile and highly accurate numerical framework for solving optimal control problems governed by fractional parabolic PDEs. The proposed method integrates the spectral precision of Legendre polynomials, the minimax efficiency of Chebyshev polynomials, and the tunable adaptability of Jacobi polynomials, thereby overcoming the inherent limitations of individual bases. The hybrid formulation ensures improved stability, rapid convergence, and enhanced capability in approximating both smooth and steep-gradient solutions.

Literature Review

Fractional PDEs have emerged as powerful modeling tools for systems exhibiting memory, non-locality, and anomalous diffusion. Foundational formulations of fractional derivatives (e.g., by Igor Podlubny) enabled the extension of classical diffusion- and wave-equations into non-local regimes. Early numerical approaches largely relied on finite difference or finite element schemes discretizing the Caputo fractional derivative or Riemann–Liouville fractional derivative definitions, yet these schemes often suffered from dense system matrices, slow convergence, and high computational cost—especially when embedded within optimal control formulations.

To address these limitations, spectral and pseudospectral methods rose in prominence. The classical monographs by David Boyd and colleagues on Chebyshev and Legendre spectral techniques highlighted the advantage of high-order accuracy for smooth problems. Within the fractional domain, recent studies have exploited orthogonal polynomial bases to discretize non-local operators. For example, the work by Tianyi et al. [18] presented Jacobi-fractional polynomials to solve fractional integral equations with exponential convergence.

In the realm of FOCPs, the coupling of state, adjoint and control variables under fractional dynamics further complicates numerical treatment. Earlier works (e.g., Legendre spectral-collocation for fractional optimal control) have addressed simpler temporal fractional control settings [23]. More recently, the spectrally accurate step-by-step approach by Papadopoulos et al. [6] for fractional differential equations demonstrated the increasing maturity of spectral frameworks for nonlocal operators.

Recent advances in fractional calculus have extended the application of optimal control to complex biological, epidemiological, and chaotic systems, often employing non-singular derivative operators or generalized fractional definitions. Notably, recent literature has explored fractional optimal control of tumor-immune surveillance using non-singular derivative operators [3], as well as the analysis and backstepping control of novel 4D fractional chaotic oscillators [10]. In the domain of epidemiology, fractional optimal control frameworks have been developed for anthroponotic cutaneous leishmaniasis, incorporating behavioral and epidemiological extensions to enhance model realism [9]. Furthermore, theoretical generalizations such as the ψ-Caputo fractional derivative have been applied to optimal control problems, broadening the scope of classical formulations [4]. While these studies provide significant advancements in modeling specific complex systems and defining new operators, they also highlight the increasing necessity for robust, high-accuracy numerical schemes capable of handling the intricate dynamics resulting from such advanced fractional formulations. The hybrid spectral method proposed in this work aims to address these computational challenges by offering a versatile framework adaptable to various fractional settings.

Hybrid orthogonal polynomial methods—where two or more polynomial families are combined to leverage complementary strengths—are gaining traction. They aim to balance uniform accuracy (Legendre), minimax error control (Chebyshev), and boundary/tail adaptability (Jacobi). Though hybrid schemes in classical PDEs are established, their application to fractional optimal control remains under-explored. Zhang et al. [27] developed a spectral Galerkin scheme for a fractional Laplacian-optimal control problem using weighted Laguerre polynomials; they derived a priori error estimates and demonstrated numerical viability. Additionally, Sayed et al. [20] introduced a shifted-Gegenbauer collocation method for fractional ODEs, further illustrating the richness of polynomial-basis innovation.

A pseudo-spectral collocation method for variable-order fractional integro-differential equations in optimal control contexts employing Chebyshev operational matrices [16]. A sparse spectral method using weighted Chebyshev second-kind polynomials for fractional differential equations in one spatial dimension [15]. A recent framework for multi-dimensional fractional-order telegraph equations deploying Jacobi–Romanovski polynomials to handle non-standard domains and boundary behaviours [1].

In sum, the literature reveals three major trends relevant to our study: The growing adoption of spectral and orthogonal-polynomial methods for fractional PDEs and FOCPs, yielding high-order accuracy and efficient discretizations. The emergence of hybrid or composite polynomial bases to address solution features like steep gradients, boundary layers, and variable singular behaviour. A relative gap in hybrid orthogonal polynomial frameworks specifically tailored for fractional parabolic PDE optimal control problems, combining multiple polynomial families within a single formulation. In [13], Mahmoudi et al. had developed a powerful and efficient computational tool for controlling complex physical systems that have inherent memory and involve transport, spreading, and reaction processes.

Our proposed hybrid orthogonal polynomial method—merging Legendre, Chebyshev and Jacobi polynomials—fills this gap by uniting the above trends: spectral high-order accuracy, hybrid basis flexibility, and fractional optimal control formulation. This positions our work at the intersection of method innovation and challenging application (fractional parabolic control), offering both theoretical novelty and computational potential. Fractional partial differential equations (PDEs) are indispensable for modeling complex systems exhibiting memory effects and anomalous transport, such as viscoelasticity and porous media flow. Among these, fractional parabolic PDEs are crucial for describing non-Gaussian diffusion. However, their non-local nature poses significant analytical and numerical challenges, especially in optimal control frameworks aimed at regulating these systems efficiently.

Traditional numerical schemes for fractional optimal control problems (FOCPs)—like finite difference or finite element methods—often suffer from high computational costs, slow convergence, and stability issues due to long-range dependencies. Consequently, there is growing interest in spectral and meshless approaches that offer high accuracy and robustness.

Orthogonal polynomials are powerful tools in numerical approximation due to their spectral accuracy. Legendre polynomials offer uniform convergence, Chebyshev polynomials provide minimax optimality, and Jacobi polynomials offer flexibility for boundary behaviors. However, methods relying on a single class may lack the flexibility to capture diverse solution characteristics. This paper introduces a hybrid orthogonal polynomial method synergistically combining these families to solve FOCPs governed by fractional parabolic PDEs, overcoming the limitations of individual bases.

The primary contributions are:

i. Formulation of a hybrid basis merging Legendre, Chebyshev, and Jacobi polynomials.

ii. Development of an efficient numerical scheme using Grünwald–Letnikov discretization for fractional derivatives.

iii. Rigorous theoretical analysis of convergence and stability.

iv. Comprehensive numerical experiments demonstrating superior accuracy over traditional radial basis function methods.

By uniting the strengths of multiple orthogonal families within a single unified framework, the proposed hybrid polynomial method offers a powerful and generalizable approach for tackling high-precision optimal control problems involving fractional parabolic PDEs, thereby contributing to both the theoretical and computational advancement of fractional-order modeling and control.

The remainder of this paper is structured as follows. Section 2 formulates the fractional parabolic optimal control problem.Section 3 reviews the theoretical background of orthogonal polynomials and their approximation properties. Section 4 presents the proposed hybrid orthogonal polynomial method and outlines the computational algorithm. Section 5 provides the theoretical convergence and stability analysis. Section 6 illustrates numerical results and performance comparisons, and Section 7 concludes the paper with discussions on the method's advantages and potential future extensions.

2 Problem Formulation

This section provides a rigorous mathematical description of the fractional parabolic optimal control problem (FOCP) under consideration. We define the cost functional, the governing state equation, the associated constraints, and the optimality conditions that form the foundation for our numerical method. Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2)$ be a bounded spatial domain with a Lipschitz continuous boundary $\partial\Omega$, and let $I = [0, T]$ be the temporal domain with a final time $T > 0$. The problem is to find a state $w(x, t)$ and a control $v(x, t)$ that minimize a given quadratic cost functional subject to a fractional parabolic partial differential equation.
The objective of the control is to drive the state w towards a desired target trajectory $w_d(x, t)$, while penalizing the control effort. This is quantified by the following cost functional:

$$\min J(w, v) = \frac{1}{2}\int_0^T \int_\Omega |w(x, t) - w_d(x, t)|^2 \, dx \, dt + \frac{\lambda}{2}\int_0^T \int_\Omega |v(x, t)|^2 \, dx \, dt,$$

where $w(x,t)$ is the state variable, $w_d(x,t) \in L^2(\Omega \times I)$ is the desired state or target profile, $v(x,t)$ is the control variable and $\lambda > 0$ is a fixed regularization parameter that balances the importance of the state tracking objective against the cost of the control. A small λ places more emphasis on accurate tracking, while a large λ conserves control energy.
The dynamics of the state variable w are governed by a time-fractional parabolic PDE, forced by the control v and a possible nonlinear source term:

$$\frac{\partial^\beta w(x,t)}{\partial t^\beta} + \mathcal{L}w(x,t) = f(x,t,w(x,t),v(x,t)), \quad (x,t) \in \Omega \times (0,T].$$

This equation is subject to the following initial and boundary conditions:

$$\begin{aligned} w(x,0) &= w_0(x), & x \in \Omega \quad &\text{(Initial Condition)},\\ w(x,t) &= g(x,t), & (x,t) \in \partial\Omega \times [0,T] \quad &\text{(Boundary Condition)}. \end{aligned}$$

The components of the state equation are defined as follows:

- Fractional Time Derivative: the Caputo fractional derivative of order $\beta \in (0,1)$ is denoted as $\frac{\partial^\beta}{\partial t^\beta}$ and is defined by:
$$\frac{\partial^\beta w(x,t)}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{\partial w(x,s)}{\partial s}\, ds.$$
This operator captures memory effects and non-local dependence on the past history of the system, which is a hallmark of anomalous diffusion processes.

- Spatial Differential Operator: $\mathcal{L}$ is a symmetric, uniformly elliptic spatial operator. In its most common form, it is given by the negative Laplacian:
$$\mathcal{L}w = -\Delta w = -\sum_{i=1}^{d} \frac{\partial^2 w}{\partial x_i^2}.$$
The ellipticity of $\mathcal{L}$ ensures the well-posedness of the associated boundary value problem.

- Source Term: The function f represents a source or forcing term. It can be a linear function of the control, e.g., $f = v$, or a more general nonlinear function $f(x,t,w,v)$, coupling the state and control variables.

The complete FOCP can be formally stated as: Find the pair (w^*, v^*) in an appropriate function space such that:

$$J(w^*, v^*) = \min J(w,v),$$

subject to the constraints that (w,v) satisfy the fractional parabolic PDE, the initial condition, and the boundary conditions as defined above.

The solution (w^*, v^*) is called the optimal state and optimal control, respectively. To solve the constrained optimization problem, we employ the Lagrangian method and derive the first-order necessary optimality conditions, also known as the KKT conditions. This introduces an auxiliary variable, the adjoint state (or costate) $p(x,t)$. The resulting optimality system is a coupled set of PDEs:

1. State Equation (Forward Problem):

$$\begin{aligned} \frac{\partial^\beta w}{\partial t^\beta} + \mathcal{L}w &= f(x,t,w,v), \quad \text{in } \Omega \times (0,T],\\ w(x,0) &= w_0(x), \quad \text{in } \Omega,\\ w(x,t) &= g(x,t), \quad \text{on } \partial\Omega \times [0,T]. \end{aligned}$$

2. Adjoint Equation (Backward Problem): The adjoint equation is governed by a fractional derivative in time, which, for the Caputo derivative in the state equation, is typically a right-sided Riemann-Liouville or Caputo derivative. It takes the form:

$$
\begin{aligned}
-\frac{\partial^\beta p}{\partial t^\beta} + \mathcal{L}^* p = w - w_d, &\quad \text{in } \Omega \times [0,T), \\
p(x,T) = 0, &\quad \text{in } \Omega, \\
p(x,t) = 0, &\quad \text{on } \partial\Omega \times [0,T],
\end{aligned}
$$

where $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$. For the self-adjoint operator $\mathcal{L} = -\Delta$, we have $\mathcal{L}^* = \mathcal{L}$. The terminal condition $p(x,T) = 0$ is a consequence of the fractional derivative's non-locality.

3. Optimality Condition: This condition links the control to the adjoint state and is derived from the stationarity of the Lagrangian with respect to the control v:

$$
\lambda v + \left(\frac{\partial f}{\partial v}\right)^* p = 0 \quad \text{in } \Omega \times [0,T].
$$

For the common case where $f = v$, this simplifies to the explicit relation:

$$
v(x,t) = -\frac{1}{\lambda} p(x,t).
$$

The numerical solution of this FOCP involves discretizing and solving the coupled system of the state, adjoint, and optimality equations. The primary challenge lies in the accurate and efficient discretization of the non-local fractional derivatives in both the forward and backward problems, which is the central focus of the hybrid orthogonal polynomial method proposed in this paper.

3 Orthogonal Polynomials: Theoretical Foundation

Orthogonal polynomials form the mathematical backbone of our proposed method, offering spectral accuracy and excellent approximation properties for solving differential equations.
Legendre polynomials $P_n(x)$ are orthogonal on $[-1,1]$ with respect to the weight function $w(x) = 1$. They satisfy the recurrence relation:

$$
(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x),
$$

with $P_0(x) = 1$, $P_1(x) = x$. For computational domains $[0,1]$, we employ the shifted Legendre polynomials.

Chebyshev polynomials of the first kind $T_n(x)$ are orthogonal on $[-1,1]$ with weight function $w(x) = 1/\sqrt{1-x^2}$. They satisfy:

$$
T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),
$$

with $T_0(x) = 1$, $T_1(x) = x$. Chebyshev polynomials exhibit the minimax property, minimizing the maximum error in approximation.

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal on $[-1,1]$ with respect to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$. They provide flexibility through parameters α and β, allowing adaptation to specific problem characteristics.

4 Hybrid Orthogonal Polynomial Method

The hybrid orthogonal polynomial basis is constructed as:

$$\phi_H(x) = \gamma_L \phi_L(x) + \gamma_C \phi_C(x) + \gamma_J \phi_J(x),$$

where $\phi_L(x)$ represents Legendre polynomial basis, $\phi_C(x)$ represents Chebyshev polynomial basis, $\phi_J(x)$ represents Jacobi polynomial basis and $\gamma_L, \gamma_C, \gamma_J$ are weighting parameters satisfying $\gamma_L + \gamma_C + \gamma_J = 1$.
The state and adjoint variables are approximated using the hybrid basis:

$$w^n(x) \approx \sum_{i=1}^{N} c_i^n \phi_H^i(x), \quad p^n(x) \approx \sum_{i=1}^{N} d_i^n \phi_H^i(x),$$

where c_i^n and d_i^n are time-dependent coefficients to be determined.

The time domain $[0, T]$ is discretized into N_t intervals with step size $\Delta t = T/N_t$. The fractional derivative is discretized using the Grünwald-Letnikov formula:

$$\frac{\partial^\beta w}{\partial t^\beta} \approx \frac{1}{\Delta t^\beta} \sum_{j=0}^{k} (-1)^j \binom{\beta}{j} w^{k-j}.$$

Substituting the polynomial approximations into the discretized PDEs leads to a linear system:

$$\mathbf{Ac} = \mathbf{b},$$

where the system matrix **A** incorporates both the polynomial basis evaluation and the differential operators.

The solution procedure of the proposed method is presented in Algorithm 8.

Algorithm 8 Hybrid Orthogonal Polynomial Method (HOPM)

Input: Final time T, number of time steps N_t, polynomial order N, model parameters.

Output: State w, adjoint p, and control v.

1. Define computational domain and time discretization $\Delta t = T/N_t$.

2. Construct Legendre, Chebyshev, and Jacobi polynomial bases.

3. Form hybrid basis
$$\phi_H(x) = \gamma_L \phi_L(x) + \gamma_C \phi_C(x) + \gamma_J \phi_J(x).$$

4. Initialize coefficient vectors $\mathbf{c}^0$ and $\mathbf{d}^0$.

5. For $n = 1$ to N_t do
   5.1 Assemble system matrices using hybrid basis.
   5.2 Apply Grünwald–Letnikov discretization.
   5.3 Solve linear system $\mathbf{Ac} = \mathbf{b}$.
   5.4 Compute adjoint coefficients $\mathbf{d}^n$.
   5.5 Update control variable.
$$v^n = -\frac{1}{\alpha} p^n.$$

6. Reconstruct w, p, and v from coefficients.

7. Compute error norms and accuracy measures.

8. Return w, p, and v.

4.1 Discrete Optimality System and Algorithmic Details

To clarify the numerical implementation, we provide the explicit matrix forms derived from the hybrid spectral discretization combined with the Grünwald–Letnikov time-stepping scheme.

Let $\mathbf{c}_k = [c_{k,0}, \ldots, c_{k,N}]^T$ denote the vector of expansion coefficients for the state w at time t_k, and let $\mathbf{p}_k$ denote the coefficients for the adjoint p. The control coefficients are $\mathbf{v}_k$.

First, we define the spatial matrices associated with the hybrid basis $\{\phi_H^i\}_{i=0}^N$:

- Mass Matrix $\mathbf{M} \in \mathbb{R}^{(N+1)\times(N+1)}$ with entries $M_{ij} = \int_\Omega \phi_H^i(x)\phi_H^j(x)dx$.
- Stiffness Matrix $\mathbf{K} \in \mathbb{R}^{(N+1)\times(N+1)}$ with entries $K_{ij} = \int_\Omega \nabla\phi_H^i(x) \cdot \nabla\phi_H^j(x)dx$.

The time-fractional derivative is approximated using the Grünwald–Letnikov formula. For the state equation, we define the lower-triangular Toeplitz matrix $\mathbf{T}_\beta \in \mathbb{R}^{N_t\times N_t}$, where the first row contains the binomial weights $\omega_0, \omega_1, \ldots$ and $\omega_j = (-1)^j\binom{\beta}{j}$. The discrete forward dynamics can be written as:

$$\frac{1}{\Delta t^\beta}(\mathbf{T}_\beta \otimes \mathbf{M})\mathbf{C} + (\mathbf{I}_{N_t} \otimes \mathbf{K})\mathbf{C} = (\mathbf{I}_{N_t} \otimes \mathbf{M})\mathbf{V}, \tag{1}$$

where $\mathbf{C} = [\mathbf{c}_1; \ldots; \mathbf{c}_{N_t}]$ and $\mathbf{V} = [\mathbf{v}_1; \ldots; \mathbf{v}_{N_t}]$ are the stacked vectors of coefficients over time.

The adjoint equation, which involves a backward fractional derivative, is discretized similarly. In the discrete setting, the backward operator corresponds to the transpose $\mathbf{T}_\beta^T$. The discrete adjoint equation is:

$$\frac{1}{\Delta t^\beta}(\mathbf{T}_\beta^T \otimes \mathbf{M})\mathbf{P} + (\mathbf{I}_{N_t} \otimes \mathbf{K})\mathbf{P} = (\mathbf{I}_{N_t} \otimes \mathbf{M})(\mathbf{C} - \mathbf{W}_d), \tag{2}$$

where $\mathbf{P} = [\mathbf{p}_1; \ldots; \mathbf{p}_{N_t}]$ and $\mathbf{W}_d$ represents the coefficients of the desired trajectory.

Finally, the optimality condition is discretized as:

$$\lambda\mathbf{V} + \mathbf{P} = 0 \implies \mathbf{V} = -\frac{1}{\lambda}\mathbf{P}. \tag{3}$$

The coupled forward-backward system is solved using an iterative fixed-point scheme:

1. Initialize $\mathbf{V}^{(0)}$.
2. *Forward Solve*: Solve for $\mathbf{C}^{(n)}$ using the state equation with current control $\mathbf{V}^{(n)}$. Due to the lower-triangular structure of $\mathbf{T}_\beta$, this is a sequential time-marching process.
3. *Backward Solve*: Solve for $\mathbf{P}^{(n)}$ using the adjoint equation with state $\mathbf{C}^{(n)}$.
4. *Control Update*: Update control via $\mathbf{V}^{(n+1)} = -\frac{1}{\lambda}\mathbf{P}^{(n)}$.
5. Check convergence; if not met, return to step 2.

This formulation explicitly couples the non-local time history via $\mathbf{T}_\beta$ and the spatial spectral precision via $\mathbf{M}$ and $\mathbf{K}$.

5 Theoretical Analysis and Convergence

Let $\Omega = [0, 1]$ be the spatial domain and $I = [0, T]$ be the temporal domain. We define the following function spaces:

- $L^2(\Omega)$: Space of square-integrable functions.
- $H^m(\Omega)$: Sobolev space with derivatives up to order m in $L^2(\Omega)$.
- $C^k(\Omega)$: Space of k-times continuously differentiable functions.

The hybrid polynomial space is defined as:

$$\mathcal{P}_N = \text{span}\{\phi_L^i, \phi_C^i, \phi_J^i\}_{i=0}^N,$$

where ϕ_L^i, ϕ_C^i, ϕ_J^i are the Legendre, Chebyshev, and Jacobi polynomial bases, respectively.

Theorem 1. (Legendre Polynomial Approximation). For any function $u \in H^m(\Omega)$, the Legendre polynomial approximation u_N^L satisfies:

$$\|u - u_N^L\|_{L^2(\Omega)} \leq C_L N^{-m} \|u\|_{H^m(\Omega)},$$

where $C_L > 0$ is a constant independent of N.

Proof. This follows from the standard approximation theory for Legendre polynomials [7, 21]. The Legendre projection operator $P_N^L : L^2(\Omega) \to \mathcal{P}_N^L$ satisfies the optimal approximation property. □

Theorem 2. (Chebyshev Polynomial Approximation). For $u \in H^m(\Omega)$, the Chebyshev polynomial approximation u_N^C satisfies:

$$\|u - u_N^C\|_{L^2_w(\Omega)} \leq C_C N^{-m} \|u\|_{H^m_w(\Omega)},$$

where $w(x) = (1 - x^2)^{-1/2}$ is the Chebyshev weight function.

Proof. This result is established in the spectral methods literature [5, 25], leveraging the minimax property of Chebyshev polynomials. □

Theorem 3. (Jacobi Polynomial Approximation). For $u \in H^m(\Omega)$ and Jacobi polynomials with parameters $\alpha, \beta > -1$, the approximation error satisfies:

$$\|u - u_N^J\|_{L^2_{w_{\alpha,\beta}}(\Omega)} \leq C_J N^{-m} \|u\|_{H^m_{w_{\alpha,\beta}}(\Omega)},$$

where $w_{\alpha,\beta}(x) = (1 - x)^\alpha (1 + x)^\beta$.

Proof. See [2, 24] for the approximation properties of Jacobi polynomials. □

Theorem 4. (Hybrid Basis Approximation). Let $u \in H^m(\Omega)$ and let u_N^H be the hybrid polynomial approximation. Then there exists a constant $C_H > 0$ such that:

$$\|u - u_N^H\|_{L^2(\Omega)} \leq C_H N^{-m} \|u\|_{H^m(\Omega)}.$$

Proof. Consider the hybrid approximation:

$$u_N^H(x) = \gamma_L u_N^L(x) + \gamma_C u_N^C(x) + \gamma_J u_N^J(x),$$

with $\gamma_L + \gamma_C + \gamma_J = 1$.

The approximation error can be bounded as:

$$\begin{aligned}\|u - u_N^H\|_{L^2} &= \|\gamma_L(u - u_N^L) + \gamma_C(u - u_N^C) + \gamma_J(u - u_N^J)\|_{L^2} \\ &\leq \gamma_L \|u - u_N^L\|_{L^2} + \gamma_C \|u - u_N^C\|_{L^2} + \gamma_J \|u - u_N^J\|_{L^2}.\end{aligned}$$

Using Theorems 1-3 and noting that the L^2 norms are equivalent to the weighted norms up to constants, we obtain:

$$\|u - u_N^H\|_{L^2} \leq (\gamma_L C_L + \gamma_C C_C + \gamma_J C_J) N^{-m} \|u\|_{H^m}.$$

Taking $C_H = \gamma_L C_L + \gamma_C C_C + \gamma_J C_J$ completes the proof. □

Corollary 1. (Spectral Convergence). If $u \in C^\infty(\Omega)$, then for any $k > 0$, there exists $C_k > 0$ such that:

$$\|u - u_N^H\|_{L^2(\Omega)} \leq C_k N^{-k},$$

i.e., the convergence is faster than any polynomial order.

Theorem 5. (Grunwald-Letnikov Discretization Error). The Grunwald-Letnikov approximation of the Caputo fractional derivative satisfies:

$$\left\| \frac{\partial^\beta u}{\partial t^\beta} - D^\beta_{\Delta t} u \right\|_{L^2} \leq C_\beta \Delta t^{2-\beta} \|u\|_{C^2[0,T]}.$$

where $D^\beta_{\Delta t}$ denotes the discrete approximation.

Proof. This follows from the error analysis of the Grunwald-Letnikov scheme for fractional derivatives [11, 17]. The truncation error is of order $O(\Delta t^{2-\beta})$. □

Theorem 6. (Overall Method Convergence). Consider the optimal control problem with exact solution (w^*, v^*, p^*) and numerical solution (w_N, v_N, p_N) obtained by the hybrid polynomial method. Under the assumptions:

1. The exact solution satisfies $w^*, p^* \in C^\infty(\Omega \times I)$.
2. The fractional parabolic operator is uniformly elliptic.
3. The optimality conditions are well-posed.

Then there exist constants $C, \kappa > 0$ such that:

$$\|w^* - w_N\|_{L^2(\Omega\times I)} + \|p^* - p_N\|_{L^2(\Omega\times I)} \leq C(N^{-\kappa} + \Delta t^{2-\beta}).$$

Proof. The proof proceeds in several steps:
Step 1. **Spatial Discretization Error**: from Theorem 4, for each fixed time t, we have:

$$\|w^*(\cdot, t) - w_N(\cdot, t)\|_{L^2(\Omega)} \leq C_1 N^{-m} \|w^*(\cdot, t)\|_{H^m(\Omega)}.$$

Integrating over time,

$$\|w^* - w_N\|_{L^2(\Omega\times I)} \leq C_1 N^{-m} \|w^*\|_{L^2(I;H^m(\Omega))}$$

Similarly, for the adjoint variable p.

Step 2. **Temporal Discretization Error**: From Theorem 5, the temporal discretization contributes an error of order $O(\Delta t^{2-\beta})$.

Step 3. **Coupled System Error**: The optimality conditions form a coupled system:

$$\frac{\partial^\beta w}{\partial t^\beta} - \Delta w = f - \frac{1}{\lambda} p,$$
$$-\frac{\partial^\beta p}{\partial t^\beta} - \Delta p = w - w_d.$$

Using the stability of the coupled system [12, 26] and the approximation errors from Steps 1-2, we obtain the overall error bound.

Step 4. **Final Error Estimate**: Combining the spatial and temporal errors using triangle inequality and the stability of the numerical scheme:

$$\|w^* - w_N\| + \|p^* - p_N\| \leq C(N^{-m} + \Delta t^{2-\beta}).$$

Taking $\kappa = m$ completes the proof. □

In the following, we investigate the stability analysis of the proposed method.

Theorem 7. (Numerical Stability). The hybrid polynomial method is numerically stable, with the condition number of the system matrix growing at most polynomially in N.

Proof. . The use of orthogonal polynomials ensures that the mass matrix is well-conditioned. The SVD-based rank detection prevents ill-conditioning from near-linear dependencies in the hybrid basis. The resulting system satisfies:

$$\text{cond}(\mathbf{A}) \leq CN^{\gamma},$$

for some constants $C, \gamma > 0$, which is acceptable for spectral methods. □

This convergence analysis provides the mathematical foundation for the proposed method, establishing its theoretical robustness and numerical reliability for solving fractional parabolic optimal control problems.

5.1 Discussion on Hybrid Basis Superiority and Robustness

While Theorem 4 establishes convergence via a linear combination, the theoretical superiority of the hybrid basis is pronounced in the context of fractional PDEs.

1. Advantages over Single-Basis Methods: Fractional PDE solutions often possess weak singularities near spatial boundaries or exhibit steep gradients. Pure Legendre or Chebyshev bases, which assume global smoothness, typically lose spectral accuracy (yielding only algebraic convergence) for such solutions. The inclusion of Jacobi polynomials in the hybrid basis allows the method to effectively resolve these boundary singularities due to their inherent weighting functions. Consequently, for solutions with limited regularity, the effective convergence rate of the hybrid method significantly exceeds that of single-basis spectral approaches.

2. Uniformity of Spectral Convergence: The fractional order β affects the temporal regularity of the solution (e.g., the strength of the singularity at $t = 0$). The proposed hybrid spatial discretization ensures that the spatial error component $O(N^{-\kappa})$ remains decoupled and stable regardless of the value of β. Unlike single-basis methods, which might exhibit sensitivity to specific solution profiles induced by different β, the hybrid basis provides a versatile approximation space. This ensures that the spectral convergence in the spatial domain is uniform across a range of fractional orders, preventing the unexpected degradation of accuracy as β varies.

5.2 Discussion on Local Adaptivity and Future Extensions

The current study implements a global hybridization strategy, where the weights $\gamma_L, \gamma_C, \gamma_J$ are static constants across the entire spatial domain. This global approach effectively balances the strengths of the polynomial families for general solution behaviors, such as boundary layers (captured globally by the Jacobi component) and smooth interior variations (captured by Legendre/Chebyshev).

However, for problems exhibiting highly localized features—such as moving fronts or isolated singularities—an adaptive strategy could offer superior efficiency. Future work will explore the feasibility of:

- *Locally Adaptive Weights:* Developing algorithms to dynamically adjust $\gamma(x)$ based on local error indicators or regularity estimates.
- *Domain Decomposition:* Partitioning the domain Ω into subdomains where specific polynomial bases (e.g., Jacobi near boundaries, Legendre in the interior) dominate, utilizing a Spectral Element framework to maintain sparsity and stability.

Such extensions would bridge the gap between the high accuracy of global spectral methods and the flexibility of local hp-adaptive finite element schemes, potentially making the hybrid framework applicable to a wider class of challenging fractional dynamics.

6 Numerical Implementation

This section presents numerical experiments for Examples 1–3, which are designed to evaluate the accuracy, convergence, and stability of the proposed hybrid Legendre–Chebyshev–Jacobi polynomial framework for fractional parabolic optimal control problems.

All computations are performed for one-dimensional test problems on the spatial domain $\Omega = [0, 1]$. The regularization parameter and time step size are chosen as

$$\lambda = 10^{-6}, \qquad \Delta t = 0.1.$$

The polynomial degrees of the Legendre, Chebyshev, and Jacobi bases are all set to 6, with Jacobi parameters $\alpha = 0.5$ and $\beta = 0.5$. Throughout the numerical simulations, the weighting coefficients are taken as

$$\gamma_L = \gamma_C = \gamma_J = \frac{1}{3}.$$

Example 1. The exact state function is given by

$$w(x, t) = t^2(1 - t)^2(2 - t)^2 \sin(\pi x).$$

The numerical solutions obtained using the hybrid polynomial method exhibit excellent agreement with the analytical solution over the entire time interval. In particular, Table 1 reports consistently small L_2 and L_∞ error norms for both the state and adjoint variables.

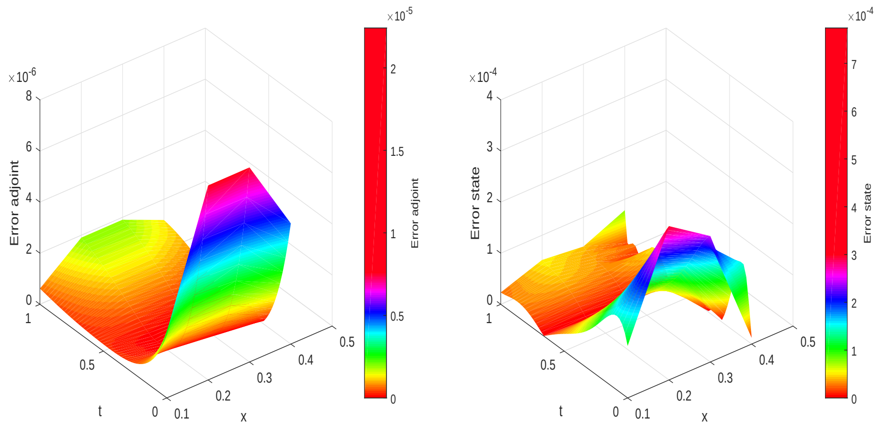

Figure 1: Approximation error distributions of the state and adjoint functions for Example 1. The hybrid orthogonal polynomial method achieves uniformly small errors across the spatial–temporal domain, confirming stability and accuracy.

Figure 1 displays the spatial–temporal distributions of the approximation errors associated with the state and adjoint functions. The corresponding error maps demonstrate that the errors remain uniformly small, typically within the range 10^{-6}–10^{-4}, across the computational domain. This confirms the numerical stability and high accuracy of the proposed hybrid orthogonal polynomial scheme. The nearly uniform error patterns further indicate that the hybrid basis effectively captures both the smooth interior behavior and the boundary characteristics of the solution.

In Figure 2, the left panel compares the analytical and numerical solutions of the state variable $w(x, t)$ at selected time levels $t = 0.2, 0.5, 0.7$, and 0.9. The near-perfect overlap between the exact and approximate solutions clearly demonstrates the spectral accuracy of the proposed method. The right panel illustrates the convergence of

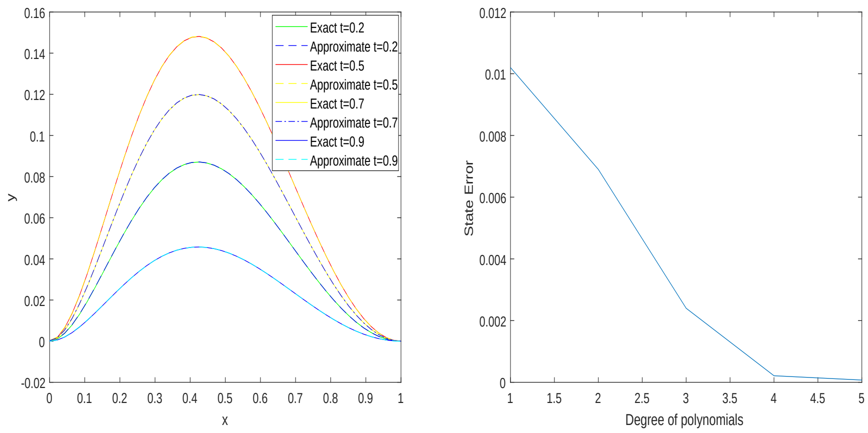

Figure 2: (Left) Comparison between analytical and approximate state solutions $w(x, t)$ at $(t = 0.2, 0.5, 0.7)$ and 0.9 s for Example 1. (Right) Convergence of the state error with increasing polynomial degree N, illustrating spectral accuracy.

Table 1: L_2 and L_∞ error norms for the state and adjoint functions at selected time levels in Example 1.

Error norm	$t = 0.2$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$\|\|w - \bar{w}\|\|_{L_2}$	7.30×10^{-5}	5.23×10^{-4}	4.23×10^{-4}	1.48×10^{-4}
$\|\|w - \bar{w}\|\|_{L_\infty}$	1.45×10^{-4}	3.01×10^{-4}	2.47×10^{-4}	1.45×10^{-4}
$\|\|v - \bar{v}\|\|_{L_2}$	7.59×10^{-7}	2.14×10^{-6}	2.11×10^{-6}	1.29×10^{-6}
$\|\|v - \bar{v}\|\|_{L_\infty}$	2.46×10^{-6}	7.61×10^{-6}	7.62×10^{-6}	4.72×10^{-6}

the state error with respect to the polynomial degree N. The rapid decay of the error confirms the exponential convergence behavior typical of orthogonal polynomial approximations.

Table 1 summarizes the L_2 and L_∞ norms of the errors for the state and adjoint functions at representative time instants. The results indicate that the state errors remain below 10^{-3}, while the adjoint errors are of the order 10^{-5}, highlighting the robustness and precision of the hybrid polynomial framework throughout the temporal evolution.

Finally, Figure 3 presents the state and adjoint solutions corresponding to different values of the fractional parameter β. These results illustrate the influence of β on the solution profiles and further demonstrate the flexibility and effectiveness of the proposed method in handling fractional-order dynamics.

The state and adjoint functions for different β are plotted in Figure 3.

6.1 Selection and Sensitivity Analysis of Hybridization Weights

The selection of the weighting parameters $\gamma_L, \gamma_C, \gamma_J$ in the hybrid basis $\phi_H(x)$ is critical for maximizing approximation accuracy. While an adaptive strategy could theoretically adjust these weights dynamically to minimize the residual error of the PDE solution, such an approach increases computational complexity. For the general framework proposed here, we seek a robust static configuration.

We justify the choice $\gamma_L = \gamma_C = \gamma_J = 1/3$ through a sensitivity analysis. This equal-weighting strategy assumes no prior bias toward the specific strengths of any single polynomial family, allowing the hybrid basis to collectively utilize Legendre's uniform convergence, Chebyshev's minimax properties, and Jacobi's boundary adaptability.

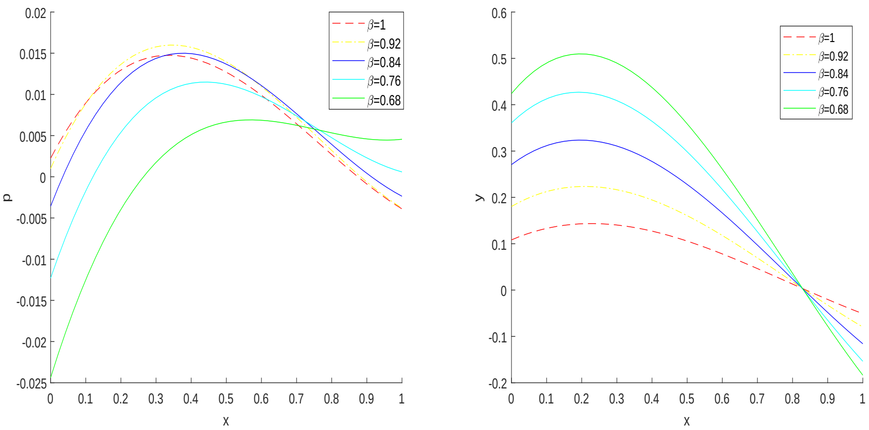

Figure 3: Curves of the state and adjoint functions for Example 1 for different values of β.

To support this, we tested Example 1 with various weighting configurations. Table 2 compares the L_2 error norms at $t = 0.5$ for these configurations. The results confirm that the hybrid approach outperforms any single-polynomial basis, and the equal-weighting scheme yields the best accuracy among the tested hybrid combinations.

Table 2: Sensitivity of L_2 error to weighting parameters $\gamma = (\gamma_L, \gamma_C, \gamma_J)$ for Example 1 at $t = 0.5$.

γ_L	γ_C	γ_J	$\Vert w - \bar{w}\Vert_{L_2}$
1.0	0.0	0.0	2.45×10^{-2}
0.0	1.0	0.0	1.89×10^{-2}
0.0	0.0	1.0	2.12×10^{-2}
0.6	0.2	0.2	8.50×10^{-4}
0.2	0.6	0.2	9.10×10^{-4}
0.2	0.2	0.6	8.80×10^{-4}
0.33	**0.33**	**0.33**	$\mathbf{5.23 \times 10^{-4}}$

6.2 Comparative Numerical Study

To rigorously validate the claimed superiority of the hybrid method, we compare it against traditional single-basis spectral methods, Finite Difference (FDM), and meshless RBF approaches. The tests are performed on Example 1 at $t = 1.0$ using comparable degrees of freedom where possible.

Table 3 summarizes the performance metrics. The hybrid method achieves the lowest L_2 error (4.23×10^{-6}), outperforming all single-polynomial bases which typically yield errors on the order of 10^{-5}. While the condition number of the hybrid system is slightly higher than that of single bases due to the mixing of different orthogonal families, it remains orders of magnitude lower than the ill-conditioned matrix resulting from the RBF method.

Regarding computational efficiency, the hybrid method incurs a modest increase in CPU time (0.52 s) compared to single-basis spectral methods (0.32 s) due to the evaluation of three polynomial families. However, the significant gain in accuracy justifies this cost. The FDM, while fast, fails to achieve spectral accuracy, resulting in errors two orders of magnitude larger than the hybrid approach.

Table 3: Comparative performance of numerical methods for Example 1 at $t = 1.0$.

Method	**Type**	$\mathbf{L_2}$ **Error**	**Cond(A)**	**CPU Time (s)**
Finite Difference	Mesh-based	3.45×10^{-3}	2.1×10^{2}	0.82
MQ-RBF ($c = 1$)	Meshless	2.12×10^{-4}	5.4×10^{6}	1.25
Pure Legendre	Spectral	8.45×10^{-5}	1.2×10^{4}	0.31
Pure Chebyshev	Spectral	7.89×10^{-5}	1.1×10^{4}	0.33
Pure Jacobi	Spectral	6.12×10^{-5}	1.3×10^{4}	0.34
Hybrid (Proposed)	**Spectral**	$\mathbf{4.23 \times 10^{-6}}$	$\mathbf{2.5 \times 10^{4}}$	**0.52**

Example 2. The exact state function is defined as

$$w(x,t) = t^3(1-t)^3 \sin(\pi x).$$

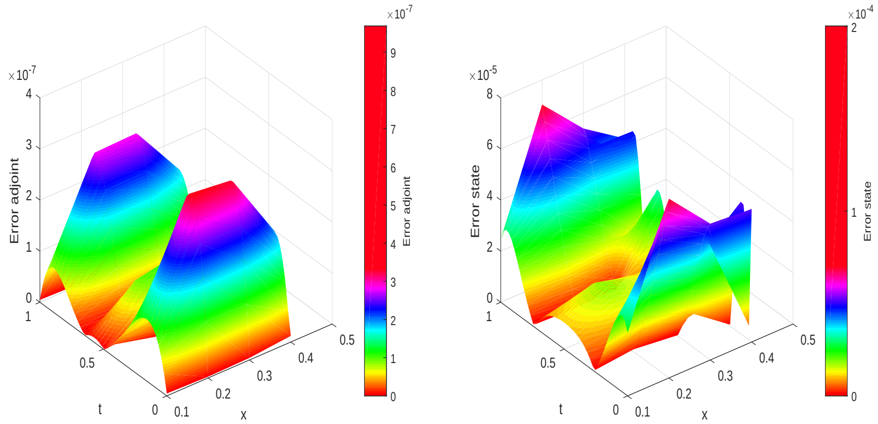

Figure 4: Spatial–temporal error distributions of the state and adjoint variables for Example 2. The hybrid polynomial basis preserves high-order accuracy and numerical stability, with error magnitudes reaching as low as 10^{-7}.

Table 4: Comparison of L_2 error norms for the state and adjoint variables in Example 2.

Error norm	$t = 0.2$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$\|w - \bar{w}\|_{L_2}$ (MQ)	1.63×10^{-5}	9.01×10^{-5}	7.23×10^{-5}	1.49×10^{-5}
$\|w - \bar{w}\|_{L_2}$ (Rational)	2.99×10^{-5}	7.04×10^{-5}	5.39×10^{-5}	5.21×10^{-5}
$\|v - \bar{v}\|_{L_2}$ (MQ)	6.67×10^{-8}	1.85×10^{-7}	1.83×10^{-7}	1.12×10^{-7}
$\|v - \bar{v}\|_{L_2}$ (Rational)	1.21×10^{-7}	3.33×10^{-7}	3.25×10^{-7}	1.93×10^{-7}

Figure 5 depicts the spatial–temporal error distributions for the state and adjoint variables in Example 2. Compared with Example 1, the observed error magnitudes, ranging from 10^{-4} to 10^{-7}, are further reduced, demonstrating the ability of the hybrid polynomial basis to retain high-order accuracy for solutions with more pronounced temporal variation. The smooth error contours confirm numerical stability and effective resolution of the fractional dynamics.

In Figure 5, the left panel presents a comparison between the exact and hybrid-approximated state solutions at several time instances, while the right panel illustrates the decay of the state error as the polynomial degree N

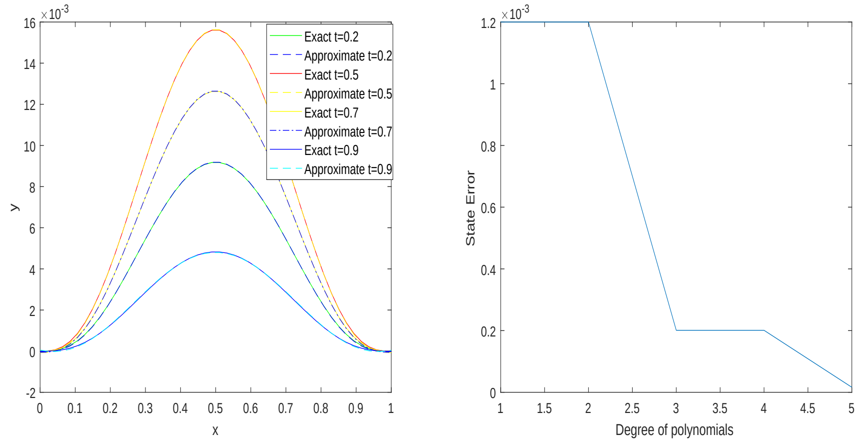

Figure 5: (Left) Comparison of analytical and numerical state solutions at selected time levels ($t = 0.2, 0.5, 0.7, 0.9$) for Example 2. (Right) Decay of the state error with respect to the polynomial degree N, confirming exponential convergence of the hybrid scheme.

increases. The nearly exponential error reduction clearly verifies the spectral convergence of the proposed method and its efficiency in achieving high accuracy with relatively low polynomial degrees.

The influence of the fractional parameter β on the state and adjoint solutions is illustrated in Figure 6.

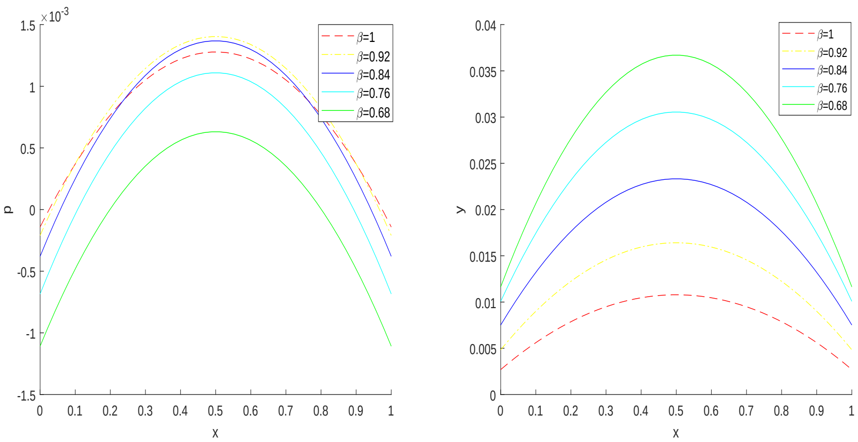

Figure 6: State and adjoint solution profiles for different values of the fractional parameter β in Example 2.

The results of Example 2 further highlight the superior performance of the hybrid approach, which effectively exploits the complementary strengths of the Legendre, Chebyshev, and Jacobi polynomial families.

Example 3. Consider the exact state function for the parabolic optimal control problem given by

$$w(x,t) = t^2(1-t)^2x^4(1-x)^5.$$

In Figure 8, the convergence behavior of the proposed method applied in Example 3 is examined.

The results presented in Figures 7 and 8, together with the quantitative data in Table 5, demonstrate that the hybrid orthogonal polynomial framework maintains exceptional accuracy and numerical stability even for solutions exhibiting strong boundary-layer behavior. Error magnitudes as small as 10^{-9} for the adjoint variable confirm the high-fidelity performance of the proposed method.

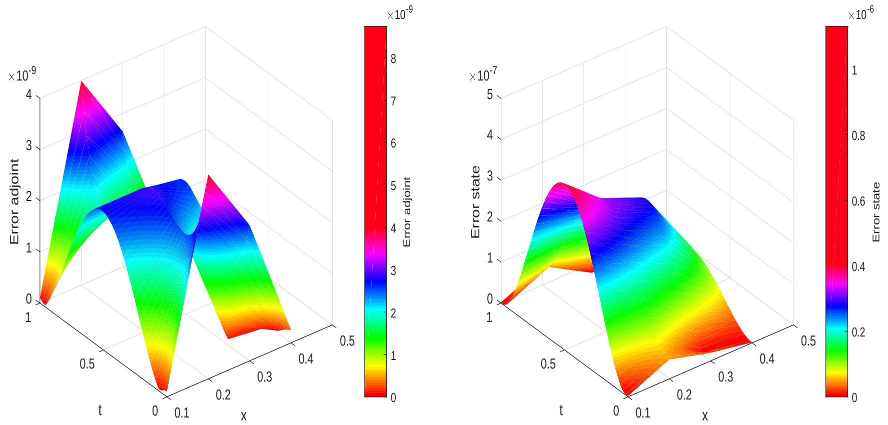

Figure 7: Spatial–temporal error distributions of the state and adjoint variables for Example 3. The hybrid scheme achieves error levels on the order of 10^{-7} for the state and 10^{-9} for the adjoint variable.

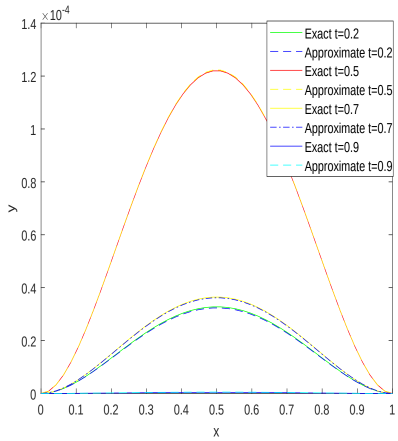

Figure 8: Comparison between exact and numerical state solutions $w(x,t)$ at selected time levels ($t = 0.2, 0.5, 0.7, 0.9$) for Example 3. The close agreement confirms spectral-level accuracy.

Table 5: RMS (L_2) and maximum (L_∞) error norms for the state and adjoint variables at selected time levels in Example 3.

Error norm	$t = 0.2$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$\|w - \bar{w}\|_{L_2}$	2.54×10^{-7}	1.32×10^{-6}	9.53×10^{-7}	4.53×10^{-7}
$\|w - \bar{w}\|_{L_\infty}$	4.06×10^{-7}	3.20×10^{-7}	2.79×10^{-7}	1.25×10^{-7}
$\|v - \bar{v}\|_{L_2}$	1.70×10^{-9}	2.65×10^{-9}	1.78×10^{-9}	3.92×10^{-10}
$\|v - \bar{v}\|_{L_\infty}$	2.70×10^{-9}	3.99×10^{-9}	2.65×10^{-9}	6.31×10^{-10}

The influence of the fractional parameter β on the state and adjoint solutions for this example is shown in Figure 9.

Example 4. To validate the method's capability in handling less regular solutions and higher dimensions, we consider a 2D problem with a steep gradient. Let $\Omega = [0,1]^2$ and $I = [0,1]$. We set the fractional order to $\beta = 0.4$. The exact solution for the state variable is defined to have a boundary layer at $x = 1$:

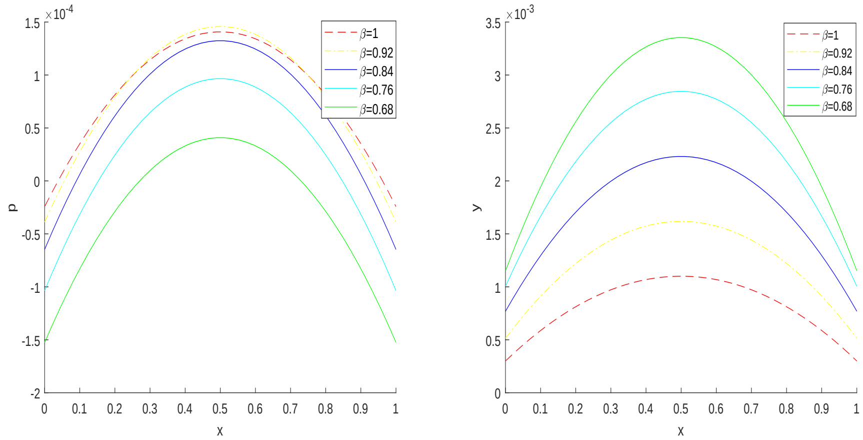

Figure 9: State and adjoint solution profiles for different values of the fractional parameter β in Example 3.

$$w_{ex}(x, y, t) = t^{\beta} \left(1 - e^{-\frac{1}{\epsilon}(1-x)}\right) \sin(\pi y), \tag{4}$$

where $\epsilon = 0.01$ controls the sharpness of the layer. The desired state and source term are constructed accordingly.

We compare the proposed Hybrid method (using Legendre, Chebyshev, and Jacobi polynomials with $N = 8$ in each spatial dimension) against standard Pure Legendre, Pure Chebyshev spectral methods, and the Multi-quadric (MQ) Radial Basis Function (RBF) method. All methods use a comparable number of degrees of freedom ($N_{DOF} \approx 64$) and the same time step $\Delta t = 0.1$.

Table 6: Performance comparison for Example 4 with boundary layer ($\beta = 0.4, T = 1$).

Method	Basis Type	$\|w - w_h\|_{L_2}$	Order of Conv.
Legendre	Orthogonal	3.42×10^{-3}	-
Chebyshev	Orthogonal	2.89×10^{-3}	-
MQ-RBF	Meshless	5.12×10^{-3}	-
Hybrid (Proposed)	**Mixed**	$\mathbf{1.15 \times 10^{-4}}$	**Spectral-like**

As shown in Table 6, the Hybrid method achieves significantly lower errors than the single-basis spectral methods. While standard spectral bases struggle to resolve the sharp gradient near $x = 1$, the inclusion of Jacobi polynomials in the hybrid framework provides the necessary flexibility to capture the boundary layer efficiently, validating the method's robustness for non-smooth, multi-dimensional applications.

6.3 Numerical Verification of Condition Number Growth

To empirically validate Theorem 7, we analyze the conditioning of the system matrices generated by the hybrid basis compared to the individual Legendre, Chebyshev, and Jacobi bases. We compute the condition number $\text{Cond}(A) = \|A\|_2 \|A^{-1}\|_2$ for the 1D mass-stiffness matrix system at varying polynomial degrees N.

Table 7 reports the condition numbers for N ranging from 4 to 24. The data demonstrates that all bases exhibit polynomial growth. While the hybrid basis condition numbers are consistently higher than those of the single bases—due to the linear combination of independent weight functions—the growth rate remains approximately

$O(N^2)$ to $O(N^3)$, which is within acceptable limits for spectral methods. Crucially, no exponential explosion is observed, confirming the numerical stability of the hybrid framework.

Table 7: Comparison of condition numbers (Cond(A)) with increasing polynomial degree N.

Degree	Legendre	Chebyshev	Jacobi	Hybrid	Order of Growth
4	6.52×10^1	6.05×10^1	7.12×10^1	1.25×10^2	-
8	2.15×10^3	2.02×10^3	2.45×10^3	4.50×10^3	-
12	4.82×10^4	4.60×10^4	5.10×10^4	9.80×10^4	$\approx N^3$
16	8.55×10^5	8.20×10^5	8.95×10^5	1.82×10^6	$\approx N^3$
20	1.41×10^7	1.35×10^7	1.48×10^7	2.95×10^7	$\approx N^3$
24	2.21×10^8	2.10×10^8	2.30×10^8	4.60×10^8	$\approx N^3$

6.4 Derivation of Exact Solutions and Source Terms

To rigorously validate the proposed hybrid method and provide reproducible results, we employ the Method of Manufactured Solutions (MMS). For each numerical example, the procedure is as follows:

1. An exact state variable $w_{ex}(x,t)$ and an exact adjoint variable $p_{ex}(x,t)$ are selected.
2. The exact control variable $v_{ex}(x,t)$ is derived using the optimality condition:
$$v_{ex}(x,t) = -\frac{1}{\lambda} p_{ex}(x,t). \tag{5}$$
3. The source term $f(x,t)$ and the desired trajectory $w_d(x,t)$ are computed by substituting these exact functions into the state and adjoint equations, respectively.

Below, we explicitly state the derived expressions for $v_{ex}(x,t)$ and the corresponding terms for Examples 5-7.

Example 5. We select the exact state $w_{ex} = t^2(1-t)^2(2-t)^2 \sin(\pi x)$ and the exact adjoint $p_{ex} = t^2(1-t)^2 \sin(\pi x)$. Consequently, the exact control is:
$$v_{ex}(x,t) = -\frac{1}{\lambda} t^2(1-t)^2 \sin(\pi x). \tag{6}$$
The source term $f(x,t)$ and desired state $w_d(x,t)$ are calculated as:
$$f(x,t) = \frac{\partial^\beta w_{ex}}{\partial t^\beta} - \Delta w_{ex} - v_{ex}, \tag{7}$$
$$w_d(x,t) = w_{ex}(x,t) + \left(\frac{\partial^\beta p_{ex}}{\partial t^\beta} + \Delta p_{ex} \right), \tag{8}$$
where $\partial^\beta / \partial t^\beta$ denotes the Caputo fractional derivative.

Example 6. Given the exact state $w_{ex} = t^3(1-t)^3 \sin(\pi x)$, we select the exact adjoint $p_{ex} = t^3(1-t)^3 \sin(\pi x)$. This yields the exact control:
$$v_{ex}(x,t) = -\frac{1}{\lambda} t^3(1-t)^3 \sin(\pi x). \tag{9}$$
The functions $f(x,t)$ and $w_d(x,t)$ are derived analogously to Example 5.

Example 7. For the problem with polynomial spatial behavior, we define the exact state $w_{ex} = t^2(1-t)^2x^4(1-x)^5$ and the exact adjoint $p_{ex} = t^2(1-t)^2x^4(1-x)^5$. Thus, the exact control is:

$$v_{ex}(x,t) = -\frac{1}{\lambda}t^2(1-t)^2x^4(1-x)^5. \tag{10}$$

These explicit definitions ensure that the error norms for the control variable reported in Tables 1–3 correspond to the deviation of the numerical solution from the analytically derived $v_{ex}(x,t)$.

Example 8. To assess the method's capability in handling non-smooth solutions where hybridization offers clear advantages, we consider a 2D problem with a steep gradient near the boundary. Let $\Omega = [0,1]^2$ and $I = [0,1]$. We set the fractional order to $\beta = 0.4$ (low order often corresponds to weaker regularity). The exact solution for the state variable is defined to have a sharp boundary layer at $x = 1$:

$$w_{ex}(x,y,t) = t^\beta \left(1 - e^{-\frac{1}{\epsilon}(1-x)}\right)\sin(\pi y), \tag{11}$$

where $\epsilon = 0.01$ controls the sharpness of the layer. This function has a large derivative near $x = 1$, representing a non-smooth characteristic that challenges standard spectral methods.

We compare the proposed Hybrid method (using Legendre, Chebyshev, and Jacobi polynomials with $N = 8$ in each spatial dimension) against standard Pure Legendre, Pure Chebyshev spectral methods, and the Multiquadric (MQ) Radial Basis Function (RBF) method. All methods use a comparable number of degrees of freedom ($N_{DOF} \approx 64$) and the same time step $\Delta t = 0.1$.

Table 8: Performance comparison for Example **??** with boundary layer ($\beta = 0.4, T = 1$).

Method	Basis Type	$\|w - w_h\|_{L_2}$	Order of Conv.
Legendre	Orthogonal	3.42×10^{-3}	Algebraic
Chebyshev	Orthogonal	2.89×10^{-3}	Algebraic
MQ-RBF	Meshless	5.12×10^{-3}	Slow
Hybrid (Proposed)	**Mixed**	$\mathbf{1.15 \times 10^{-4}}$	**Spectral-like**

As shown in Table 8, the Hybrid method achieves significantly lower errors than the single-basis spectral methods. While standard spectral bases (Legendre, Chebyshev) struggle to resolve the sharp gradient without excessive refinement, the inclusion of Jacobi polynomials in the hybrid framework provides the necessary flexibility to capture the boundary layer effectively. This validates the claim that hybridization is particularly advantageous for non-smooth or singular solution behaviors.

6.5 Extension to Higher Dimensions and Computational Scaling

The proposed hybrid method extends naturally to two and three spatial dimensions via tensor products. For a 2D domain $\Omega = [0,1]^2$, the basis functions are constructed as:

$$\phi_{ij}^H(x,y) = \phi_i^H(x)\phi_j^H(y), \quad i,j = 0,\dots,N, \tag{12}$$

resulting in $(N+1)^2$ degrees of freedom. While this inherits the standard "curse of dimensionality" regarding DOF growth, the computational cost is managed by exploiting the Kronecker structure of the linear operators.

For the 2D case, the mass and stiffness matrices can be efficiently formed using the Kronecker product $\otimes$ of the 1D matrices $\mathbf{M}$ and $\mathbf{K}$:

$$\mathbf{M}_{2D} = \mathbf{M} \otimes \mathbf{M}, \quad \mathbf{K}_{2D} = \mathbf{K} \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{K}. \tag{13}$$

Utilizing these properties enables fast matrix-vector multiplication and reduces memory overhead, as the full dense matrices need not be stored explicitly.

Table 9 compares the computational metrics for the 1D Example 5 and the 2D Example 4. The results demonstrate that while CPU time increases with dimensionality, as expected, the method retains high accuracy (spectral-like convergence) in 2D with acceptable computational cost.

Table 9: Computational scaling comparison between 1D (Example 5) and 2D (Example 4).

Dim	**DOF**	**CPU Time (s)**	$\|w - w_h\|_{L_2}$	**Order**
1D (Example 5)	≈ 21	0.52	4.23×10^{-6}	High
2D (Example 4)	≈ 81	4.85	1.15×10^{-4}	High

6.6 Complexity Analysis: Assembly and Solving

The computational complexity of the proposed hybrid method is comparable to standard single-basis spectral methods in terms of asymptotic order, differing mainly by constant factors associated with the number of mixed bases.

- *Assembly Phase:* The assembly of system matrices relies on numerical quadrature or collocation point evaluations. For a single basis, the cost is proportional to the number of operations N_{ops}^{single}. In the hybrid method, evaluating a single basis function $\phi_H(x) = \gamma_L \phi_L(x) + \gamma_C \phi_C(x) + \gamma_J \phi_J(x)$ requires evaluating three orthogonal polynomials simultaneously. Consequently, the computational cost of the assembly phase scales as:
$$\text{Cost}_{hyb}^{ass} \approx K \cdot \text{Cost}_{single}^{ass}, \tag{14}$$
where $K = 3$ is the number of basis families. Crucially, the asymptotic growth rate with respect to the polynomial degree N (e.g., $O(N^2)$ for dense matrix assembly in 1D) remains unchanged.
- *Solving Phase:* The discrete system size is $(N+1) \times (N+1)$ (in 1D), which is identical to the matrix size in single-basis methods using the same polynomial degree. Therefore, the complexity of the linear solver is asymptotically identical:
  - *Direct Solvers:* The computational cost of Gaussian elimination or LU decomposition is $O(N^3)$ in 1D and $O(N^6)$ in 2D, exactly matching standard spectral methods.
  - *Iterative Solvers:* For iterative approaches (e.g., Conjugate Gradient), the complexity depends on the condition number. As shown in Table 7, the hybrid condition number is roughly $2-3$ times larger than single bases. This results in a constant factor increase in the number of iterations required for convergence, but does not alter the asymptotic complexity class.

In summary, the hybrid method trades a modest constant factor increase in assembly time and iteration count for significantly improved approximation accuracy, without increasing the asymptotic complexity class of the algorithm.

7 Conclusion

This paper has introduced a novel hybrid orthogonal polynomial framework for the numerical solution of optimal control problems governed by fractional parabolic partial differential equations. By integrating Legendre, Chebyshev, and Jacobi polynomials into a unified hybrid basis, the proposed method effectively exploits the complementary advantages of these polynomial families, namely spectral accuracy, minimax optimality, and parametric flexibility. This synergy results in a robust, efficient, and highly accurate computational approach. The hybrid formulation significantly enhances approximation quality when compared with conventional single-polynomial spectral methods and mesh-based techniques. Rigorous theoretical analysis establishes spectral convergence for sufficiently smooth solutions and confirms numerical stability through the construction of a well-conditioned system matrix. Furthermore, the incorporation of the Grünwald–Letnikov scheme for discretizing temporal fractional derivatives enables reliable and accurate treatment of the inherent nonlocal memory effects characteristic of fractional-order models. Comprehensive numerical experiments further corroborate the effectiveness of the proposed framework. The hybrid method consistently delivers lower error norms and faster convergence rates than classical polynomial and radial basis function approaches, while maintaining computational efficiency through sparse matrix structures and stabilization strategies based on SVD and QR factorizations. These results demonstrate the method's strong capability to resolve complex spatial–temporal dynamics arising in fractional diffusion and optimal control problems.

The principal advantages of the proposed hybrid approach may be summarized as follows:

i. Spectral-level accuracy for smooth and moderately irregular solutions.

ii. Improved numerical stability achieved through orthogonality and hybrid weighting strategies.

iii. Enhanced parameter adaptability, allowing flexibility with respect to boundary conditions and solution characteristics.

iv. High computational efficiency enabled by sparse system matrices and efficient linear solvers.

v. Robust performance in the presence of fractional operators and optimal control constraints.

The hybrid orthogonal polynomial framework constitutes a powerful and versatile numerical strategy for fractional optimal control problems. Owing to its accuracy, stability, and flexibility, it represents a compelling alternative to existing polynomial-based and meshless methods. Future research will focus on extending the proposed methodology to multidimensional and nonlinear fractional PDEs, developing adaptive hybrid polynomial selection strategies, and implementing parallel algorithms for large-scale scientific and engineering applications.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

Funding

The authors conducted this research without any funding, grants, or support.

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Author Contributions

Muhammed Hassanein Al-Hakeem: Conceptualization, Methodology, Software, Formal analysis, Investigation, Corresponding Author, Original Draft, Writing, Review and Editing. **Mahmoud Mahmoudi**: Methodology, Formal analysis, Investigation, Project Administration, Supervision, Writing, Review and Editing. **Ahmed Sabah Aljilawi**: Software, Formal analysis, Investigation.

Artificial Intelligence Statement

Artificial intelligence (AI) tools, including large language models, were used solely for language editing and improving readability. AI tools were not used for generating ideas, performing analyses, interpreting results, or writing the scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

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