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Parameter Estimation and Prediction for the Basic Gompertz Distribution Based on Record Statistics: Frequentist and Bayesian Approaches

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Abstract. We consider the estimation of model parameters and prediction of unobserved records based on record statistics for the Basic Gompertz distribution (BGD) with parameter λ using frequentist and Bayesian analysis. In frequentist analysis, we give the moment generating function of the mth record, the maximum likelihood estimation (MLE) of λ, the moment-based estimate (MBE) of λ, the confidence interval for λ, and the prediction of future records. Exactly, we show that

$$\hat{\lambda}_{\text{MBE}} = \frac{m(m+1)}{2\sum_{i=1}^{m}(e^{Y_i}-1)}, \quad \hat{\lambda}_{\text{MLE}} = \frac{m}{e^{y_m}-1},$$

where $m \geq 1$ and $Y_m = \max(\min)\{X_1, \ldots, X_m\}$. In Bayesian analysis, we obtain the Bayesian sample-based estimation and prediction. Exactly, we show that under the squared error loss (SEL) function,

$$\hat{\lambda}_{BS} = \frac{m+a}{e^{y_m}+b-1}$$

and under the LINEX loss function,

$$\hat{\lambda}_{BL} = -\frac{m+a}{c}\ln\left(\frac{e^{y_m}+b-1}{e^{y_m}+(b+c)-1}\right).$$

Based on Monte Carlo simulations, the performances of the different methods of estimation and prediction are compared via MSEs and Biases. Finally, a real dataset has been analyzed for illustrative purposes.

Keywords. Basic Gompertz distribution, Record statistics, Bayesian estimation, Maximum likelihood estimation (MLE), Moment-based estimation (MBE), prediction, LINEX loss function.

MSC. 62F10; 62F15; 62F25.

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1 Introduction

The Gompertz distribution is a continuous probability distribution introduced by Benjamin Gompertz in 1825 [9] to express the law of human mortality. It has since become a standard tool in demography and statistics for describing the distribution of adult lifespans, and has found broad application in survival analysis, biology, and gerontology. More recently, the distribution has been adopted in reliability engineering to model the failure rate of software systems, and at the individual level to quantify customer lifetime value in marketing science [9].

The one-parameter, or Basic Gompertz distribution (BGD), with shape parameter $\lambda > 0$, denoted $G(\lambda)$, has probability density function (pdf), cumulative distribution function (cdf), and hazard rate function given by

$$\begin{aligned}
f(x) &= \lambda\, e^{x} \exp\Big(-\lambda(e^{x}-1)\Big), && x>0,\ \lambda>0,\\
F(x) &= 1-\exp\Big(-\lambda(e^{x}-1)\Big), && x>0,\ \lambda>0,\\
H(x) &= \lambda\, e^{x}, && x>0,\ \lambda>0.
\end{aligned}$$

The standard Gompertz distribution is recovered as the special case $\lambda = 1$. Figure 1 displays the pdf and hazard rate function of the BGD for representative values of λ.

The Gompertz distribution has attracted considerable attention in the reliability and life-testing literature. Gordon [10] derived maximum likelihood estimates for a mixture of two Gompertz distributions under censoring. Chen and Shao [8] constructed exact joint confidence regions and marginal confidence intervals for the distribution parameters. Jaheen [11] studied Bayesian estimation based on record values using continuous priors and the Laplace approximation. Wu et al. [21] and Soliman et al. [16] derived exact confidence intervals under first-failure and progressive first-failure censoring plans, respectively, while Wu and Li [22] established exact confidence intervals and joint confidence regions under doubly type-II censoring. Kumar and Vaish [14] examined a stress-strength reliability model by characterizing the relationship between the distributional parameters of stress and strength components. Chacko and Mohan [7] analyzed inference for the Gompertz distribution under progressive type-II censored data with binomial removals. Jha et al. [12] addressed the estimation of multicomponent stress-strength reliability under progressive type-II censoring when both stress and strength follow unit Gompertz distributions with a common scale parameter. Wang et al. [20] considered parameter estimation for the two-parameter Gompertz distribution under both frequentist and Bayesian frameworks when only record data are available. Asgharzadeh et al. [4] studied parameter estimation and prediction of future records for the Lindley distribution based on record statistics.

The present paper extends these results to the Basic Gompertz distribution. A distinctive feature of the proposed approach is that, in contrast to much of the existing literature, many of the derived results are analytical in form and comparatively less involved (see Section 3). In particular, the construction of the pivotal statistic S introduced in Subsection 2.1 differs fundamentally from those employed in related works.

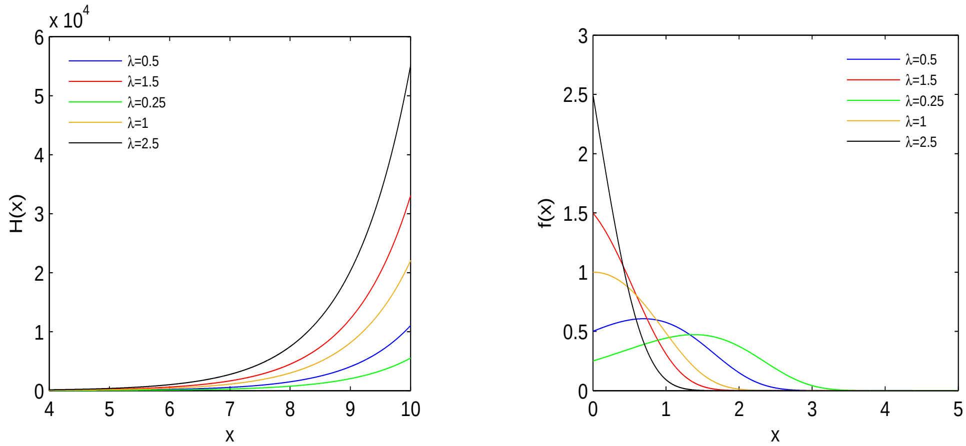

Figure 1: Hazard rate function (left) and probability density function (right) of the BGD for selected values of λ.

Consider a sequence of iid random variables $X_1, X_2, \ldots$ with cdf $F(x \mid \theta)$ and pdf $f(x \mid \theta)$. Define

$$Y_m = \max(\min)\{X_1, \ldots, X_m\}, \qquad m \geq 1.$$

Then X_j is called an upper (lower) record value if $X_j > (<)\, Y_{j-1}$ for $j > 1$. The sequence of record times $\{U(m),\, m \geq 1\}$ is defined by

$$U(1) = 1, \qquad U(m) = \min\{j : j > U(m-1),\ X_j > X_{U(m-1)}\},$$

and the corresponding sequence of upper record statistics is $\{X_{U(m)},\, m \geq 1\}$. It is worth noting that knowledge of certain distributional properties of the record value sequence suffices to characterize the underlying population distribution; for instance, the sequence of expected record values uniquely determines the common distribution of the X_i's [3]. For further applications of record statistics, the reader is referred to [1, 13, 18].

The remainder of the paper is organized as follows. Section 2 covers the frequentist analysis of the BGD: the MBE and MLE of λ are derived in Subsections 2.1 and 2.2, a pivotal-based confidence interval is constructed in Subsection 2.3, and prediction of future record values is treated in Subsection 2.4. Section 3 is devoted to Bayesian analysis, where point estimators under squared error and LINEX loss functions, HPD credible intervals, and Bayesian predictors are obtained. Monte Carlo simulation results comparing the finite-sample performance of all estimators and predictors are presented in Section 4. A real data example is analyzed in Section 5 to illustrate the practical applicability of the proposed methods. Section 6 collects the conclusions.

2 Frequentist Analysis

This section addresses the frequentist estimation of the parameter λ. In particular, the MBE and the MLE of λ are derived, along with predictors for future record values.

2.1 Moment-Based Estimator of λ

Let $W_i = -\log(1 - F(X_i \mid \lambda))$, so that $W_i \sim \text{Exp}(1)$. Observe that

$$-\log\big(1 - F(Y_i \mid \lambda)\big) = \lambda\,(e^{Y_i} - 1) \overset{D}{=} W_i^*,$$

where W_i^* denotes the ith upper record value arising from a sequence of iid random variables $W_i \sim \text{Exp}(1)$.

Arnold et al. [3] established that the spacings $E_1 = W_1^*,\ E_2 = W_2^* - W_1^*,\ \ldots,\ E_m = W_m^* - W_{m-1}^*$ are iid $\text{Exp}(1)$ random variables. Consequently, the statistic

$$S(\lambda, \mathbf{Y}) = \sum_{i=1}^{m} W_i^* = \sum_{i=1}^{m} (m - i + 1)\, E_i$$

is a weighted sum of the iid quantities E_i, with mean $\frac{m(m+1)}{2}$ and variance $\frac{m(m+1)(2m+1)}{6}$.

Let $\xrightarrow{P}$ denote convergence in probability [6]. For any $\varepsilon > 0$, Chebyshev's inequality gives

$$\begin{aligned} P\left(\left| \frac{S(\lambda, \mathbf{Y}) - \dfrac{m(m+1)}{2}}{\dfrac{m(m+1)}{2}} \right| > \varepsilon \right) \quad &\leq \quad \frac{1}{\varepsilon^2}\,\text{Var}\left(\frac{S(\lambda, \mathbf{Y})}{\dfrac{m(m+1)}{2}} \right) \\ &= \quad \frac{2m+1}{m(m+1)\,\varepsilon^2} \quad \longrightarrow \quad 0, \end{aligned} \tag{1}$$

as $m \to \infty$, which establishes that $\hat{\lambda}_{\text{MBE}}$ is a consistent and asymptotically unbiased estimator of λ. Indeed,

$$\frac{S(\lambda, \mathbf{Y})}{\dfrac{m(m+1)}{2}} = \frac{\lambda \displaystyle\sum_{i=1}^{m} (e^{Y_i} - 1)}{\dfrac{m(m+1)}{2}} \xrightarrow{P} 1,$$

from which the MBE of λ follows immediately as

$$\hat{\lambda}_{\text{MBE}} = \frac{m(m+1)}{2 \displaystyle\sum_{i=1}^{m} (e^{Y_i} - 1)}.$$

2.2 Maximum Likelihood Estimator of λ

Given the observed record vector $\mathbf{y} = (y_1, \ldots, y_m)$, the joint density of the upper record values $Y_1 = y_1, Y_2 = y_2, \ldots, Y_m = y_m$ is given by [3, Eq. (2.3.9)]

$$\begin{aligned} L(\lambda \mid \mathbf{y}) &= f(y_m \mid \lambda) \prod_{i=1}^{m-1} \frac{f(y_i \mid \lambda)}{1 - F(y_i \mid \lambda)} \\ &= \lambda^m \, e^{\sum_{i=1}^{m} y_i} \, e^{-\lambda\,(e^{y_m}-1)}. \end{aligned} \tag{2}$$

The corresponding log-likelihood function is

$$\ell := \ln L(\lambda \mid \mathbf{y}) = m \ln \lambda - \lambda\,(e^{y_m} - 1) + \sum_{i=1}^{m} y_i. \tag{3}$$

Differentiating with respect to λ yields $\frac{d\ell}{d\lambda} = \frac{m}{\lambda} - (e^{y_m} - 1)$. Setting this expression equal to zero and solving, the MLE of λ is

$$\hat{\lambda}_{\text{MLE}} = \frac{m}{e^{y_m} - 1}.$$

Since $\frac{d^2\ell}{d\lambda^2} = -\frac{m}{\lambda^2}$, the Fisher information is

$$I(\lambda) = -\mathbb{E}\left(\frac{d^2\ell}{d\lambda^2}\right) = \frac{m}{\lambda^2},$$

and the asymptotic variance of the MLE is accordingly $\operatorname{var}(\hat{\lambda}_{\text{MLE}}) \approx \dfrac{\lambda^2}{m}$.

By [3], the density of the mth upper record $X_{U(m)}$ is

$$\begin{aligned} f_m(x) = f_{X_{U(m)}}(x) &= \frac{1}{\Gamma(m)} \big[-\log(1 - F(x \mid \lambda))\big]^{m-1} f(x \mid \lambda) \\ &= \frac{\lambda^m}{\Gamma(m)} e^x (e^x - 1)^{m-1} e^{-\lambda(e^x - 1)}, \quad x > 0,\ m = 1, 2, \ldots. \end{aligned}$$

The moment generating function of $X_{U(m)}$ is

$$m_{X_{U(m)}}(t) = \mathbb{E}\big(e^{tX_{U(m)}}\big) = \frac{\lambda^m}{\Gamma(m)} \int_0^\infty (z+1)^t \, z^{m-1} \, e^{-\lambda z} \, dz,$$

and the kth moment of the mth record value follows as

$$\mu_m^{(k)} = \frac{\lambda^m}{\Gamma(m)} \int_0^\infty \log^k(z+1) \, z^{m-1} \, e^{-\lambda z} \, dz. \tag{4}$$

The mean and variance of $X_{U(m)}$ are obtained as special cases of (4) with $k = 1$ and $k = 2$, respectively. Selected values of $\mu_m^{(1)}$ for representative choices of m and λ are reported in Table 1.

Table 1: Values of $\mu_m^{(1)}$ for $m = 1, \ldots, 4$ and different values of λ, computed numerically via equation (4) using the integration method.

	λ							
m	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
1	1.3409	0.9229	0.7205	0.5963	0.5110	0.4483	0.3999	0.3613
2	2.0057	1.4615	1.1801	1.0000	0.8722	0.7759	0.7001	0.6387
3	2.4226	1.8268	1.5078	1.2982	1.1465	1.0302	0.9374	0.8613
4	2.7212	2.0993	1.7592	1.5321	1.3656	1.2363	1.1323	1.0462

2.3 Confidence Interval for λ

Observe that $S(\lambda, \mathbf{Y})$ admits the representation $S(\lambda, \mathbf{Y}) = \sum_{i=1}^{m} V_i$, where $V_1, \ldots, V_m$ are independent exponential random variables with respective rates $\lambda_i = \frac{1}{m-i+1}$ (see [15] for details). The density of the pivotal quantity $S(\lambda, \mathbf{Y})$ is given by [15, p. 310]

$$f_{S(\lambda,\mathbf{Y})}(t) = \sum_{i=1}^{m} \left(\prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{\lambda_j}{\lambda_j - \lambda_i} \right) \lambda_i \, e^{-\lambda_i t}, \qquad t > 0,$$

and the corresponding survival function is

$$P(S(\lambda, \mathbf{Y}) > t) = \sum_{i=1}^{m} \left(\prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{\lambda_j}{\lambda_j - \lambda_i} \right) e^{-\lambda_i t}.$$

A $100(1-\gamma)\%$ confidence interval for λ is then constructed from the pivotal relationship

$$P\left(s_{\gamma/2} < \lambda \sum_{i=1}^{m} (e^{Y_i} - 1) < s_{1-\gamma/2} \right) = 1 - \gamma,$$

where $s_{\gamma/2}$ and $s_{1-\gamma/2}$ are the lower and upper $\gamma/2$ percentage points of $S(\lambda, \mathbf{Y})$, respectively. Rearranging the inequalities with respect to λ yields the explicit interval

$$P\left(\frac{s_{\gamma/2}}{\sum_{i=1}^{m} (e^{Y_i} - 1)} < \lambda < \frac{s_{1-\gamma/2}}{\sum_{i=1}^{m} (e^{Y_i} - 1)} \right) = 1 - \gamma.$$

Selected 90th and 95th percentage points of $S(\lambda, \mathbf{Y})$ for various values of m, computed via simulation and numerical integration, are reported in Table 2.

Table 2: 90th and 95th percentage points of the pivotal quantity S.

m	2	3	4	5	6	7
2.5%	0.3515	1.1308	2.6638	4.5027	6.8977	10.3838
5%	0.4903	1.4717	3.2200	5.3762	8.4894	12.1340
95%	7.1015	13.1761	20.4572	29.4937	39.1709	50.2215
97.5%	8.4626	15.0916	23.6537	33.3506	44.2719	55.8134
m	8	9	10	15	25	35
2.5%	14.2135	18.9979	24.2127	62.7115	199.7386	403.4039
5%	16.6834	21.8511	27.9026	70.1711	217.1680	430.2990
95%	62.1805	76.3059	90.7467	183.5682	447.7010	710.9923
97.5%	69.1709	84.6695	99.0043	198.3151	473.4896	734.7813

2.4 Prediction of Future Records

Let $\mathbf{Y} = (Y_1, \ldots, Y_m)$ denote the first m observed upper records, and let $V = Y_n$ $(n > m)$ be the future record to be predicted on the basis of $\mathbf{Y}$. The conditional density of V given $Y_m = y_m$ is [3]

$$\begin{aligned} f_{V|Y_m}(v) &= \frac{\big[H(v) - H(y_m)\big]^{n-m-1}}{\Gamma(n-m)} \frac{f(v \mid \lambda)}{1 - F(y_m \mid \lambda)} \\ &= \frac{\big[\lambda(e^v - e^{y_m})\big]^{n-m-1}}{\Gamma(n-m)} \lambda\, e^{v - \lambda(e^v - e^{y_m})}, \qquad v > y_m, \end{aligned} \tag{5}$$

where $H(\cdot) = -\ln(1 - F(\cdot))$ denotes the cumulative hazard function. It follows directly from (5) that

$$\lambda\,(e^V - e^{Y_m}) \mid Y_m = y_m \;\sim\; \Gamma(n-m,\,1).$$

Let $\phi(z)$ denote the density of $\lambda(e^V - e^{Y_m}) \mid Y_m = y_m$. An interval $[a, b]$ is called a highest conditional density (HCD) prediction interval (PI) of content $1-\gamma$ $(0 < \gamma < 1)$ if

$$[a, b] = \big\{z : z \in [0, \infty),\ \phi(z) \geq j\big\}, \qquad \int_a^b \phi(z)\, dz = 1 - \gamma,$$

for some $j \geq 0$. Two cases arise depending on whether $n - m$ exceeds unity.

Case 1 $(n > m+1)$: The density ϕ is unimodal and attains its maximum at $n-m-1 \in (0, \infty)$. The HCD method requires finding a and b as the $100\gamma_1$th and $100(1-\gamma+\gamma_1)$th percentiles of the $\Gamma(n-m, 1)$ distribution, respectively, subject to $\int_a^b \phi(z)\, dz = 1 - \gamma$ and $\phi(a) = \phi(b)$. The lower and upper prediction limits L and U then satisfy

$$e^{\lambda(e^L - e^{y_m})} = e^a, \qquad e^{\lambda(e^U - e^{y_m})} = e^b,$$

yielding

$$L = \ln\left(e^{y_m} + \frac{a}{\lambda}\right), \qquad U = \ln\left(e^{y_m} + \frac{b}{\lambda}\right).$$

CASE 2 ($n = m+1$): The density ϕ is strictly decreasing on $[0, \infty)$ with $\phi(0) = 1$ and $\phi(\infty) = 0$, so the HCD interval reduces to the one-sided form $[0, U]$. Since $\lambda(e^V - e^{Y_m}) \mid Y_m = y_m$ follows an exponential distribution with unit rate, the upper prediction limit is

$$U = \ln\left(e^{y_m} - \frac{\log \gamma}{\lambda}\right).$$

3 Bayesian Analysis

The $Gamma(a, b)$ prior is chosen as the natural conjugate prior for the exponential family likelihood (2). Then,

$$\lambda|\mathbf{Y} \sim Gamma\left(m + a, e^{y_m} + b - 1\right).$$

Now, the Bayes estimate (BE) of λ under squared error loss (SEL) function is the posterior mean [2]

$$\hat{\lambda}_{BS} = \frac{m+a}{e^{y_m} + b - 1}.$$

The BE of λ under LINEX loss function $L(\hat{\eta}, \eta) = e^{c(\hat{\eta}-\eta)} - c(\hat{\eta} - \eta) - 1$ for $c \neq 0$ is

$$\hat{\lambda}_{BL} = -\frac{1}{c} \ln\left(\mathbb{E}_{\lambda|\mathbf{Y}}(e^{-c\lambda})\right) = -\frac{m+a}{c} \ln\left(\frac{e^{y_m} + b - 1}{e^{y_m} + (b+c) - 1}\right).$$

The size and sign of c represent the degree and orientation of asymmetry, respectively. When $c > 0$ overestimation is more costly than underestimation and vice versa. As c approaching to zero, the LINEX loss function behaves approximately like the symmetric SE loss function [19]. Based on the above values of λ, a $100(1-\alpha)\%$ highest posterior density (HPD) interval, as proposed by Chen and Shao [8], can be constructed as follows. First, generate T samples of λ from the Gamma$\left(m + a, e^{y_m} + b - 1\right)$, and sort them in ascending order as $\lambda_{(1)}, \ldots, \lambda_{(T)}$. Next, form all possible $100(1-\alpha)\%$ credible intervals for λ as

$$(\lambda_{(1)}, \lambda_{([T(1-\alpha)])}), \ldots, (\lambda_{([T\alpha])}, \lambda_{(T)}),$$

where $[T]$ denotes the greatest integer less than or equal to T. The HPD credible interval is identified as the one with the smallest length among these intervals.

Assume $n > m$. The posterior predictive density (PPD) of V given $\mathbf{y}$ is

$$p(v|\mathbf{y}) = \int_0^\infty f(v|y_m, \lambda)\pi(\lambda|\mathbf{y})d\lambda.$$

From (5), the Bayes predictive density function is

$$
\begin{aligned}
p(v|\mathbf{y}) &= \frac{e^{v}(e^{v}-e^{y_m})^{n-m-1}}{\Gamma(n-m)\Gamma(m+a)}\int_0^\infty \lambda^{n+a-1}e^{-\lambda(e^{v}+b-1)}d\lambda \\
&= \frac{e^{v}(e^{v}-e^{y_m})^{n-m-1}}{B(n-m,m+a)(e^{v}+b-1)^{n+a}},
\end{aligned}
$$

where $B(\cdot,\cdot)$ is the beta function. It is obvious that the PPD is not the well-known distribution and the computation of the $\mathbb{E}(V|\mathbf{y})$ is not an easy task. Following Asgharzadeh et al. [4], the MC samples are applied to generate samples from the predictive distributions. The Bayes predictor of V, under the SEL function, is given by

$$
\begin{aligned}
\hat{V}_{BS} &= \mathbb{E}_{Posterior}(V|\mathbf{y}) \\
&= \int_{y_m}^{\infty} vp(v|\mathbf{y})dv \\
&= \frac{1}{B(n-m,m+a)}\int_0^\infty \frac{\ln(t+e^{y_m})t^{n-m-1}}{(e^{y_m}+t+b-1)^{n+a}}dt,
\end{aligned}
$$

and under the LINEX function,

$$
\begin{aligned}
\hat{V}_{BL} &= -\frac{1}{c}\ln(\mathbb{E}_{Posterior}(e^{-cV}|\mathbf{y})) \\
&= -\frac{1}{c}\ln\left(\int_{y_m}^{\infty} e^{-cv}p(v|\mathbf{y})dv\right) \\
&= -\frac{1}{c}\ln\left(\frac{1}{B(n-m,m+a)}\int_0^\infty \frac{t^{n-m-1}}{(e^{y_m}+t)^{c}(e^{y_m}+t+b-1)^{n+a}}dt\right).
\end{aligned}
$$

Based on MC samples $\{\lambda_j : j = 1, 2, \ldots, N\}$,

$$
\hat{p}(v|\mathbf{y}) = \frac{1}{N}\sum_{j=1}^{N} f(v|y_m, \lambda_j),
$$

where $\hat{p}(v|\mathbf{y})$ is the simulation consistent estimator of $p(v|\mathbf{y})$. Therefore,

$$
\begin{aligned}
\hat{V}_{BS} &= \frac{1}{N}\sum_{j=1}^{N}\frac{\lambda_j^{n-m}e^{\lambda_j e^{y_m}}}{\Gamma(n-m)}\int_{y_m}^{\infty} v(e^{v}-e^{y_m})^{n-m-1}e^{v-\lambda_j e^{v}}dv \\
&= \frac{1}{N}\sum_{j=1}^{N}\frac{\lambda_j^{n-m}e^{\lambda_j e^{y_m}}}{\Gamma(n-m)}\int_0^\infty \ln(e^{y_m}+u)u^{n-m-1}e^{-\lambda_j(e^{y_m}+u)}du
\end{aligned}
$$

and

$$
\hat{V}_{BL} = -\frac{1}{c}\ln\left(\frac{1}{N}\sum_{j=1}^{N}\frac{\lambda_j^{n-m}e^{\lambda_j e^{y_m}}}{\Gamma(n-m)}\int_{y_m}^{\infty} e^{-cv}(e^{v}-e^{y_m})^{n-m-1}e^{v-\lambda_j e^{v}}dv\right)
$$

$$= -\frac{1}{c}\ln\left(\frac{1}{N}\sum_{j=1}^{N}\frac{\lambda_j^{n-m}e^{\lambda_j e^{y_m}}}{\Gamma(n-m)}\int_0^{\infty}(e^{y_m}+u)^{-c}u^{n-m-1}e^{-\lambda_j(e^{y_m}+u)}du\right).$$

With the above results, we can present a two-sided Bayesian prediction interval for $V = Y_n$ where $n = m+1, m+2, \ldots$. To do this, we need to obtain the simultaneous numerical solutions $L(\mathbf{Y})$ and $U(\mathbf{Y})$ through the following equations for the lower and upper bounds:

$$P(V > L(\mathbf{y})|\mathbf{y}) = \int_{L(\mathbf{y})}^{\infty} p(v|\mathbf{y})dv = 1 - \frac{\alpha}{2},$$

and

$$P(V > U(\mathbf{y})|\mathbf{y}) = \int_{U(\mathbf{y})}^{\infty} p(v|\mathbf{y})dv = \frac{\alpha}{2}.$$

4 Numerical Study

In this section, following standard practice in Monte Carlo simulation studies, the performance of different estimation and prediction methods based on Monte Carlo simulations are compared. This is done through two measures of biases and mean square errors (MSEs). In this simulation, the model parameter value $\lambda = 2$ is considered. This value is representative. Different values have been considered and same results have been obtained. We use the following two priors to compute Bayesian estimators and predictors:

$$\text{Prior 1} : a = 2, b = 1,$$
$$\text{Prior 2} : a = 4, b = 2.$$

As we see, the means of both priors are stated to be equal. For Prior 1: $\lambda = a/b = 2 = 2/1$. For Prior 2: $\lambda = 2 = 4/2$. So, Prior 2 provides more information than prior 1 because the variance of prior 2 is smaller than that of prior 1, even though the two priors have the same means. Table 3 shows the average biases and MSEs of estimates of λ over 5000 replications for different record samples. The results of the comparison of pivotal and HPD confidence intervals in terms of average confidence lengths (ACLs) and coverage probabilities (CPs) are reported in Table 4. Table 3 yields lower MSE than MBE. Also, BEs of λ are more efficient under Prior 2. On the other hand, for small values of c, the estimators under the LINEX and SEL loss functions are very close to each other and have lower MSEs under Prior 2 than the other estimators. Table 4 clearly shows that pivotal intervals are wider than the HPD intervals. As m increases, the ALs and all MSEs decrease, while the CPs increase. This is shown in Tables 3 and 4.

To investigate the prediction problem, we randomly generate the upper record sample $Y_1, Y_2, \ldots, Y_m$ from the Basic Gompertz distribution. Next, we compute the Bayesian point and interval predictors for $V = Y_n$ when $n > m$. Table 5 shows the mean square prediction errors (MSPEs) and biases of the BPs in 1000 replications for different choices of m and n. In Table 6, the CPs of the lengths of 95% PIs and mean values are reported. From Table 5, it is clear that BPs under Prior 2 are more efficient. Table 6 shows that Bayesian PIs generally yield longer interval lengths and lower coverage probabilities than HCD PIs. These PIs provide the longest confidence length with the lowest CP. For fixed n, as m increases, the confidence lengths decrease while the CPs increase. Also, for fixed n, the CPs and confidence lengths increase and decrease, respectively as m increases.

Table 3: Biases and MSEs of the MLE and Bayes estimators of λ.

m		MLE	MBE	Prior 1				Prior 2			
				BS	BL			BS	BL		
	c				-1	-0.5	0.25		-1	-0.5	0.25
5	MSE	2.2286	3.0824	0.5353	1.8365	0.9202	0.4295	0.2643	0.6404	0.3954	0.2252
	Bias	0.0538	0.2610	0.4378	0.1168	0.1136	0.4235	0.7322	0.2035	0.5856	0.4948
6	MSE	1.8279	2.5178	0.5427	1.5681	0.8707	0.4457	0.2847	0.6245	0.4081	0.2457
	Bias	0.4341	0.5313	0.2050	0.6467	0.3944	0.1255	0.1249	0.4112	0.2550	0.0673
7	MSE	1.1334	1.4501	0.4537	1.1130	0.6821	0.3824	0.2595	0.5245	0.3584	0.2273
	Bias	0.3253	0.3951	0.1745	0.5332	0.3331	0.1062	0.1117	0.3624	0.2270	0.0601
8	MSE	0.8595	1.1838	0.4013	0.8850	0.5767	0.3443	0.2437	0.4623	0.3270	0.2157
	Bias	0.2699	0.3580	0.1578	0.4631	0.2954	0.0974	0.1055	0.3297	0.2095	0.0584
9	MSE	0.7300	1.0207	0.3779	0.7780	0.5286	0.3267	0.2386	0.4357	0.3164	0.2116
	Bias	0.2680	0.3238	0.1718	0.4452	0.2966	0.1164	0.1224	0.3296	0.2191	0.0782
10	MSE	0.7238	0.9235	0.3845	0.7434	0.5220	0.3066	0.2286	0.4313	0.3112	0.2027
	Bias	0.2459	0.3048	0.1613	0.4065	0.2741	0.1107	0.1163	0.3063	0.2055	0.0753
15	MSE	0.3945	0.5344	0.2709	0.4218	0.3334	0.2472	0.2004	0.2974	0.2410	0.1850
	Bias	0.1519	0.1976	0.1148	0.2686	0.1878	0.0808	0.0905	0.2212	0.1530	0.0611
25	MSE	0.2484	0.3113	0.2109	0.2252	0.1936	0.1586	0.1834	0.1850	0.1603	0.1330
	Bias	0.0927	0.1056	0.0733	0.1901	0.1440	0.0796	0.0652	0.1699	0.1280	0.0691

5 Data Analysis

Consider a dataset of failure times for a truncated airplane windshield, presented by Tahir, et al. [17]. Airplane windshields are made up of multiple layers of material, all of which are laminated under high temperature and pressure. This is therefore a complex item. The type of failure in this item is not structural. It should be noted that the failure of this item will not result

Table 4: Average lengths (ALs) and coverage probabilities (CPs) of 95% confidence intervals of λ.

	Pivotal Method		HPD Method			
			Prior 1		Prior 2	
m	Length	C.P.	Length	C.P.	Length	C.P.
5	3.9223	0.890	3.0907	0.928	2.6426	0.935
6	3.1687	0.895	2.9059	0.935	2.5242	0.941
7	3.1264	0.896	2.7112	0.940	2.4004	0.943
8	3.0037	0.898	2.5628	0.942	2.2983	0.948
9	2.7964	0.900	2.4676	0.943	2.2288	0.948
10	2.6652	0.900	2.3557	0.947	2.1437	0.950
15	2.0327	0.902	1.9528	0.947	1.8289	0.952
25	1.5317	0.905	1.4710	0.950	1.4702	0.955

in damage to the aircraft, but will result in the windshield being replaced. The values in this dataset are:

$$1.866,\ 2.385,\ 3.443,\ 1.876,\ 2.481,\ 3.467,\ 1.899,\ 2.610,\ 3.478,\ 1.911,\ 2.625,\ 3.578,$$
$$1.912,\ 2.632,\ 3.595,\ 1.070,\ 1.914,\ 2.646,\ 3.699,\ 1.124,\ 1.981,\ 2.661,\ 3.779,\ 1.248,$$
$$2.010,\ 2.688,\ 3.924,\ 1.281,\ 2.038,\ 2.820,\ 3.000,\ 3.000,\ 1.281,\ 2.085,\ 2.890,\ 1.303,$$
$$2.089,\ 2.902,\ 1.432,\ 2.097,\ 2.934,\ 1.480,\ 2.135,\ 2.962,\ 1.505,\ 2.154,\ 2.964,\ 1.506,$$
$$2.190,\ 3.000,\ 1.568,\ 2.194,\ 3.103,\ 1.615,\ 2.223,\ 3.114,\ 1.619,\ 2.224,\ 3.117,\ 1.652,$$
$$2.229,\ 3.166,\ 1.652,\ 2.300,\ 3.344,\ 1.757,\ 2.324,\ 3.376.$$

The Kolmogorov-Smirnov (K-S) test is used to check the validity of using the Basic Gompertz distribution to fit this dataset. The K-S statistic (based on the parameter $\lambda = 0.0756$ and obtained by MLE) is equal to 0.1417 and the corresponding p-value is also equal to 0.1183. This means that it is reasonable to fit the Basic Gompertz distribution to this dataset. As we see, the K-S test uses MLE of λ parameter from the full dataset while MLE of λ from the regards is completely different. So, we explicitly stat that the K-S test should use the parameter estimated from the complete dataset for the goodness-of-fit test.

In Figure 2, the empirical distribution function and the PP plot are plotted. The record extracted from the above data set is as follows:

$$2.385,\ 3.000,\ 3.699,\ 3.779,\ 3.924.$$

It is also possible to fit a distribution directly based on observed record data [5]. For $\lambda = 0.07$, a plot of the five observed records above against their expected values shows that there is

Table 5: MSPEs and Biases of Bayesian predictors for future records.

			Prior 1				Prior 2			
m	n		BS	BL			BS	BL		
		c		-1	-0.5	0.25		-1	-0.5	0.25
5	6	MSPE	0.0235	0.0323	0.0277	0.0227	0.0203	0.0216	0.0209	0.0202
		Bias	0.0582	0.1085	0.0796	0.0514	0.0170	0.0347	0.0253	0.0137
	7	MSPE	0.0452	0.0933	0.0596	0.0412	0.0305	0.0388	0.0329	0.0288
		Bias	0.1395	0.2453	0.1803	0.1227	0.0699	0.1061	0.0868	0.0412
6	7	MSPE	0.0169	0.0286	0.0200	0.0154	0.0127	0.0141	0.0133	0.0126
		Bias	0.0685	0.1184	0.0847	0.0605	0.0279	0.0441	0.0348	0.0237
	8	MSPE	0.0379	0.0677	0.0487	0.0345	0.0270	0.0314	0.0291	0.0263
		Bias	0.1060	0.1907	0.1449	0.0933	0.0421	0.0708	0.0561	0.0367
7	8	MSPE	0.0127	0.0207	0.0148	0.0121	0.0107	0.0109	0.0108	0.0107
		Bias	0.0615	0.1041	0.0781	0.0560	0.0225	0.0362	0.0284	0.0201
	9	MSPE	0.0347	0.0523	0.0472	0.0319	0.0222	0.0268	0.0232	0.0202
		Bias	0.1369	0.1785	0.1720	0.1267	0.0849	0.1032	0.0880	0.0724
8	9	MSPE	0.0123	0.0168	0.0135	0.0120	0.0091	0.0100	0.0097	0.0090
		Bias	0.0425	0.0811	0.0564	0.0371	0.0068	0.0179	0.0122	0.0043
	10	MSPE	0.0230	0.0448	0.0302	0.0204	0.0125	0.0156	0.0142	0.0125
		Bias	0.1125	0.1795	0.1389	0.1026	0.0547	0.0760	0.0650	0.0505
9	10	MSPE	0.0099	0.0143	0.0119	0.0096	0.0074	0.0082	0.0077	0.0075
		Bias	0.0554	0.0853	0.0687	0.0515	0.0229	0.0322	0.0278	0.0210
	11	MSPE	0.0278	0.0493	0.0360	0.0262	0.0159	0.0196	0.0179	0.0152
		Bias	0.1314	0.1915	0.1572	0.1250	0.0792	0.0977	0.0873	0.0751
10	11	MSPE	0.0082	0.0121	0.0094	0.0077	0.0061	0.0066	0.0062	0.0060
		Bias	0.0478	0.0762	0.0584	0.0430	0.0157	0.0244	0.0201	0.0139
	12	MSPE	0.0188	0.0328	0.0237	0.0171	0.0113	0.0133	0.0123	0.0108
		Bias	0.0954	0.1481	0.1157	0.0871	0.0451	0.0612	0.0524	0.0412

Table 6: Average lengths (ALs) and coverage probabilities (CPs) of prediction intervals (PIs) for future records.

		Bayesian Method				HPD Method			
		Prior 1		Prior 2		Prior 1		Prior 2	
m	n	Length	C.P.	Length	C.P.	Length	C.P.	Length	C.P.
5	6	1.6027	0.897	1.3045	0.899	1.4736	0.900	1.2632	0.903
	7	2.0424	0.890	1.5715	0.893	1.9505	0.895	1.5114	0.897
6	7	1.5908	0.908	1.2995	0.910	1.4590	0.912	1.2565	0.917
	8	1.8952	0.903	1.4992	0.907	1.7663	0.907	1.4472	0.912
7	8	1.5529	0.913	1.2749	0.917	1.4298	0.920	1.2370	0.924
	9	1.8572	0.910	1.4794	0.915	1.7204	0.913	1.4193	0.920
8	9	1.4957	0.920	1.2468	0.923	1.3873	0.928	1.2129	0.935
	10	1.8467	0.917	1.4408	0.920	1.6764	0.922	1.3841	0.931
9	10	1.4548	0.934	1.2210	0.937	1.3522	0.942	1.1913	0.945
	11	1.7876	0.929	1.4106	0.935	1.6243	0.930	1.3545	0.940
10	11	1.4450	0.940	1.2011	0.945	1.3447	0.949	1.1871	0.953
	12	1.7418	0.931	1.3852	0.937	1.5823	0.940	1.3324	0.945

Table 7: Point estimators and 95% HPD interval for λ.

			BL Loss Function			
MLE	**MBE**	**BS**	$c=-1$	$c=-0.5$	$c=0.25$	**HPD Interval**
0.1008	0.0933	0.1007	0.1018	0.1013	0.1005	$(0.0267,\ 0.1952)$

Table 8: Point predictors and 95% prediction intervals (PIs) of Y_6 and Y_7 based on Bayesian and HPD methods.

Variable	**BS**	BL Loss Function			**HPD Interval**	**Bayesian Interval**
		$c=-1$	$c=-0.5$	$c=0.25$		
Y_6	0.8887	0.8983	0.8906	0.8849	$(0.6174,\ 1.0068)$	$(0.6237,\ 1.0425)$
Y_7	0.9885	1.0040	1.0029	0.9858	$(0.6561,\ 1.1704)$	$(0.6700,\ 1.2137)$

a very strong correlation and therefore it is quite reasonable to assume that these record values come from the BG distribution (correlation coefficient as high a 0.9844). Based on the above upper records, we obtained all estimators as well as the HPD interval for λ. We have listed these results in Table 7 assuming that for calculating BEs and HPD intervals, the prior λ is is improper $(a = b = 0)$ because we have no prior information. The LINEX loss function has been used under different values of c $(c = -1,\ -0.5,\ 0.25)$. Finally, we computed the Bayesian point

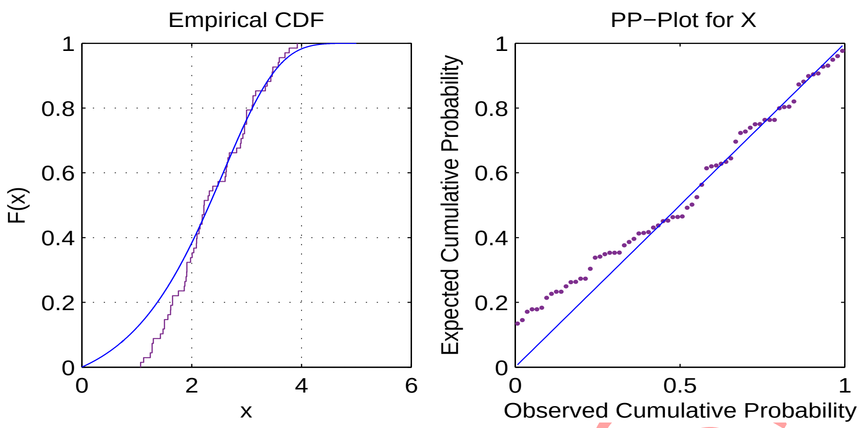

Figure 2: Empirical distribution function and PP-plot for the real data set.

predictors and 95% Bayesian prediction intervals (BPIs) of Y_6 and Y_7 based on the above five observed records. The results are presented in Table 8.

6 Conclusion

This paper examined parameter estimation and prediction of future record values for the Basic Gompertz distribution $G(\lambda)$ under both frequentist and Bayesian frameworks. A moment-based estimator and the MLE of λ were derived in closed form, and exact confidence intervals were constructed via the pivotal statistic $S(\lambda, \mathbf{Y})$. Prediction intervals for future records were obtained through the highest conditional density method. Under a gamma conjugate prior, Bayesian point estimators were derived in closed form under squared error and LINEX loss functions, and HPD credible and prediction intervals were computed by Monte Carlo sampling. Simulation results confirmed that Bayesian estimators, especially under Prior 2, outperform their frequentist counterparts in MSE and bias, a finding supported by the real data analysis.

Limitations. The framework is restricted to complete upper record sequences from the one-parameter BGD; censored or partial records, sensitivity to hyperparameter choice, and robustness to model misspecification were not addressed.

Future Research. Natural extensions include inference for the two-parameter Gompertz model, progressively censored record data, non-conjugate Bayesian priors, and the stress-strength reliability $R = P(X < Y)$ based on BGD records.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published article.

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Conflict of Interest

The author declares no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Author Contributions

Ramin Kazemi: Conceptualization, Methodology, Software, Formal Analysis, Investigation, Writing – Original Draft, Supervision, Writing – Review and Editing. **Akram Kohansal**: Conceptualization, Methodology, Validation, Resources, Writing – Original Draft, Writing – Review and Editing.

Artificial Intelligence Statement

AI tools, including large language models, were used solely for language editing and improving readability. They were not used for generating ideas, performing analyses, interpreting results, or writing scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

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