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A Two-Stage Network DEA Model Under Hybrid Disposability Technology: An Application to Healthcare Centers

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Abstract. Two-stage network Data Envelopment Analysis (DEA) models under variable returns to scale (VRS) suffer from a well-known pitfall: efficiency score decomposition and frontier projection can be mutually inconsistent, undermining both the theoretical foundations and practical interpretability of the results. A further limitation is the universal assumption of strong disposability for all inputs and outputs, which is unrealistic when variables are structurally or statistically interdependent—as is common in healthcare settings. This paper addresses both issues simultaneously by developing a novel two-stage network DEA model under hybrid disposability (HD) technology, which allows selective strong or weak disposability for subsets of closely related inputs, intermediate measures, and outputs. We formally derive the efficiency decomposition and frontier projection under HD technology, establish theoretical consistency between the envelopment and multiplier forms, and prove that the proposed model yields Pareto-efficient targets. The model captures synergistic scale effects across stages and preserves structural dependencies between them, thereby providing a more realistic representation of multi-stage production systems. The practical relevance and advantages of the proposed framework are demonstrated through an empirical case study involving 32 Iranian healthcare centers operating under a two-stage network structure with interdependent variables.

Keywords. Data envelopment analysis; Two-stage network DEA; Hybrid disposability; Variable returns to scale; Efficiency decomposition; Frontier projection; Healthcare performance evaluation.

MSC. 90C05; 90B30; 90C90 .

1 Introduction

Data Envelopment Analysis (DEA), first introduced by Charnes et al. [2], is a well-established nonparametric approach for evaluating the relative efficiency of decision-making units (DMUs) that consume multiple inputs to produce multiple outputs. Over the past four decades, DEA has evolved into a rich and comprehensive analytical framework, finding applications across a wide range of industries and public-sector organizations [11]. A particularly significant extension of classical DEA is *network DEA*, which moves beyond the traditional "black-box" view of production by explicitly modeling the internal structure of DMUs [24].

Among the various network topologies studied in the literature, *two-stage network DEA* has attracted substantial attention owing to its capacity to assess performance across interdependent subsystems. In a two-stage structure, a first stage converts primary inputs into intermediate measures, which in turn serve as inputs to a second stage that generates the final outputs. Such models allow the analyst to evaluate not only the overall efficiency of a DMU but also the stage-level efficiencies, offering granular insights into internal operations and potential improvement pathways [3, 4, 10, 32]. Two-stage network DEA models also facilitate the allocation of responsibility across organizational components, which is particularly valuable in public-sector and multi-departmental systems [34].

Despite their appeal, two-stage network DEA models suffer from a fundamental theoretical tension: *efficiency score decomposition* and *frontier projection* can fail to align under variable returns to scale (VRS), thereby undermining both the theoretical soundness and the practical interpretability of the results [5, 35]. The root cause of this misalignment lies in the synergistic effects of scale heterogeneity across stages, which distort the production possibility set (PPS) and consequently affect efficiency scores and target projections. Several studies have proposed modifications to two-stage DEA models to better capture these scale interactions and to restore consistency between the envelopment and multiplier formulations [6, 21, 25]. More specifically, Zhang et al. [36, 37] developed series, parallel, and hybrid network DEA structures; Lim and Zhu [21] established primal-dual correspondence and frontier projection results; and Guo et al. [16], Chu and Zhu [7], and Chen and Wang [6] further examined the relationships among duality, frontier projection, and efficiency decomposition from both additive and multiplicative perspectives. Collectively, these contributions have substantially strengthened the theoretical foundations of two-stage network DEA while preserving its economic interpretability. Additionally, Pourmahmoud et al. [30] extended the network DEA paradigm to a three-stage setting under decision-making uncertainty.

A separate but related limitation of most DEA models—whether formulated under constant returns to scale (CRS) or VRS—is the universal assumption of *strong disposability* for all inputs and outputs. Strong (or free) disposability implies that any input may be increased and any output may be reduced independently and without bound, while remaining within the production possibility set [13]. Although this assumption is mathematically convenient, it is often unrealistic in practice. In many real-world applications—particularly in healthcare and education—certain inputs and outputs are structurally or statistically interdependent, so that independent, unconstrained adjustments are either operationally infeasible or economically meaningless.

An important exception in the DEA literature concerns the treatment of *undesirable outputs* through *joint weak disposability*, which requires that desirable and undesirable outputs be reduced proportionally rather than independently [12, 20]. Building on this idea, Mehdiloozad and Podinovski [22] and Podinovski et al. [28] introduced *selective strong and weak disposability* technologies, in which specific subsets of inputs or outputs are assumed to be jointly weakly disposable while the remaining variables retain strong disposability. In such settings, maintaining the assumption of universal strong disposability can yield efficiency scores that are either unrealistically optimistic or practically infeasible [27].

To address this, the notion of *hybrid disposability* (HD) technology was formalized by Mehdiloo and Podinovski [23], enabling selective strong and weak disposability to coexist within a single production technology. For instance, in hospital performance assessment, the number of physicians and nurses are typically managed in a coordinated manner; treating them as independently adjustable inputs may produce clinically infeasible targets. Similarly, outputs such as occupied beds and discharged patients are inherently linked, so that modeling them as strongly disposable can distort the efficiency frontier. Applying joint weak disposability to such variable groups preserves their internal proportionality, avoids frontier distortions, and yields efficiency scores that are both theoretically sound and managerially meaningful [23, 27].

The preceding discussion reveals two distinct but interrelated gaps in the existing literature. First, although HD technology has been shown to produce more realistic efficiency assessments in single-stage DEA, its integration into multi-stage network structures remains largely unexplored. The selective disposability framework of Mehdiloo and Podinovski [23] is confined to single-stage settings and therefore cannot capture interdependencies *across* subsystems in sequential production processes. Second, although Chen and Wang [6] established theoretical consistency between efficiency score decomposition and frontier projection in two-stage network DEA under VRS, their model retains the assumption of universal strong disposability, which is insufficiently flexible for systems with structurally interrelated variables.

The present paper bridges both gaps simultaneously. We integrate HD technology into a two-stage network DEA framework, enabling selective strong and weak disposability to coexist *within and across* stages. In doing so, we establish theoretical consistency between efficiency decomposition and frontier projection under VRS and HD technology, capture the synergistic scale effects between stages, and preserve dual equivalence between the envelopment and multiplier forms. This integration creates a new theoretical bridge linking disposability assumptions, scale heterogeneity, and network duality in DEA.

The practical motivation for this work is drawn from the performance evaluation of *healthcare centers*. Healthcare delivery is inherently a multi-stage process: in the first stage, human, financial, and capital resources support diagnostic and preliminary treatment activities; in the second stage, these intermediate services generate final clinical and quality outcomes. Moreover, many healthcare variables are structurally interdependent—the number of general practitioners and nurses are jointly managed, advanced imaging devices are complementary, and pharmaceutical expenditures tend to co-vary with overall operational costs. In such an environment, the standard strong disposability assumption leads to unrealistic or biased efficiency estimates, underscoring the need for the HD framework proposed here [8, 9, 14, 15, 19].

The main contributions of this paper are as follows.

1. We develop a novel two-stage network DEA model under HD technology that accommodates selective strong and weak disposability for subsets of closely related inputs, intermediate measures, and outputs simultaneously.

2. We derive formal consistency between efficiency score decomposition and frontier projection within the HD framework, resolving the theoretical pitfall that arises under VRS in two-stage network DEA.

3. We establish dual equivalence between the envelopment and multiplier forms of the proposed model and prove that the resulting frontier projections are Pareto-efficient under HD technology.

4. We propose a corrective algorithm to handle cases in which the stage-level efficiency score exceeds unity, ensuring theoretically admissible and practically meaningful decompositions.

5. We demonstrate the practical relevance of the framework through an empirical application to 32 Iranian healthcare centers, providing stage-specific efficiency scores, actionable improvement targets, and robustness analyses.

The remainder of this paper is organized as follows. Section 2 develops the two-stage network DEA framework under HD technology, including the theoretical foundations, axioms, envelopment and multiplier models, and the corrective algorithm. Section 3 presents a numerical example based on a school-like production system to illustrate the proposed model. Section 4 applies the model to 32 Iranian healthcare centers and analyzes the efficiency results. Section 5 addresses the limited discriminating power of HD technology and proposes a super-efficiency extension to improve DMU differentiation. Section 6 concludes the paper with a summary of findings, acknowledged limitations, and directions for future research.

2 Two-stage Network DEA Under HD Technology

This section develops the modeling framework for evaluating DMU efficiency within a two-stage network structure under hybrid disposability (HD) technology. We first introduce the notation and the structural assumptions of the network, then formally state the two disposability axioms, derive the envelopment and multiplier models, establish their duality, and present a corrective algorithm for obtaining theoretically admissible efficiency decompositions.

2.1 Notation and Network Structure

Consider n observed DMUs, indexed by $j \in J = \{1, \dots, n\}$, each operating under a two-stage network structure. The data of DMU$_j$ are represented by the triple (X_j, Z_j, Y_j), where

$$\begin{aligned} X_j &= (x_{1j}, \dots, x_{mj})^\top \in \mathbb{R}^m_+ \quad \text{(inputs)}, \\ Z_j &= (z_{1j}, \dots, z_{hj})^\top \in \mathbb{R}^h_+ \quad \text{(intermediate measures)}, \\ Y_j &= (y_{1j}, \dots, y_{sj})^\top \in \mathbb{R}^s_+ \quad \text{(outputs)}. \end{aligned}$$

We assume that each DMU has at least one strictly positive component in each vector, i.e., $X_j \neq \mathbf{0}$, $Z_j \neq \mathbf{0}$, and $Y_j \neq \mathbf{0}$ for all $j \in J$. The DMU under evaluation is denoted DMU$_o$ = (X_o, Z_o, Y_o).

As illustrated in Figure 1, Stage 1 transforms the m primary inputs X_j into h intermediate measures Z_j, which subsequently serve as the sole inputs to Stage 2 in order to produce the s final outputs Y_j. Let T^1, T^2, and T denote the production possibility sets (PPS) of Stage 1, Stage 2, and the overall two-stage system, respectively.

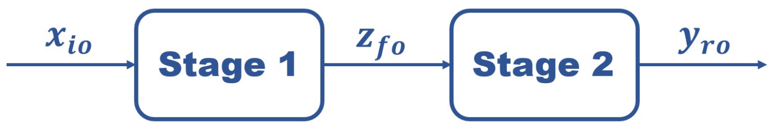

Figure 1: Two-stage network structure.

2.2 Hybrid Disposability Technology

In many real-world settings—particularly healthcare—inputs and outputs are not independently adjustable. For example, the numbers of physicians and nurses are typically managed jointly within staffing plans, so that reducing one independently of the other may yield infeasible operational targets. Similarly, the number of inpatient admissions and total hospitalization days are structurally linked, and treating them as independently reducible distorts the efficiency frontier. Mehdiloo and Podinovski [23] proposed a framework of *selective strong and weak disposability* to handle such dependencies in single-stage settings. We extend their framework to the two-stage network context.

Index-set decomposition. Let $I = \{1, \dots, m\}$, $IM = \{1, \dots, h\}$, and $O = \{1, \dots, s\}$ denote the index sets of inputs, intermediate measures, and outputs, respectively. We assume the following mutually exhaustive decompositions:

$$I = I^S \cup I^W \cup I^{SW}, \quad IM = IM^S \cup IM^W \cup IM^{SW}, \quad O = O^S \cup O^W \cup O^{SW}, \tag{1}$$

where the superscripts S, W, and SW denote *strongly disposable*, *weakly disposable*, and *selectively (both strongly and weakly) disposable* subsets, respectively. Accordingly, each DMU is represented as

$$(X, Z, Y) = \left(X^S, X^W, X^{SW},\ Z^S, Z^W, Z^{SW},\ Y^S, Y^W, Y^{SW}\right),$$

where each sub-vector collects the variables belonging to the corresponding subset in (1).

The subset $I^W \cup I^{SW}$ contains inputs that are *jointly weakly disposable*: they can be scaled proportionally by a common factor $\delta \geq 1$ (increase) or $\gamma \in [0, 1]$ (decrease), but cannot be adjusted independently. The subset $I^S \cup I^{SW}$ contains inputs that are *individually strongly disposable*: each element can be increased or decreased independently of the others. Inputs in I^{SW} satisfy both conditions simultaneously, which necessitates their separate treatment in the model. An analogous structure applies to *IM* and O.

The basic regularity conditions that define HD technology are (cf. [23]):

(a) **Inclusion of observations:** all observed DMUs belong to the technology set.

(b) **Convexity:** the technology set is convex.

(c) **Closedness:** the technology set is closed.

(d) **Selective disposability:** as specified by Axioms ASSDTN and ASWDTN below.

Axiom ASSDTN. Selective strong disposability for two-stage networks.

Stage 1: If $(X^S, X^W, X^{SW}, Z^S, Z^W, Z^{SW}) \in T^1$, then for all $\bar{X}^S \geq X^S$, $\bar{X}^{SW} \geq X^{SW}$, $\mathbf{0} \leq \bar{Z}^S \leq Z^S$, $\mathbf{0} \leq \bar{Z}^{SW} \leq Z^{SW}$:

$$\left(\bar{X}^S, X^W, \bar{X}^{SW}, \bar{Z}^S, Z^W, \bar{Z}^{SW}\right) \in T^1.$$

Stage 2: If $(Z^S, Z^W, Z^{SW}, Y^S, Y^W, Y^{SW}) \in T^2$, then for all $\bar{Z}^S \geq Z^S$, $\bar{Z}^{SW} \geq Z^{SW}$, $\mathbf{0} \leq \bar{Y}^S \leq Y^S$, $\mathbf{0} \leq \bar{Y}^{SW} \leq Y^{SW}$:

$$(\bar{Z}^S,\, Z^W,\, \bar{Z}^{SW},\, \bar{Y}^S,\, Y^W,\, \bar{Y}^{SW}) \in T^2.$$

In words, ASSDTN permits individual, unconstrained adjustments of variables in $(\cdot)^S$ and $(\cdot)^{SW}$ while leaving the jointly weakly disposable sub-vectors $(\cdot)^W$ unchanged.

Axiom ASWDTN. Selective weak disposability for two-stage networks.

Stage 1: If $(X^S, X^W, X^{SW}, Z^S, Z^W, Z^{SW}) \in T^1$, then for all $\delta \geq 1$ and $\gamma \in [0, 1]$,

$$(X^S,\, \delta X^W,\, \delta X^{SW},\, Z^S,\, \gamma Z^W,\, \gamma Z^{SW}) \in T^1.$$

Stage 2: If $(Z^S, Z^W, Z^{SW}, Y^S, Y^W, Y^{SW}) \in T^2$, then for all $\delta \geq 1$ and $\gamma \in [0, 1]$,

$$(Z^S,\, \delta Z^W,\, \delta Z^{SW},\, Y^S,\, \gamma Y^W,\, \gamma Y^{SW}) \in T^2.$$

ASWDTN requires that the jointly weakly disposable sub-vectors $(\cdot)^W$ and $(\cdot)^{SW}$ be scaled by the *same* proportional factor, thereby preserving their internal ratios.

Remark 1. Axioms ASSDTN and ASWDTN are logically independent: neither implies the other, and a technology may satisfy one without satisfying the other. Moreover, any technology satisfying universal strong disposability automatically satisfies both axioms, but the converse does not hold in general. The combination of ASSDTN and ASWDTN therefore represents a strictly weaker—and more flexible—assumption than universal strong disposability.

2.3 Illustrative Examples of the Axioms

Example 1 (ASWDTN holds; ASSDTN does not). Consider a Stage 1 technology in which a main product and a jointly produced by-product can only be reduced proportionally (e.g., due to environmental regulations). This satisfies ASWDTN. However, if regulations prohibit reducing the main product independently of the by-product, unilateral reduction is infeasible, so ASSDTN fails.

Example 2 (ASSDTN holds; ASWDTN does not). In Stage 2, suppose one input can be increased independently (ASSDTN holds). However, if a group of complementary inputs must be maintained in a fixed ratio—so that a proportional simultaneous increase is technically infeasible—then ASWDTN fails for that group.

These examples confirm that ASSDTN and ASWDTN are independent principles.

2.4 Envelopment Model Under HD Technology

Having established the HD technology axioms, we now derive the input-oriented envelopment model for measuring the radial efficiency of $\text{DMU}_o = (X_o, Z_o, Y_o)$ under HD technology in a two-stage network. The model simultaneously evaluates Stage 1 and Stage 2 and enforces the VRS normalizing constraints in both stages.

$$\min_{\boldsymbol{\lambda}^1,\boldsymbol{\mu}^1,\boldsymbol{\nu}^1,\boldsymbol{\lambda}^2,\boldsymbol{\mu}^2,\boldsymbol{\nu}^2,\theta^{\text{THD}},\theta^{2\text{-THD}}} \quad \theta_o^{\text{THD}}$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j^1 \, x_{ij}^S \leq \theta_o^{\text{THD}} \, x_{io}^S, \quad i \in I^S, && \text{(2a)}\\
& \sum_{j=1}^{n} (\lambda_j^1 + \mu_j^1) \, x_{ij}^W = \theta_o^{\text{THD}} \, x_{io}^W, \quad i \in I^W, && \text{(2b)}\\
& \sum_{j=1}^{n} (\lambda_j^1 + \mu_j^1) \, x_{ij}^{SW} \leq \theta_o^{\text{THD}} \, x_{io}^{SW}, \quad i \in I^{SW}, && \text{(2c)}\\
& \sum_{j=1}^{n} \lambda_j^1 \, z_{fj}^S \geq \theta_o^{2\text{-THD}} \, z_{fo}^S, \quad f \in IM^S, && \text{(2d)}\\
& \sum_{j=1}^{n} (\lambda_j^1 - \nu_j^1) \, z_{fj}^W = \theta_o^{2\text{-THD}} \, z_{fo}^W, \quad f \in IM^W, && \text{(2e)}\\
& \sum_{j=1}^{n} (\lambda_j^1 - \nu_j^1) \, z_{fj}^{SW} \geq \theta_o^{2\text{-THD}} \, z_{fo}^{SW}, \quad f \in IM^{SW}, && \text{(2f)}\\
& \sum_{j=1}^{n} \lambda_j^1 = 1, && \text{(2g)}\\
& \lambda_j^1 - \nu_j^1 \geq 0, \quad j \in J, && \text{(2h)}\\
& \lambda_j^1, \, \mu_j^1, \, \nu_j^1 \geq 0, \quad j \in J, \quad \theta_o^{\text{THD}} \text{ free}, && \text{(2i)}\\
& \sum_{j=1}^{n} \lambda_j^2 \, z_{fj}^S \leq \theta_o^{2\text{-THD}} \, z_{fo}^S, \quad f \in IM^S, && \text{(2j)}\\
& \sum_{j=1}^{n} (\lambda_j^2 + \mu_j^2) \, z_{fj}^W = \theta_o^{2\text{-THD}} \, z_{fo}^W, \quad f \in IM^W, && \text{(2k)}\\
& \sum_{j=1}^{n} (\lambda_j^2 + \mu_j^2) \, z_{fj}^{SW} \leq \theta_o^{2\text{-THD}} \, z_{fo}^{SW}, \quad f \in IM^{SW}, && \text{(2l)}
\end{aligned}$$

$$\sum_{j=1}^{n} \lambda_j^2\, y_{rj}^S \geq y_{ro}^S, \quad r \in O^S, \tag{2m}$$

$$\sum_{j=1}^{n} (\lambda_j^2 - \nu_j^2)\, y_{rj}^W = y_{ro}^W, \quad r \in O^W, \tag{2n}$$

$$\sum_{j=1}^{n} (\lambda_j^2 - \nu_j^2)\, y_{rj}^{SW} \geq y_{ro}^{SW}, \quad r \in O^{SW}, \tag{2o}$$

$$\sum_{j=1}^{n} \lambda_j^2 = 1, \tag{2p}$$

$$\lambda_j^2 - \nu_j^2 \geq 0, \quad j \in J, \tag{2q}$$

$$\lambda_j^2,\, \mu_j^2,\, \nu_j^2 \geq 0, \quad j \in J, \quad \theta_o^{2\text{-THD}} \text{ free.} \tag{2r}$$

In Model (2), θ_o^{THD} and $\theta_o^{2\text{-THD}}$ are the overall efficiency score and the Stage 2 efficiency score of DMU_o, respectively, under HD technology. The Stage 1 efficiency score is recovered via the multiplicative decomposition

$$\theta_o^{\text{THD}} = \theta_o^{1\text{-THD}} \times \theta_o^{2\text{-THD}}, \tag{3}$$

so that $\theta_o^{1\text{-THD}} = \theta_o^{\text{THD}}/\theta_o^{2\text{-THD}}$. Constraints (2a)–(2i) govern Stage 1 (inputs to intermediate measures), while Constraints (2j)–(2r) govern Stage 2 (intermediate measures to outputs). The intensity variables λ_j^k $(k = 1, 2)$ define the reference benchmark for each stage; μ_j^k accommodates the upward slack required by the weak-disposability equality constraints on inputs/intermediate measures; and ν_j^k accommodates the downward slack required by the weak-disposability equality constraints on intermediate measures/outputs. The variables of the model are summarized in Table 1.

Definition 1. For two vectors A and B of the same dimension, we write $A \leq B$ if every component of A is less than or equal to the corresponding component of B.

Definition 2. The *linkage reduction* of $X = (X^S, X^W, X^{SW})$ is the proportional decrease in the input vector induced by the Stage-1 optimization, quantified as $\big((1 - \theta_o^{1\text{-THD}})X^S,\ (1 - \theta_o^{1\text{-THD}})X^W,\ (1 - \theta_o^{1\text{-THD}})X^{SW}\big)$. The *maximum reducible amount* of X is $\big((1 - \theta_o^{\text{THD}})X^S,\ (1 - \theta_o^{\text{THD}})X^W,\ (1 - \theta_o^{\text{THD}})X^{SW}\big)$.

Theorem 1. If $\theta_o^{\text{THD}} \geq \theta_o^{2\text{-THD}}$, then the linkage reduction of X exceeds its maximum reducible amount. Conversely, if $\theta_o^{\text{THD}} < \theta_o^{2\text{-THD}}$, the linkage reduction does not exceed the maximum reducible amount.

Proof. Since $\theta_o^{\text{THD}} = \theta_o^{1\text{-THD}} \times \theta_o^{2\text{-THD}}$, the condition $\theta_o^{\text{THD}} \geq \theta_o^{2\text{-THD}}$ implies $\theta_o^{1\text{-THD}} \geq 1$, and hence $1 - \theta_o^{\text{THD}} \leq 1 - \theta_o^{2\text{-THD}}$, which by Definition 1 means the maximum reducible amount is component-wise no greater than the further reduction amount. Consequently, the linkage reduction exceeds the maximum reducible amount. The converse follows by reversing the inequality. □

Theorem 2. Model (2) is always feasible and attains a bounded optimal solution.

Proof. Setting $\theta_o^{\text{THD}} = \theta_o^{2\text{-THD}} = 1$, $\lambda_o^k = 1$, $\lambda_j^k = 0$ for $j \neq o$, and $\mu_j^k = \nu_j^k = 0$ for all $j \in J$ and $k = 1, 2$ yields a feasible solution. Since the objective is a minimization over a non-empty feasible set with a linear objective function, the optimal value is bounded below by zero, guaranteeing a finite optimal solution. □

Table 1: Decision variables in Models (2) and (4).

Variable	Description
$\lambda_j^k,\ \mu_j^k,\ \nu_j^k\ (k = 1, 2)$	Intensity variables for DMU$_j$ in stage k; μ_j^k handles weak-disposability input slack, ν_j^k handles weak-disposability output slack.
$\theta_o^{k\text{-THD}},\ \theta_o^{\text{THD}}\ (k = 1, 2)$	Stage-k and overall efficiency scores from the envelopment model (2).
$\phi_o^{k\text{-THD}},\ \phi_o^{\text{THD}}\ (k = 1, 2)$	Stage-k and overall efficiency scores from the multiplier model (4).
v^S, v^W, v^{SW}	Dual weight vectors for inputs (constraints (2a)–(2c)).
$\alpha^S, \alpha^W, \alpha^{SW}$	Dual weight vectors for Stage-1 intermediate measures (constraints (2d)–(2f)).
$\beta^S, \beta^W, \beta^{SW}$	Dual weight vectors for Stage-2 intermediate measures (constraints (2j)–(2l)).
u^S, u^W, u^{SW}	Dual weight vectors for outputs (constraints (2m)–(2o)).
$\Gamma_j^1,\ \Gamma_j^2$	Dual variables for the non-negativity constraints (2h) and (2q).
$\eta_o^1,\ \eta_o^2$	Scale impact factors (VRS intercepts) for Stage 1 and Stage 2; dual to constraints (2g) and (2p).

2.5 Multiplier Model and Duality

The dual (multiplier form) of Model (2) is:

$$
\begin{aligned}
\max_{\substack{v,\alpha,\beta,u,\\ \Gamma^1,\Gamma^2,\eta_o^1,\eta_o^2}} \quad & \sum_{r\in O^S} u_r^S y_{ro}^S + \sum_{r\in O^W} u_r^W y_{ro}^W + \sum_{r\in O^{SW}} u_r^{SW} y_{ro}^{SW} + \eta_o^1 + \eta_o^2 \\
\text{s.t.} \quad & \sum_{i\in I^S} v_i^S x_{io}^S + \sum_{i\in I^W} v_i^W x_{io}^W + \sum_{i\in I^{SW}} v_i^{SW} x_{io}^{SW} = 1, && \text{(4a)}\\
& \sum_{f\in IM^S} \alpha_f^S z_{fj}^S + \sum_{f\in IM^W} \alpha_f^W z_{fj}^W + \sum_{f\in IM^{SW}} \alpha_f^{SW} z_{fj}^{SW} - \sum_{i\in I^S} v_i^S x_{ij}^S - \sum_{i\in I^W} v_i^W x_{ij}^W \\
& - \sum_{i\in I^{SW}} v_i^{SW} x_{ij}^{SW} + \Gamma_j^1 + \eta_o^1 \le 0, \quad j\in J, && \text{(4b)}\\
& \sum_{f\in IM^W} \alpha_f^W z_{fj}^W + \sum_{f\in IM^{SW}} \alpha_f^{SW} z_{fj}^{SW} + \Gamma_j^1 \ge 0, \quad j\in J, && \text{(4c)}\\
& \sum_{i\in I^W} v_i^W x_{ij}^W + \sum_{i\in I^{SW}} v_i^{SW} x_{ij}^{SW} \ge 0, \quad j\in J, && \text{(4d)}\\
& \sum_{r\in O^S} u_r^S y_{rj}^S + \sum_{r\in O^W} u_r^W y_{rj}^W + \sum_{r\in O^{SW}} u_r^{SW} y_{rj}^{SW} - \sum_{f\in IM^S} \beta_f^S z_{fj}^S - \sum_{f\in IM^W} \beta_f^W z_{fj}^W \\
& - \sum_{f\in IM^{SW}} \beta_f^{SW} z_{fj}^{SW} + \Gamma_j^2 + \eta_o^2 \le 0, \quad j\in J, && \text{(4e)}\\
& \sum_{r\in O^W} u_r^W y_{rj}^W + \sum_{r\in O^{SW}} u_r^{SW} y_{rj}^{SW} + \Gamma_j^2 \ge 0, \quad j\in J, && \text{(4f)}\\
& \sum_{f\in IM^W} \beta_f^W z_{fj}^W + \sum_{f\in IM^{SW}} \beta_f^{SW} z_{fj}^{SW} \ge 0, \quad j\in J, && \text{(4g)}\\
& \sum_{f\in IM^S} \alpha_f^S z_{fj}^S + \sum_{f\in IM^W} \alpha_f^W z_{fj}^W + \sum_{f\in IM^{SW}} \alpha_f^{SW} z_{fj}^{SW} = \\
& \sum_{f\in IM^S} \beta_f^S z_{fj}^S + \sum_{f\in IM^W} \beta_f^W z_{fj}^W + \sum_{f\in IM^{SW}} \beta_f^{SW} z_{fj}^{SW}, \quad j\in J, && \text{(4h)}\\
& v_i^S,\ v_i^{SW} \ge 0, \quad v_i^W \text{ free}, \quad i\in I^S\cup I^W\cup I^{SW}, && \text{(4i)}\\
& \alpha_f^S,\ \alpha_f^{SW} \ge 0, \quad \alpha_f^W \text{ free}, \quad f\in IM, && \text{(4j)}\\
& \beta_f^S,\ \beta_f^{SW} \ge 0, \quad \beta_f^W \text{ free}, \quad f\in IM, && \text{(4k)}\\
& u_r^S,\ u_r^{SW} \ge 0, \quad u_r^W \text{ free}, \quad r\in O, && \text{(4l)}\\
& \Gamma_j^1,\ \Gamma_j^2 \ge 0, \quad \eta_o^1,\ \eta_o^2 \text{ free}, \quad j\in J. && \text{(4m)}
\end{aligned}
$$

Here, η_o^1 and η_o^2 are the VRS scale impact factors for Stage 1 and Stage 2, respectively. By strong duality of linear programming, the optimal objective values of Models (2) and (4) coincide: $\phi_o^{\text{THD}} = \theta_o^{\text{THD}}$. Following Kao and Hwang [17] and Banker et al. [1], the stage-level multiplier efficiency scores are defined as:

$$\phi_o^{2\text{-THD}} = \frac{\sum_{r\in O^S} u_r^S y_{ro}^S + \sum_{r\in O^W} u_r^W y_{ro}^W + \sum_{r\in O^{SW}} u_r^{SW} y_{ro}^{SW} + \eta_o^2}{\sum_{f\in IM^S} \beta_f^S z_{fo}^S + \sum_{f\in IM^W} \beta_f^W z_{fo}^W + \sum_{f\in IM^{SW}} \beta_f^{SW} z_{fo}^{SW}}, \tag{5}$$

$$\phi_o^{\text{THD}} = \frac{\sum_{r\in O^S} u_r^S y_{ro}^S + \sum_{r\in O^W} u_r^W y_{ro}^W + \sum_{r\in O^{SW}} u_r^{SW} y_{ro}^{SW} + \eta_o^1 + \eta_o^2}{\sum_{i\in I^S} v_i^S x_{io}^S + \sum_{i\in I^W} v_i^W x_{io}^W + \sum_{i\in I^{SW}} v_i^{SW} x_{io}^{SW}}, \tag{6}$$

$$\phi_o^{1\text{-THD}} = \frac{\phi_o^{\text{THD}}}{\phi_o^{2\text{-THD}}}. \tag{7}$$

Constraint (4h) ensures that the dual weights assigned to the intermediate measures as outputs of Stage 1 equal those assigned to them as inputs of Stage 2, thus guaranteeing internal consistency of the efficiency decomposition across stages.

If Stage 2 is treated as a standalone system, the corresponding envelopment model is:

$$\min_{\boldsymbol{\lambda}^2,\boldsymbol{\mu}^2,\boldsymbol{\nu}^2,\theta^{2\text{-THD}}} \quad \theta_o^{2\text{-THD}}$$

$$\text{s.t.} \quad \sum_{j=1}^n \lambda_j^2 z_{fj}^S \le \theta_o^{2\text{-THD}} z_{fo}^S, \quad f \in IM^S, \tag{8a}$$

$$\sum_{j=1}^n (\lambda_j^2 + \mu_j^2) z_{fj}^W = \theta_o^{2\text{-THD}} z_{fo}^W, \quad f \in IM^W, \tag{8b}$$

$$\sum_{j=1}^n (\lambda_j^2 + \mu_j^2) z_{fj}^{SW} \le \theta_o^{2\text{-THD}} z_{fo}^{SW}, \quad f \in IM^{SW}, \tag{8c}$$

$$\sum_{j=1}^n \lambda_j^2 y_{rj}^S \ge y_{ro}^S, \quad r \in O^S, \tag{8d}$$

$$\sum_{j=1}^n (\lambda_j^2 - \nu_j^2) y_{rj}^W = y_{ro}^W, \quad r \in O^W, \tag{8e}$$

$$\sum_{j=1}^n (\lambda_j^2 - \nu_j^2) y_{rj}^{SW} \ge y_{ro}^{SW}, \quad r \in O^{SW}, \tag{8f}$$

$$\sum_{j=1}^n \lambda_j^2 = 1, \tag{8g}$$

$$\lambda_j^2 - \nu_j^2 \ge 0, \quad j \in J, \tag{8h}$$

$$\lambda_j^2,\ \mu_j^2,\ \nu_j^2 \ge 0, \quad j \in J, \quad \theta_o^{2\text{-THD}} \text{ free.} \tag{8i}$$

2.6 Efficiency Definition and Frontier Projection

Definition 3. $\text{DMU}_o = (X_o, Z_o, Y_o)$ is *HD-efficient* in Stage 1, Stage 2, and the overall system if and only if $\bar{\theta}_o^{1\text{-THD}} = 1$, $\bar{\theta}_o^{2\text{-THD}} = 1$, and $\bar{\theta}_o^{\text{THD}} = 1$, respectively; otherwise it is *HD-inefficient*.

The frontier projection (efficiency target) of DMU_o under HD technology is:

$$\left(\bar{\theta}_o^{1\text{-THD}} X^S,\ \bar{\theta}_o^{1\text{-THD}} X^W,\ \bar{\theta}_o^{1\text{-THD}} X^{SW},\ \bar{\theta}_o^{2\text{-THD}} Z^S,\ \bar{\theta}_o^{2\text{-THD}} Z^W,\ \bar{\theta}_o^{2\text{-THD}} Z^{SW},\ Y^S, Y^W, Y^{SW}\right), \tag{9}$$

where $\bar{\theta}_o^{1\text{-THD}}$ and $\bar{\theta}_o^{2\text{-THD}}$ are the adjusted efficiency scores obtained from Algorithm 1.

Algorithm 1 Corrective decomposition for two-stage HD efficiency scores

Step 1: Solve Model (2) to obtain θ_o^{THD} and $\theta_o^{2\text{-THD}}$.

Step 2: Set the adjusted scores as follows:

Case (i): If $\theta_o^{\text{THD}} < \theta_o^{2\text{-THD}}$, set

$$\bar{\theta}_o^{1\text{-THD}} = \frac{\theta_o^{\text{THD}}}{\theta_o^{2\text{-THD}}}, \quad \bar{\theta}_o^{2\text{-THD}} = \theta_o^{2\text{-THD}}, \quad \bar{\theta}_o^{\text{THD}} = \theta_o^{\text{THD}}.$$

Case (ii): If $\theta_o^{\text{THD}} \geq \theta_o^{2\text{-THD}}$, set

$$\bar{\theta}_o^{1\text{-THD}} = 1, \quad \bar{\theta}_o^{2\text{-THD}} = \theta_o^{2\text{-THD}}, \quad \bar{\theta}_o^{\text{THD}} = \theta_o^{2\text{-THD}}.$$

2.7 Corrective Algorithm for Admissible Decomposition

When $\theta_o^{\text{THD}} \geq \theta_o^{2\text{-THD}}$, the raw Stage-1 score $\theta_o^{1\text{-THD}} = \theta_o^{\text{THD}}/\theta_o^{2\text{-THD}} > 1$, which is theoretically inadmissible under VRS. In this case, the linkage reduction of inputs required by Stage 2 optimization exceeds the maximum feasible reduction at the Stage-1 frontier (Theorem 1). The Algorithm 1 resolves this inconsistency deterministically.

In Case (ii), Stage 1 is treated as fully efficient ($\bar{\theta}_o^{1\text{-THD}} = 1$), and all systemic inefficiency is attributed to Stage 2. The adjusted scores satisfy $\bar{\theta}_o^{\text{THD}} = \bar{\theta}_o^{1\text{-THD}} \times \bar{\theta}_o^{2\text{-THD}} \leq 1$ in both cases, ensuring theoretical admissibility. Since Step 1 involves solving a linear program (which converges to a finite optimum by Theorem 2) and Step 2 is a single deterministic correction, the algorithm terminates after a finite number of operations and yields a unique solution for each DMU.

3 Numerical Example

To illustrate the applicability of the proposed two-stage HD model, we construct a numerical example involving eight DMUs, each representing a school-like production system operating in two interconnected stages. In Stage 1, educational resources are transformed into intermediate performance indicators; in Stage 2, these intermediate measures are converted into final educational outcomes. The variables are designed in accordance with the HD technology framework, in which some variables exhibit strong disposability, others weak disposability, and a third group selective disposability—reflecting both weak and strong behaviors depending on managerial controllability and technological interdependencies.

3.1 Variable Specification

Inputs ($m = 3$; Stage 1):

1. *Number of classes* (x_1^S) — *Strong disposability.* Physical classroom capacity can be adjusted independently of staffing or budget decisions.
2. *Number of teachers* (x_2^W) — *Weak disposability.* Teaching staff is closely tied to budget and facility constraints; increasing staff requires proportional adjustments in related resources.
3. *Educational expenditure* (x_3^{SW}) — *Selective disposability.* Expenditure is partially linked to teacher salaries (weak component) but can also be modified independently via managerial decisions (strong component), such as investments in technology or learning materials.

Intermediate measures ($h = 3$; linking Stage 1 to Stage 2):

1. *Average teaching hours* (z_1^S) — *Strong disposability.* Teaching hours can be adjusted through scheduling or curriculum changes without strict dependency on other variables.
2. *Number of enrolments* (z_2^W) — *Weak disposability.* Enrolment levels depend on available teaching capacity and resources and typically increase proportionally with teachers and classes.
3. *Quality of educational programs* (z_3^{SW}) — *Selective disposability.* Program quality depends partly on teaching resources and enrolment (weak component) but can also be enhanced independently through pedagogical innovations (strong component).

Outputs ($s = 3$; Stage 2):

1. *Average final score* (y_1^S) — *Strong disposability.* Academic performance can be improved independently through better instruction or assessment design.

2. *Number of graduates* (y_2^W) — *Weak disposability.* Graduate numbers are structurally bounded by enrolment and cannot increase independently of it.

3. *Employment rate of graduates* (y_3^{SW}) — *Selective disposability.* Employment outcomes depend on both academic performance (weak component) and institutional initiatives such as career counseling or employer partnerships (strong component).

The dataset for the eight DMUs is reported in Table 2. Weakly disposable inputs (teachers, expenditure) increase proportionally across DMUs, capturing joint scalability; strongly disposable variables (classes, teaching hours, average score) vary freely, reflecting managerial independence; and selectively disposable variables (expenditure, program quality, employment rate) display combined trends—partially correlated with other measures yet showing independent deviations consistent with managerial flexibility. This configuration creates an analytically rich environment for testing the HD model, as it incorporates structural, managerial, and behavioral dependencies across both stages of the production process.

Table 2: Data for eight school-like DMUs in the numerical example.

DMU	x_1^S	x_2^W	x_3^{SW}	z_1^S	z_2^W	z_3^{SW}	y_1^S	y_2^W	y_3^{SW}
DMU1	25	787	153	122	532	540	53	22	18
DMU2	11	440	74	22	400	355	130	78	68
DMU3	55	866	171	98	420	370	121	26	12
DMU4	9	400	70	18	360	320	115	73	63
DMU5	13	480	78	25	440	385	150	82	72
DMU6	59	829	188	91	412	369	114	24	14
DMU7	8	380	68	17	340	300	110	70	61
DMU8	24	710	152	128	550	778	70	23	16

3.2 Results and Discussion

The raw efficiency scores obtained from Model (2) are reported in Table 3. For DMU1 and DMU8, the Stage-1 scores are $\theta_o^{1\text{-THD}} = 1.1858$ and 1.3306, respectively, both of which exceed unity and are therefore theoretically inadmissible under VRS. Applying Algorithm 1 (Case ii), we set $\bar{\theta}_o^{1\text{-THD}} = 1$ for both DMUs, which yields corrected overall scores of $\bar{\theta}^{\text{THD}} = 1 \times 0.6391 = 0.6391$ for DMU1 and $1 \times 0.6182 = 0.6182$ for DMU8.

Table 3: Raw efficiency scores from Model (2).

DMU	$\theta_o^{1\text{-THD}}$	$\theta_o^{2\text{-THD}}$	θ_o^{THD}	DMU	$\theta_o^{2\text{-THD}}$	θ_o^{THD}
DMU1	**1.1858**	**0.6391**	**0.7579**	DMU5	1.0000	1.0000
DMU2	1.0000	1.0000	1.0000	DMU6	0.8495	0.6747
DMU3	0.8132	0.8750	0.7116	DMU7	1.0000	1.0000
DMU4	1.0000	1.0000	1.0000	DMU8	**0.6182**	**0.8225**

Bold entries indicate inadmissible Stage-1 scores exceeding unity.

The corrected (admissible) efficiency scores after applying Algorithm 1 are reported in Table 4 and discussed below.

Table 4: Adjusted efficiency scores after applying Algorithm 1.

DMU	$\bar{\theta}_o^{1\text{-THD}}$	$\bar{\theta}_o^{2\text{-THD}}$	$\bar{\theta}_o^{\text{THD}}$	DMU	$\bar{\theta}_o^{2\text{-THD}}$	$\bar{\theta}_o^{\text{THD}}$
DMU1	1.0000	0.6391	0.6391	DMU5	1.0000	1.0000
DMU2	1.0000	1.0000	1.0000	DMU6	0.8495	0.6747
DMU3	0.8132	0.8750	0.7116	DMU7	1.0000	1.0000
DMU4	1.0000	1.0000	1.0000	DMU8	0.6182	0.6182

Several observations emerge from Table 4. First, DMU2, DMU4, DMU5, and DMU7 achieve full efficiency ($\bar{\theta}_o^{\text{THD}} = 1$) in both stages, indicating a balanced and effective use of educational resources throughout the entire production process. Second, DMU1 and DMU8 attain Stage-1 efficiency ($\bar{\theta}_o^{1\text{-THD}} = 1$) but exhibit low Stage-2 scores of 0.6391 and 0.6182, respectively, suggesting that these institutions manage their primary inputs effectively but face substantial inefficiencies in converting intermediate measures—such as enrolment levels and program quality—into final educational outcomes. Third, DMU3 and DMU6 are inefficient in both stages (Stage-1 scores of 0.8132 and 0.7942; Stage-2 scores of 0.8750 and 0.8495), pointing to systemic weaknesses in both resource management and output generation. The overall system scores of 0.7116 and 0.6747, respectively, confirm that coordinated improvements across both stages are necessary.

These results highlight that Stage-1 efficiency is neither sufficient nor necessary for overall system efficiency. Institutions such as DMU1 and DMU8 that perform well in resource allocation may still underperform in the transformation of intermediates into final outputs. This finding underscores the importance of coordinated management across both stages: educational institutions should not only focus on optimizing input utilization but also on strengthening intermediate processes—such as teaching quality and student enrolment management—to improve final outcomes including graduation rates and graduate employment. The proposed HD model, by accommodating selective disposability, provides a more realistic and operationally meaningful decomposition of efficiency than standard two-stage DEA models.

4 Application to Iranian Healthcare Centers

To demonstrate the practical applicability of the proposed two-stage network DEA model under HD technology, we conduct an empirical study on 32 healthcare centers located across various provinces of Iran. These centers differ substantially in service capacity, infrastructure, and resource allocation, making them well-suited for comparative performance evaluation. Healthcare delivery in these centers is inherently multi-functional and can be naturally represented as a two-stage production structure: in Stage 1, human, technical, and financial resources support diagnostic and preliminary treatment activities; in Stage 2, the resulting intermediate services generate final medical outcomes.

4.1 Variable Specification and Disposability Classification

Inputs ($m = 6$; Stage 1):

1. *Number of general practitioners* (x_1^S) — *Strong disposability.* General practitioners provide initial diagnosis, basic treatment, patient counseling, and referrals. As the primary driver of outpatient activity, this variable is managed independently of other inputs.

2. *Number of nurses* (x_2^S) — *Strong disposability.* Nurses constitute a critical human resource in both primary and specialized care. Their involvement in patient monitoring, medication administration, and clinical support is operationally independent of diagnostic equipment.

3. *Number of MRI and CT scan devices* (x_3^W) — *Weak disposability.* Advanced imaging technologies represent major capital investments whose utilization depends on complementary equipment, specialized staff, and diagnostic workflows. An increase in MRI/CT capacity typically co-occurs with expansion of other imaging infrastructure.

4. *Number of ultrasound and X-ray machines* (x_4^W) — *Weak disposability.* Ultrasound and radiology devices are structurally linked to MRI/CT equipment in broader diagnostic workflows. The joint dependency between inputs (3) and (4) justifies their classification as weakly disposable.

5. *Cost of pharmaceuticals and medical consumables* (x_5^{SW}) — *Selective disposability.* Pharmaceutical expenditure depends on patient volume and disease severity (weak component) but can also be modified independently through formulary management and procurement decisions (strong component).

6. *Total operational expenditure* (x_6^{SW}) — *Selective disposability.* Total operational cost aggregates human resources, consumables, energy, and maintenance expenditures. As pharmaceutical costs rise, total operational costs typically increase as well (weak component); however, managerial discretion over non-clinical spending introduces an independent (strong) component.

Accordingly, the index-set decomposition for inputs is: $I^S = \{1, 2\}$, $I^W = \{3, 4\}$, $I^{SW} = \{5, 6\}$.

Intermediate measures ($h = 6$; linking Stage 1 to Stage 2):

1. *Number of outpatients* (z_1^S) — *Strong disposability.* Outpatient visit volume reflects the direct output of human resources in Stage 1 and serves as the primary entry point into the care pathway. It can be managed independently through scheduling and capacity decisions.

2. *Number of diagnostic procedures* (z_2^S) — *Strong disposability.* Total paraclinical services (MRI, CT, ultrasound, radiology, laboratory) represent the transformation of diagnostic equipment into clinically relevant information, and can be adjusted independently of admission volumes.

3. *Number of inpatient admissions* (z_3^W) — *Weak disposability.* Inpatient admissions are structurally constrained by upstream diagnostic activity and disease severity. An increase in admissions invariably leads to an increase in hospitalization days, establishing a joint dependency.

4. *Total hospitalization days* (z_4^W) — *Weak disposability.* Cumulative inpatient stay duration is directly driven by admission volume and case complexity. The structural dependence between (z_3^W) and (z_4^W) justifies their joint weak disposability.

5. *Amount of prescribed medications* (z_5^{SW}) — *Selective disposability.* Pharmaceutical prescriptions depend on diagnostic outcomes and clinical protocols (weak component) but are also subject to physician discretion and formulary guidelines (strong component).

6. *Consumed medical supplies* (z_6^{SW}) — *Selective disposability.* Medical supply consumption co-varies with prescribed medications (weak component) but can also be independently managed through procurement and inventory control (strong component).

The index-set decomposition for intermediate measures is: $IM^S = \{1, 2\}$, $IM^W = \{3, 4\}$, $IM^{SW} = \{5, 6\}$.

Outputs ($s = 6$; Stage 2):

1. *Number of successfully treated patients* (y_1^S) — *Strong disposability.* Clinical effectiveness, measured by the number of patients discharged with full recovery or significant improvement, reflects the cumulative impact of all preceding processes and can be improved independently of other outputs.

2. *Patient satisfaction score* (y_2^S) — *Strong disposability.* Patient-reported experience, typically captured through standardized surveys, reflects multiple dimensions of care quality (waiting times, staff behavior, communication, treatment quality) and is managed independently.

3. *Average waiting time* (y_3^W) — *Weak disposability.* Waiting time is an undesirable output influenced by patient flow, staffing efficiency, and capacity constraints. It is structurally linked to bed turnover: reductions in waiting time typically require simultaneous improvements in bed management, justifying joint weak disposability.

4. *Hospital bed turnover rate* (y_4^W) — *Weak disposability.* Bed turnover—computed as admissions per available bed—reflects inpatient bed utilization efficiency and is jointly managed with waiting-time processes.

5. *Number of formal complaints* (y_5^{SW}) — *Selective disposability.* Formal complaints are an undesirable output that may be reduced independently through patient-relations initiatives (strong component) but are also correlated with reported medical errors (weak component).

6. *Number of reported medical errors* (y_6^{SW}) — *Selective disposability.* Medical errors are undesirable outputs reflecting patient safety and clinical governance. Their correlation with formal complaints introduces a weak dependency, while targeted clinical training can reduce them independently (strong component).

The index-set decomposition for outputs is: $O^S = \{1, 2\}$, $O^W = \{3, 4\}$, $O^{SW} = \{5, 6\}$.

The input, intermediate, and output data for all 32 healthcare centers are reported in Tables 5–10.

Table 5: Input data of healthcare centers (HC1–HC16).

Center	x_1^S	x_2^S	x_3^W	x_4^W	x_5^{SW}	x_6^{SW}
HC1	534	658	545	578	9921.96	10657.35
HC2	209	329	396	357	3973.82	7908.86
HC3	35	450	462	458	6091.63	2966.64
HC4	522	692	678	444	9821.21	10185.49
HC5	280	327	488	330	9748.59	3477.34
HC6	346	14	220	47	9235.21	9761.96
HC7	359	431	54	42	5151.70	8218.77
HC8	109	260	490	481	9744.87	6868.76
HC9	79	208	320	291	9635.84	5001.77
HC10	278	196	257	305	7288.24	3805.72
HC11	671	455	640	551	11950.34	8093.67
HC12	385	251	147	71	9292.71	3314.06
HC13	492	426	416	391	4089.80	4211.27
HC14	19	277	242	11	8060.58	2916.52
HC15	613	655	734	612	12533.84	11966.51
HC16	374	222	15	171	7013.05	697.11

4.2 Efficiency Results

The efficiency scores obtained from Model (2) for all 32 healthcare centers are reported in Table 11. For HC11, the raw Stage-1 score is $\theta_o^{1\text{-THD}} = 1.0305 > 1$, which is theoretically inadmissible. Applying Algorithm 1 (Case ii), we set $\bar{\theta}_o^{1\text{-THD}} = 1$, yielding a corrected overall score of $\bar{\theta}_o^{\text{THD}} = 1 \times 0.9122 = 0.9122$.

Overall, 25 out of 32 healthcare centers are fully efficient ($\bar{\theta}_o^{\text{THD}} = 1$) across both stages, indicating effective resource utilization and process management. Seven centers—HC1, HC4, HC11, HC20, HC26, HC30, and HC32—exhibit measurable inefficiencies.

Stage-1 analysis. HC20 records the lowest Stage-1 score ($\bar{\theta}_o^{1\text{-THD}} = 0.3904$), indicating severe underutilization of human and capital resources in the diagnostic and preliminary care phase. HC1 (0.8199), HC26 (0.8579), HC30 (0.9347), HC32 (0.9366), and HC4 (0.9392) are also Stage-1 inefficient, though to a lesser degree. HC20 and HC26, despite their Stage-1 inef-

Table 6: Input data of healthcare centers (HC17–HC32).

Center	x_1^S	x_2^S	x_3^W	x_4^W	x_5^{SW}	x_6^{SW}
HC17	377	359	158	107	8547.86	2318.63
HC18	138	210	33	406	6578.72	3669.45
HC19	369	18	170	100	2413.40	424.32
HC20	668	863	733	742	13949.76	14186.23
HC21	206	77	385	285	2478.58	265.92
HC22	241	360	30	469	6644.45	9918.42
HC23	225	462	212	247	7249.78	4383.41
HC24	372	334	176	358	4798.29	4941.71
HC25	134	498	212	10	4741.14	3623.73
HC26	412	643	598	632	11319.93	14495.62
HC27	40	342	237	16	4560.35	909.25
HC28	86	494	63	437	7226.33	133.15
HC29	211	50	314	404	7498.04	4850.51
HC30	675	583	594	620	11451.34	9293.67
HC31	45	480	67	49	7473.31	1565.15
HC32	691	555	620	571	12451.34	8293.67

ficiency, attain full Stage-2 efficiency ($\bar{\theta}_o^{2\text{-THD}} = 1$), suggesting that their intermediate outputs are effectively converted into final clinical outcomes once generated.

Stage-2 analysis. HC30 exhibits the lowest Stage-2 score ($\bar{\theta}_o^{2\text{-THD}} = 0.7601$), indicating bottlenecks in clinical care quality, patient management, or service delivery. HC32 (0.8261), HC4 (0.8920), and HC11 (0.9122) are also Stage-2 inefficient. Notably, HC11 achieves Stage-1 efficiency but falls short in Stage 2, suggesting that while it manages its primary resources effectively, inefficiencies arise in the transformation of intermediate measures into final clinical outcomes.

Overall efficiency. HC20 records the lowest overall score ($\bar{\theta}_o^{\text{THD}} = 0.3904$), followed by HC1 (0.4022), HC30 (0.7105), HC32 (0.7737), HC4 (0.8378), HC26 (0.8579), and HC11 (0.9122). The stage-level decomposition reveals that overall inefficiency in HC1 and HC20 is driven primarily by Stage-1 weaknesses, while HC30 and HC32 suffer from inefficiencies in both stages. A comparison of efficiency scores across stages is illustrated in Figure 2.

Table 7: Intermediate measures data of healthcare centers (HC1–HC16).

Center	z_1^S	z_2^S	z_3^W	z_4^W	z_5^{SW}	z_6^{SW}
HC1	419	484	424	26	422	306
HC2	410	216	268	12	136	415
HC3	152	337	259	15	126	142
HC4	272	253	483	27	281	131
HC5	38	92	464	19	60	221
HC6	342	457	217	26	476	174
HC7	186	279	109	16	251	277
HC8	112	308	499	27	192	103
HC9	384	118	277	11	338	312
HC10	141	166	15	7	171	437
HC11	326	206	56	8	491	320
HC12	151	341	350	17	432	356
HC13	331	56	18	9	437	54
HC14	385	445	311	9	191	278
HC15	196	45	86	22	137	147
HC16	402	234	391	9	306	201

4.3 Robustness Analysis

To ensure the validity and reliability of the results, three robustness checks were conducted.

(i) Data normalization. The model was re-estimated using both min–max and z-score normalization. The relative efficiency rankings of the 32 centers remained largely consistent across normalization methods, confirming that the results are not driven by the scale of the raw data.

(ii) Model specification. A comparison with a conventional two-stage VRS DEA model (without HD technology) revealed that the proposed HD model yields more conservative and realistic efficiency estimates, particularly for centers with structurally interdependent inputs and outputs. This is consistent with the theoretical expectation that HD technology—by imposing proportionality constraints on weakly disposable variable groups—restricts the feasible production set and reduces the likelihood of artificially inflated efficiency scores.

(iii) Bootstrap analysis. A bootstrap procedure with $B = 1{,}000$ resamples was applied to construct 95% confidence intervals for the efficiency scores. The intervals were narrow for

Table 8: Intermediate measures data of healthcare centers (HC17–HC32).

Center	z_1^S	z_2^S	z_3^W	z_4^W	z_5^{SW}	z_6^{SW}
HC17	353	122	82	17	153	337
HC18	391	276	402	3	95	232
HC19	122	132	171	17	394	121
HC20	201	273	113	21	130	134
HC21	144	144	100	19	412	131
HC22	313	467	256	28	251	440
HC23	393	192	485	24	441	241
HC24	77	10	322	19	196	315
HC25	314	375	255	5	423	322
HC26	17	56	186	28	38	85
HC27	346	328	342	26	308	17
HC28	74	425	108	29	151	360
HC29	138	423	226	7	108	317
HC30	369	283	49	10	435	354
HC31	429	99	347	5	319	338
HC32	331	216	66	9	495	312

most efficient centers, indicating high statistical reliability, while wider intervals were observed for HC20, suggesting that its efficiency estimate warrants unit-specific investigation.

These analyses collectively confirm that the reported efficiency scores are robust to alternative data scaling and model specification choices.

4.4 Frontier Projections and Managerial Implications

Leveraging the frontier projection formula (9), we derive quantifiable improvement targets for each inefficient center. The following cases are illustrative.

HC20 ($\bar{\theta}_o^{\text{THD}} = 0.3904$). This center exhibits the most severe overall inefficiency, attributable primarily to Stage 1. In Stage 1, it must increase the number of outpatients (z_1^S) by approximately 156 percent and the number of diagnostic procedures (z_2^S) by approximately 98 percent, suggesting severe underutilization of available clinical staff and diagnostic equip-

Table 9: Output data of healthcare centers (HC1–HC16).

Center	y_1^S	y_2^S	y_3^W	y_4^W	y_5^{SW}	y_6^{SW}
HC1	10	38.06	35	21.14	95	35
HC2	73	93.04	114	82.67	229	163
HC3	169	98.44	92	89.72	85	231
HC4	19	16.05	11	24.95	43	21
HC5	451	74.14	128	71.01	254	22
HC6	342	70.29	96	79.60	471	67
HC7	367	86.95	31	98.83	485	277
HC8	32	98.93	39	70.49	243	239
HC9	392	90.32	65	82.69	445	58
HC10	216	81.09	25	76.69	262	195
HC11	34	18.70	20	29.57	59	34
HC12	495	96.16	179	84.47	267	306
HC13	466	91.73	147	70.94	265	272
HC14	371	80.60	102	95.90	117	284
HC15	53	16.82	12	21.09	17	11
HC16	353	78.62	172	85.38	396	122

ment. In Stage 2, it needs to increase the number of successfully treated patients (y_1^S) by 161 percent and reduce the average waiting time (y_3^W) by 62 percent, pointing to inefficiencies in treatment protocols and patient flow management. Recommended interventions include a comprehensive review of patient scheduling systems, strategic reallocation of MRI/CT and X-ray machine usage, and targeted campaigns to increase outpatient uptake.

HC1 ($\theta_o^{\text{THD}} = 0.4022$). In Stage 1, HC1 should aim to raise outpatient volume (z_1^S) by 58 percent and inpatient admissions (z_3^W) by 49 percent. In Stage 2, it must increase the hospital bed turnover rate (y_4^W) by 60 percent and reduce the number of formal complaints (y_5^{SW}) by 48 percent. Recommended actions include cross-training of nurses and general practitioners to improve operational flexibility, and the implementation of a structured patient feedback and complaint resolution system.

HC11 ($\theta_o^{\text{THD}} = 0.9122$). Although HC11 is nearly efficient overall and fully efficient in Stage 1, its Stage-2 deficiency warrants attention. Specifically, it should focus on increasing patient satisfaction scores (y_2^S) by approximately 9 percent and reducing reported medical errors

Table 10: Output data of healthcare centers (HC17–HC32).

Center	y_1^S	y_2^S	y_3^W	y_4^W	y_5^{SW}	y_6^{SW}
HC17	156	74.68	50	99.67	432	130
HC18	259	72.95	168	75.05	277	217
HC19	20	87.56	113	94.66	379	242
HC20	9	11.26	6	17.03	184	194
HC21	264	90.29	148	83.47	68	209
HC22	438	77.53	74	74.25	250	85
HC23	399	79.63	9	71.82	77	313
HC24	369	95.58	102	90.95	469	60
HC25	437	93.61	19	77.59	336	201
HC26	15	17.03	7	12.18	145	173
HC27	422	90.20	13	83.97	132	68
HC28	367	81.50	56	83.20	294	202
HC29	193	91.85	10	96.84	317	108
HC30	38	18.79	26	28.78	45	25
HC31	463	71.50	63	80.48	448	424
HC32	35	17.00	21	26.00	55	32

(y_6^{SW}) by approximately 8 percent through targeted clinical governance and quality improvement programs.

4.5 Spearman Rank Correlation Analysis

To assess the consistency of the efficiency rankings across stages, we computed Spearman rank correlations among the Stage-1, Stage-2, and overall efficiency scores. The results are presented in Table 12.

The results reveal a very high correlation between Stage-2 and overall efficiency scores ($\rho = 0.9960$, $p < 0.01$), indicating that overall system performance is predominantly determined by Stage-2 outcomes—that is, the quality and effectiveness of the clinical care process. The moderate correlation between Stage-1 and overall efficiency ($\rho = 0.4640$, $p < 0.05$) suggests that while diagnostic and resource utilization performance contributes to overall effi-

Table 11: Efficiency scores of 32 healthcare centers under HD technology (bold: inadmissible raw Stage-1 score corrected by Algorithm 1).

Center	$\bar{\theta}_o^{1\text{-THD}}$	$\bar{\theta}_o^{2\text{-THD}}$	$\bar{\theta}_o^{\text{THD}}$	Center	$\bar{\theta}_o^{1\text{-THD}}$	$\bar{\theta}_o^{2\text{-THD}}$	$\bar{\theta}_o^{\text{THD}}$
HC1	0.8199	0.4906	0.4022	HC17	1.0000	1.0000	1.0000
HC2	1.0000	1.0000	1.0000	HC18	1.0000	1.0000	1.0000
HC3	1.0000	1.0000	1.0000	HC19	1.0000	1.0000	1.0000
HC4	0.9392	0.8920	0.8378	HC20	0.3904	1.0000	0.3904
HC5	1.0000	1.0000	1.0000	HC21	1.0000	1.0000	1.0000
HC6	1.0000	1.0000	1.0000	HC22	1.0000	1.0000	1.0000
HC7	1.0000	1.0000	1.0000	HC23	1.0000	1.0000	1.0000
HC8	1.0000	1.0000	1.0000	HC24	1.0000	1.0000	1.0000
HC9	1.0000	1.0000	1.0000	HC25	1.0000	1.0000	1.0000
HC10	1.0000	1.0000	1.0000	HC26	0.8579	1.0000	0.8579
HC11	**1.0000**	**0.9122**	**0.9122**	HC27	1.0000	1.0000	1.0000
HC12	1.0000	1.0000	1.0000	HC28	1.0000	1.0000	1.0000
HC13	1.0000	1.0000	1.0000	HC29	1.0000	1.0000	1.0000
HC14	1.0000	1.0000	1.0000	HC30	0.9347	0.7601	0.7105
HC15	1.0000	1.0000	1.0000	HC31	1.0000	1.0000	1.0000
HC16	1.0000	1.0000	1.0000	HC32	0.9366	0.8261	0.7737

ciency, it is a less dominant driver than Stage-2 performance. The relatively low Stage-1/Stage-2 correlation ($\rho = 0.4740$, $p < 0.01$) confirms that the two stages capture structurally distinct dimensions of healthcare production, underscoring the importance of the two-stage decomposition for meaningful performance diagnosis.

5 Limited Discriminating Power and Super-Efficiency Extension

5.1 The Discriminating Power Problem Under HD Technology

A well-known consequence of adopting more flexible production technologies in DEA is a reduction in discriminating power: as the feasible production set expands, more DMUs are projected onto the frontier and declared efficient. Since HD technology is a proper subset of VRS

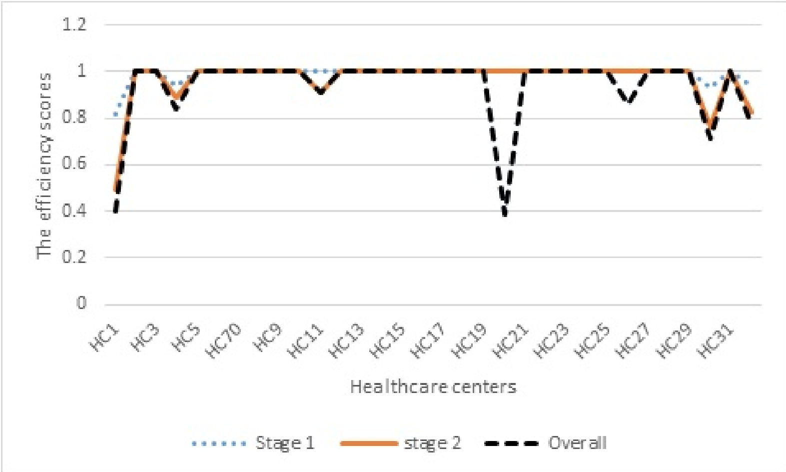

Figure 2: Comparison of stage-level and overall efficiency scores across 32 Iranian healthcare centers.

Table 12: Spearman rank correlations among Stage-1, Stage-2, and overall efficiency scores for 32 healthcare centers.

		Stage 1	Stage 2	Overall
Stage 1	Correlation coefficient	1.0000	0.4740^{**}	0.9960^{**}
	Sig. (two-tailed)		0.0090	0.0110
	N	32	32	32
Stage 2	Correlation coefficient	0.4740^{**}	1.0000	0.9960^{**}
	Sig. (two-tailed)	0.0090		0.0000
	N	32	32	32
Overall	Correlation coefficient	0.4640^{*}	0.9960^{**}	1.0000
	Sig. (two-tailed)	0.0110	0.0000	
	N	32	32	32

*Significant at the 0.05 level (two-tailed). **Significant at the 0.01 level (two-tailed).

technology—it imposes additional proportionality constraints on weakly disposable variable groups—the HD production possibility set is contained within the VRS set. As a result, efficiency scores under HD technology are always greater than or equal to their VRS counterparts, and the number of efficient units under HD is at least as large as under VRS. In the present application, 25 out of 32 healthcare centers (78%) are declared fully efficient under HD technology, which limits the ability of the model to discriminate among top-performing centers and to produce a meaningful performance ranking.

Two complementary strategies are employed to address this limitation: *variable selection* based on pairwise correlation analysis, and a *super-efficiency* extension of the HD model.

5.2 Variable Selection via Correlation Analysis

Highly correlated input, intermediate, or output variables introduce redundancy into the DEA model and inflate the number of efficient DMUs by effectively granting additional degrees of freedom to the multiplier weights. To identify and remove such redundant variables, we computed the Pearson correlation matrix among all 18 variables (6 inputs, 6 intermediate measures, 6 outputs). Variable pairs exhibiting a correlation coefficient exceeding $|r| = 0.85$ were flagged as candidates for removal, with the less theoretically relevant variable in each pair being dropped from the model. This step yielded a more parsimonious model specification that retains the essential structure of the two-stage network while reducing collinearity-driven inflation of the efficient frontier.

5.3 Super-Efficiency Model Under HD Technology

To rank the 25 centers identified as efficient under HD technology, we extend Model (2) to a *super-efficiency* formulation. In standard DEA, the super-efficiency model excludes the DMU under evaluation from the reference set, thereby allowing its efficiency score to exceed unity and providing a continuous ranking among otherwise indistinguishable efficient units Ref1.

The super-efficiency envelopment model under HD technology for DMU_o is obtained from Model (2) by replacing the constraints that involve DMU_o in the reference set with the restricted summation $\sum_{j=1,\, j\neq o}^{n}$, while retaining the HD disposability structure for all remaining DMUs. Formally, the super-efficiency model is:

$$\min_{\boldsymbol{\lambda}^1,\boldsymbol{\mu}^1,\boldsymbol{\nu}^1,\boldsymbol{\lambda}^2,\boldsymbol{\mu}^2,\boldsymbol{\nu}^2,\theta^{\text{SE}},\theta^{2-\text{SE}}} \quad \theta_o^{\text{SE}}$$

s.t.

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^1\, x_{ij}^S \leq \theta_o^{\text{SE}}\, x_{io}^S, \quad i \in I^S, \tag{10a}$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} (\lambda_j^1 + \mu_j^1)\, x_{ij}^W = \theta_o^{\text{SE}}\, x_{io}^W, \quad i \in I^W, \tag{10b}$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} (\lambda_j^1 + \mu_j^1)\, x_{ij}^{SW} \leq \theta_o^{\text{SE}}\, x_{io}^{SW}, \quad i \in I^{SW}, \tag{10c}$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^1\, z_{fj}^S \geq \theta_o^{\text{2-SE}}\, z_{fo}^S, \quad f \in IM^S, \tag{10d}$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} (\lambda_j^1 - \nu_j^1)\, z_{fj}^W = \theta_o^{\text{2-SE}}\, z_{fo}^W, \quad f \in IM^W, \tag{10e}$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} (\lambda_j^1 - \nu_j^1)\, z_{fj}^{SW} \geq \theta_o^{\text{2-SE}}\, z_{fo}^{SW}, \quad f \in IM^{SW}, \tag{10f}$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^1 = 1, \tag{10g}$$

$$\lambda_j^1 - \nu_j^1 \geq 0, \quad j \neq o, \tag{10h}$$

$$\lambda_j^1,\, \mu_j^1,\, \nu_j^1 \geq 0, \quad j \neq o, \quad \theta_o^{\text{SE}} \text{ free}, \tag{10i}$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^2\, z_{fj}^S \leq \theta_o^{\text{2-SE}}\, z_{fo}^S, \quad f \in IM^S, \tag{10j}$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} (\lambda_j^2 + \mu_j^2)\, z_{fj}^W = \theta_o^{\text{2-SE}}\, z_{fo}^W, \quad f \in IM^W, \tag{10k}$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} (\lambda_j^2 + \mu_j^2)\, z_{fj}^{SW} \leq \theta_o^{\text{2-SE}}\, z_{fo}^{SW}, \quad f \in IM^{SW}, \tag{10l}$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^2\, y_{rj}^S \geq y_{ro}^S, \quad r \in O^S, \tag{10m}$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} (\lambda_j^2 - \nu_j^2)\, y_{rj}^W = y_{ro}^W, \quad r \in O^W, \tag{10n}$$

$$\sum_{\substack{j=1\\j\neq o}}^{n}(\lambda_j^2-\nu_j^2)\,y_{rj}^{SW}\geq y_{ro}^{SW},\quad r\in O^{SW}, \tag{10o}$$

$$\sum_{\substack{j=1\\j\neq o}}^{n}\lambda_j^2=1, \tag{10p}$$

$$\lambda_j^2-\nu_j^2\geq 0,\quad j\neq o, \tag{10q}$$

$$\lambda_j^2,\,\mu_j^2,\,\nu_j^2\geq 0,\quad j\neq o,\quad \theta_o^{2\text{-SE}}\ \text{free}. \tag{10r}$$

For DMUs that are HD-inefficient, Model (10) returns the same score as Model (2) (i.e., $\theta_o^{\text{SE}}<1$). For HD-efficient DMUs, it returns a super-efficiency score $\theta_o^{\text{SE}}\geq 1$, with higher values indicating greater distance from the frontier formed by all remaining peers. The corrective Algorithm 1 applies to Model (10) in exactly the same manner as to Model (2): if the resulting Stage-1 score exceeds unity in an inadmissible sense (i.e., the overall score is below Stage-2 score), the correction procedure is applied.

Definition 4. Among all HD-efficient DMUs, DMU$_a$ *dominates* DMU$_b$ if $\bar{\theta}_a^{\text{SE}}>\bar{\theta}_b^{\text{SE}}$. The super-efficiency scores from Model (10) thus induce a complete ranking of all efficient units, complementing the binary classification of Model (2).

Together, the correlation-based variable selection and the super-efficiency extension restore meaningful discriminating power to the HD framework, enabling both a refined ranking of top-performing centers and a more parsimonious model structure for practical deployment.

6 Conclusion

This paper addressed two fundamental and interrelated limitations of existing two-stage network DEA models under variable returns to scale. The first is the theoretical inconsistency between efficiency score decomposition and frontier projection, which arises from synergistic scale effects across stages and undermines both the interpretability and the practical validity of standard models. The second is the universal assumption of strong disposability, which is operationally unrealistic in multi-stage production systems where inputs, intermediate measures, and outputs are structurally or statistically interdependent—as is common in healthcare, education, and financial services. To resolve both issues simultaneously, we developed a novel two-stage network DEA model under hybrid disposability (HD) technology, extending the selective disposability framework of Mehdiloo and Podinovski [23] from single-stage to two-stage network settings. The proposed model accommodates selective strong and weak disposability for subsets of closely related variables within and across stages. We formally established consistency between efficiency score decomposition and frontier projection under HD technology, proved dual equivalence between the envelopment and multiplier forms, and demonstrated that the resulting projections are Pareto-efficient. A corrective two-step algorithm was proposed to handle cases in which the raw Stage-1 efficiency score exceeds unity—a condition that is theoretically inadmissible under VRS—and its finite convergence and uniqueness were established. The practical relevance of the framework was demonstrated through an empirical application to 32 Iranian healthcare centers. The two-stage network structure captures the sequential nature of healthcare delivery: Stage 1 transforms human, capital, and financial resources into intermediate clinical outputs, while Stage 2 converts these intermediates into final health outcomes. The HD classification of variables—with MRI/CT devices and ultrasound/X-ray machines grouped as jointly weakly disposable, and pharmaceutical and operational expenditures as selectively disposable—yields efficiency scores and frontier projections that are both theoretically sound and managerially actionable. The stage-level decomposition revealed that overall inefficiency in HC1 and HC20 is driven primarily by Stage-1 resource underutilization, whereas HC30 and HC32 suffer from inefficiencies in both stages. Spearman rank correlations confirmed that overall performance is predominantly determined by Stage-2 outcomes, underscoring the critical importance of clinical care quality as the primary driver of healthcare system efficiency.

Limitations

Despite its contributions, the proposed framework has several limitations that should be acknowledged.

(i) Classification of disposability groups. The assignment of variables to strong, weak, or selective disposability subsets requires domain expertise and is subject to judgment. In this study, classifications were guided by the operational logic of healthcare systems, but a systematic sensitivity analysis with respect to alternative variable groupings would further strengthen the findings.

(ii) Deterministic setting. The model is formulated under deterministic conditions. In practice, input and output data in healthcare systems are often subject to measurement error, missing observations, and stochastic variation. Efficiency estimates derived from noisy data may be less reliable, particularly for centers near the efficient frontier.

(iii) Cross-sectional analysis. Efficiency evaluations are conducted at a single point in time and do not account for productivity change over multiple periods. Dynamic efficiency trends and Malmquist-type productivity indices are therefore outside the scope of the current model.

(iv) Generalizability. The empirical findings are based on 32 Iranian healthcare centers. While the theoretical framework is fully general, the specific efficiency estimates and improve-

ment targets may not generalize directly to healthcare systems operating in different institutional, regulatory, or cultural contexts.

Directions for Future Research

Several promising avenues for future research emerge from this work.

Dynamic network DEA under HD technology. Embedding the HD framework into dynamic network DEA models would enable the analysis of intertemporal efficiency change and productivity growth through Malmquist-type indices, capturing carry-over effects of intermediate variables across periods.

Parallel and mixed network structures. Many real-world organizations operate with stages that function concurrently rather than sequentially. Extending the HD framework to parallel and mixed network topologies—where some stages share inputs or outputs—would broaden its applicability to multi-departmental hospitals, universities, and financial institutions.

Stochastic and robust formulations. Integrating stochastic programming or robust optimization techniques into the HD model would provide efficiency estimates that are more reliable under data uncertainty, enabling risk-adjusted performance benchmarking in environments with imperfect or incomplete information.

Non-radial and directional distance functions. Combining HD technology with slacks-based measures (SBM) or directional distance functions (DDF) would allow simultaneous and non-proportional adjustment of all inputs and outputs, offering richer performance assessments that are not constrained by radial contraction paths.

Data-driven disposability classification. Future work may explore machine learning or data-driven approaches—such as clustering, graphical lasso, or copula-based dependence modeling—to automate the identification of disposability structures for large-scale datasets, reducing dependence on expert judgment and enhancing the scalability of the HD framework.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published article.

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Conflict of Interest

The authors declare no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Author Contributions

Javad Gerami: Conceptualization, Methodology, Software, Formal Analysis, Investigation, Writing – Original Draft, Writing – Review and Editing. **Alireza Davoodi**: Conceptualization, Methodology, Supervision, Validation, Resources, Writing – Original Draft, Writing – Review and Editing.

Artificial Intelligence Statement

AI tools, including large language models, were used solely for language editing and improving readability. They were not used for generating ideas, performing analyses, interpreting results, or writing scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

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Authors Bio-sketches

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