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Optimization-Based Bayesian and Robust Divergence Estimation for Continuous Parametric Models: Applications to Lifetime and Proportion Models

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Abstract. Parametric modeling of continuous data plays a fundamental role in reliability analysis, survival studies, and proportion modeling. This paper develops a unified optimization-based Bayesian and robust divergence estimation framework for continuous parametric models under both censored and complete data settings. The proposed approach integrates hierarchical Bayesian and E-Bayesian inference with divergence-based robust estimation—specifically the L2E method—to enhance efficiency and stability against model misspecification and potential data contamination. For lifetime data, the framework is applied to the two-parameter Weibull distribution under progressive Type-II censoring schemes; for proportion data, it is applied to the Beta distribution under complete sampling. For the Beta distribution, the proposed E-Bayesian methodology is investigated under the simplified setting in which the second shape parameter is assumed known and fixed ($\beta = 1$). This assumption is adopted to illustrate the proposed optimization-based inferential framework while keeping the methodological development analytically tractable. Bayesian and E-Bayesian estimators are derived under both symmetric and asymmetric loss functions, including squared error, entropy, and Linex losses. The divergence-based estimator is incorporated as a robust alternative within the same optimization structure. Extensive Monte Carlo simulations are conducted to evaluate estimator performance in terms of absolute bias and mean squared error across various scenarios. Numerical illustrations further demonstrate the practical applicability of the proposed framework. The results indicate that the integrated methodology provides accurate and robust parameter estimation across different data structures, offering a flexible inferential tool for continuous parametric modeling.

Keywords. E-Bayesian estimation, Hierarchical Bayesian estimation, L2E estimation, Progressive Type-II censoring, Monte Carlo simulation.

MSC. 62F15; 62N01; 62F35.

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1 Introduction

Parametric modeling of continuous data plays a fundamental role in statistical inference across a wide range of scientific disciplines. In reliability engineering and survival analysis, lifetime distributions are commonly employed to describe failure mechanisms, while in proportion and bounded data analysis, flexible continuous distributions are required to model observations restricted to the unit interval. Among these, the Weibull distribution has been extensively used in lifetime modeling due to its flexibility in representing various hazard rate behaviors, whereas the Beta distribution serves as a natural model for proportion and rate data.

The Weibull distribution is widely applied across diverse scientific fields, including survival analysis, reliability engineering, and weather forecasting. For further details and additional applications, the reader is referred to Han and Shapiro [18] and Johnson et al. [26]. A variety of estimation methods have been proposed for determining its parameters. Balakrishnan and Kateri [7] initially derived maximum likelihood estimates (MLEs) for the Weibull parameters under both complete and censored datasets, while Bayesian estimates were provided by Canavos and Taokas [12] and Guure et al. [17]. Makhdoom and Nasiri [29] obtained the MLE of the exponential distribution under Type-II censoring from imprecise data.

The concept of a hierarchical Bayesian prior was first proposed by Lindley and Smith [28] and further developed by Han [21], who introduced both E-Bayesian and hierarchical Bayesian estimation techniques. These methods have since been applied to various distributions, including the exponential distribution and the parameter ratio of the binomial distribution [22], the parameters and reliability function of the Type-12 Burr distribution under progressively Type-II censored samples [25], and the parameters of the Pascal distribution [33]. Yaghoobzade Shahrastani and Makhdoom [34] developed E-Bayesian and hierarchical Bayesian estimators for the reliability parameter of the Weibull distribution. In the present study, we focus on estimating the scale parameter of the Weibull distribution under the assumption that the shape parameter is known, employing E-Bayesian and hierarchical Bayesian methods within a progressively Type-II censoring scheme. The probability density function of the two-parameter Weibull distribution is given by:

$$f(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}}, \quad x>0, \quad \alpha>0, \quad \beta>0. \tag{1}$$

The Weibull distribution is among the most widely used lifetime models, with extensive applications in survival analysis, reliability engineering, and weather forecasting. Its versatility stems from its ability to represent a broad range of failure-time behaviors. Figures 1 and 2 display the probability density function and hazard rate function, respectively,

$$h(x) = \frac{f(x)}{1-F(x)},$$

for several parameter configurations, thereby illustrating this flexibility.

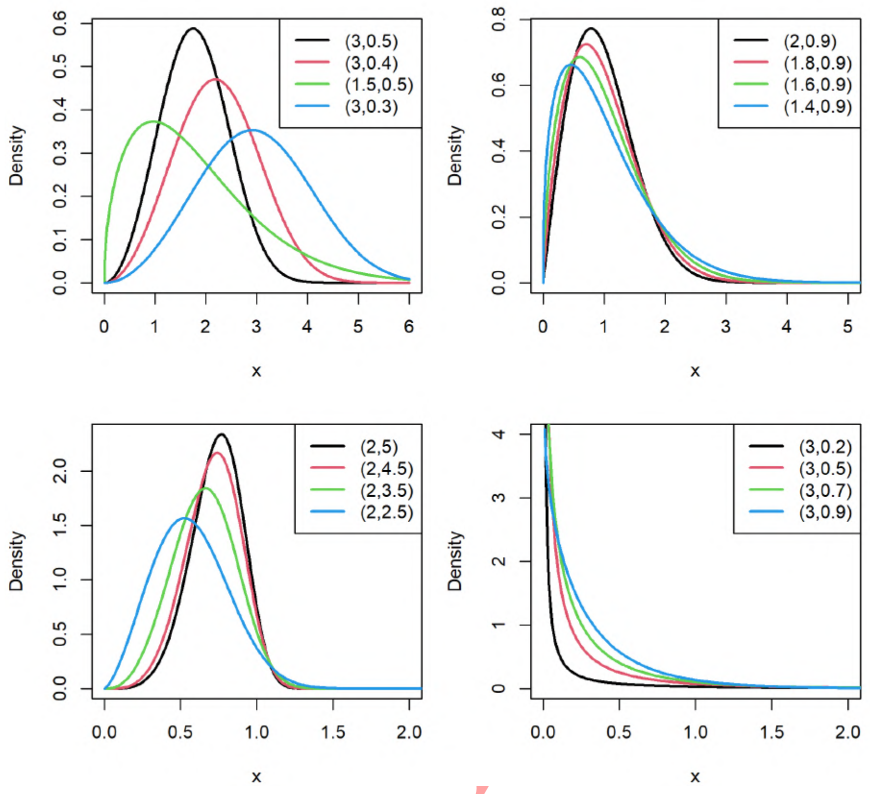

Figure 1: Probability density functions of the two-parameter Weibull distribution for different parameter settings.

Cho et al. [13] obtained the entropy estimator of the Weibull distribution defined by Equation (1) under a generalized hybrid progressive censoring scheme. Censoring is a common feature in lifetime experiments and reliability investigations. In a progressively Type-II censoring framework, suppose that R_1 surviving units are withdrawn at the first failure time, R_2 units at the second, and this procedure continues until the m-th failure occurs. The remaining $R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m$ units are then censored [5]. In this setting, the observed failure times are random variables, whereas the removal quantities R_i are predetermined fixed values. The resulting m ordered failure observations are known as progressively Type-II censored order statistics [3].

Balakrishnan and colleagues documented considerable research on inference methods for progressively Type-II censored data across a series of foundational works [4, 5, 6]. More recently, Kouadria et al. [27] introduced a new two-parameter Weibull–Lindley distribution and studied its properties.

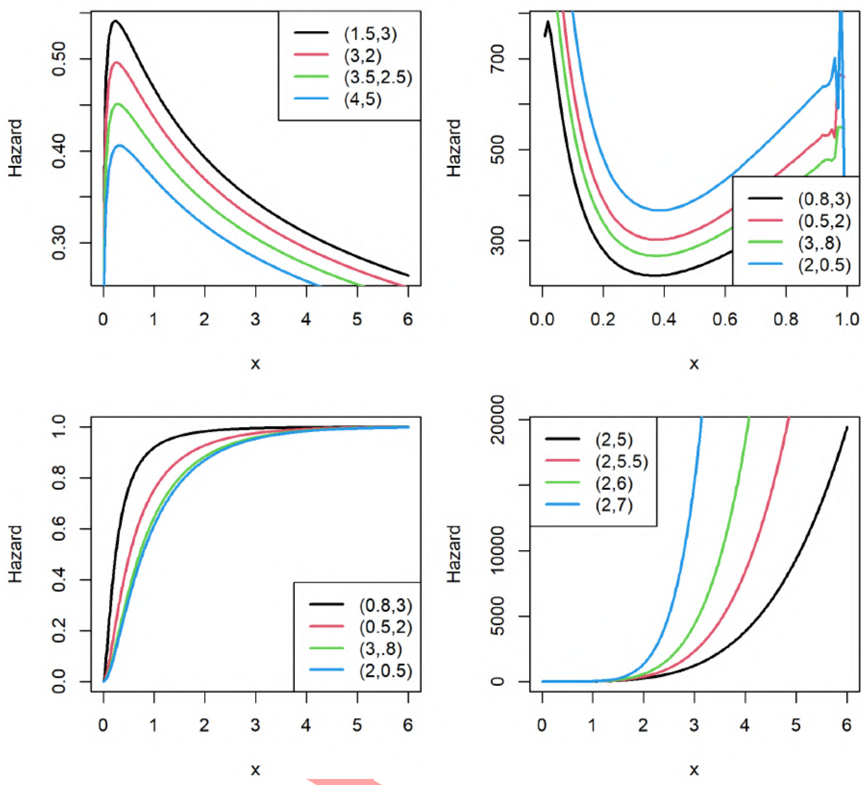

Figure 2: Hazard rate functions of the two-parameter Weibull distribution for different parameter settings.

Classical estimation methods, in particular maximum likelihood estimation (MLE), are widely applied for parameter inference in continuous models. However, MLE is known to lack robustness in the presence of model misspecification or outliers. To address this limitation, divergence-based approaches have been developed. Scott [32] introduced the L2E method, which provides a robust alternative within the general class of M-estimators by minimizing the integrated squared error between the assumed model and the empirical distribution.

Bayesian inference has attracted considerable attention as a flexible framework for incorporating prior knowledge and handling complex data structures, including censored observations. Hierarchical Bayesian estimation, originally proposed by Lindley and Smith [28], and its extension through the E-Bayesian (expectation of Bayesian) approach developed by Han and coauthors [21, 19], have been successfully applied in various inferential problems, especially in reliability contexts involving small samples and censoring schemes. These approaches offer improved stability and efficiency under suitable prior specifications.

Han and Ding [19] proposed the E-Bayesian estimation framework as an alternative inferential methodology. Dan et al. [14] later examined hierarchical Bayesian estimation for failure probability and showed that the E-Bayesian estimator converges asymptotically to its hierarchical Bayesian counterpart. Han [20, 22] applied this approach to failure probability estimation, particularly for censored or truncated data with small sample sizes. Dan et al. [14] further established that the limiting values of both E-Bayesian and hierarchical Bayesian estimators for state probability equal zero. These contributions motivate the estimation strategies considered in the present study. Accordingly, we employ these approaches for parameter estimation in continuous parametric models, including the two-parameter Weibull distribution under progressive Type-II censoring and the Beta distribution under complete sampling.

In recent years, Bayesian analysis of censored lifetime data has attracted increasing attention. Boukeloua et al. [10] developed a robust Bayesian framework based on ϕ-divergence for censored lifetime observations, whereas Elshahhat and Nassar [16] proposed Bayesian inference procedures for improved adaptive progressive Type-II censored data using MCMC techniques. Brito et al. [11] studied Bayesian and classical inference methods for the Very Flexible Weibull distribution under progressive Type-II censoring, highlighting the growing importance of computational Bayesian methods in reliability analysis.

Despite these important developments, existing studies typically focus either on a specific distribution or on a single estimation paradigm. Iqbal et al. [24] investigated E-Bayesian estimation for a hierarchical Poisson–Gamma model under restricted and unrestricted parameter spaces, demonstrating the capability of E-Bayesian methods in hierarchical modeling and uncertainty quantification. Okasha et al. [30] developed E-Bayesian procedures for estimating the inverse Weibull distribution rate parameter under different loss functions, highlighting the importance of Bayesian decision-theoretic approaches in lifetime analysis. Elbatal et al. [15] proposed Bayesian and E-Bayesian reliability analysis for improved adaptive Type-II progressive censored inverted Lindley data, emphasizing the applicability of advanced Bayesian techniques for complex censoring mechanisms. Prakash et al. [31] studied parameter estimation for a reduced Type-I heavy-tailed Weibull distribution under progressive Type-II censoring and demonstrated the effectiveness of censoring-based inference methods for flexible lifetime models. From a robust inference perspective, Anum and Pokojovy [1] introduced a hybrid density power divergence minimization method for robust estimation of location and scale parameters, illustrating the potential of divergence-based approaches to improve estimator stability under model uncertainty and data contamination.

Although these contributions have significantly advanced Bayesian, E-Bayesian, censoring-based, and robust divergence estimation methods, most existing studies have treated these methodologies separately and mainly within a single probability distribution or a specific data structure. As a result, a unified optimization-based framework that simultaneously incorporates Bayesian, hierarchical Bayesian, E-Bayesian, and divergence-based robust estimation procedures for different classes of continuous parametric models remains an important open research problem.

A unified framework that integrates optimization-based Bayesian inference with divergence-based robust estimation and that is applicable to different types of continuous data structures remains relatively unexplored. In particular, the joint treatment of censored lifetime data and complete proportion data within a common methodological structure has not been systematically investigated.

Although several studies have investigated Bayesian inference for the Weibull distribution under progressive Type-II censoring—including the works of Jaheen and Okasha [25] and Guure et al. [17]—their primary objective was to develop inference procedures for a specific probability distribution. To clarify the methodological contribution of the present work, Table 1 summarizes the principal differences between these representative studies and the proposed unified optimization-based framework.

As shown in Table 1, the novelty of the present study does not lie in proposing a new Bayesian estimator for the Weibull distribution. Rather, its main contribution is the development of a unified optimization-based inferential framework that integrates Bayesian, hierarchical Bayesian, E-Bayesian, and robust divergence estimation within a common methodological perspective and demonstrates its applicability to different classes of continuous parametric models.

Motivated by these considerations, this paper develops a unified optimization-based Bayesian and robust divergence estimation framework for continuous parametric models. Unlike previous studies, which have primarily investigated these paradigms separately and for individual probability distributions, the proposed framework provides a common inferential structure that enables their systematic study and comparison across different continuous data models. The proposed methodology combines hierarchical Bayesian and E-Bayesian estimation under various loss functions with the L2E divergence approach within a shared optimization structure. To demonstrate its flexibility and applicability, the framework is implemented for the two-parameter Weibull distribution under progressive Type-II censoring and for the Beta distribution under complete sampling.

The selection of the Weibull and Beta distributions is deliberate and serves to illustrate the generality of the proposed framework. These two models represent fundamentally different classes of continuous data encountered in practice. The Weibull distribution is one of the most widely used lifetime models in reliability and survival analysis and is naturally associated with progressive Type-II censoring schemes. In contrast, the Beta distribution is the standard model for bounded continuous data and proportion-type observations, where censoring is typically absent but robustness against model deviations and outliers is an important concern. By considering these two representative settings, the proposed framework demonstrates that the same optimization-based inferential structure can accommodate substantially different statistical models and data-generating mechanisms without requiring a distribution-specific estimation philosophy. A schematic overview of the proposed framework is presented in Figure 3.

Table 1: Comparison of the proposed framework with representative Bayesian studies for the Weibull distribution under progressive Type-II censoring.

Aspect	**Representative Studies (Jaheen & Okasha [25], Guure et al. [17])**	**Present Study**
Main objective	Develop Bayesian estimation procedures for the Weibull distribution under progressive Type-II censoring.	Develop a unified optimization-based inferential framework for continuous parametric models.
Methodological scope	Distribution-specific Bayesian inference.	Unified inferential framework applicable to different classes of continuous parametric models.
Statistical models	Weibull distribution only.	Both Weibull and Beta distributions.
Data structure	Lifetime data under progressive Type-II censoring.	Lifetime data under progressive Type-II censoring, together with bounded continuous data under complete sampling.
Inferential methods	Bayesian estimation procedures for the Weibull distribution.	Bayesian, hierarchical Bayesian, and E-Bayesian estimation for the Weibull distribution, together with robust L2E estimation for the Beta distribution.
Target of innovation	Development of estimation procedures for a specific probability model.	Development of a unified optimization-based inferential framework integrating different estimation paradigms across continuous parametric models.
Framework generality	Limited to Weibull-specific inference.	Applicable to different classes of continuous parametric models while preserving a common inferential philosophy.

It should be emphasized that, for the Beta distribution, the proposed E-Bayesian methodology is investigated under the simplified setting in which the second shape parameter is assumed known and fixed ($\beta = 1$). This assumption is adopted to provide a transparent illustration of the proposed optimization-based inferential framework while keeping the analytical development tractable. Extending the proposed methodology to the general two-parameter Beta model constitutes an important direction for future research.

The remainder of this paper is organized as follows. Section 2 describes the proposed unified optimization-based estimation framework, which establishes the common inferential foundation for the methods developed throughout this study. Sections 3 and 4 present the E-Bayesian and hierarchical Bayesian estimation procedures for the Weibull model, respectively. Sections 5 and 6 present the L2E estimation approaches for the simple Beta distribution and the Beta mixture model. Section 7 discusses the E-Bayesian estimation method for the Beta distribution. The performance of the proposed methods is examined through extensive Monte Carlo simulation experiments in Section 8, and their practical performance is demonstrated using two real data applications in Section 9. Section 10 provides concluding remarks, highlights the limitations of the proposed framework, and outlines possible directions for future research. The computational algorithm and representative R code are provided in the Appendix.

2 A Unified Optimization-Based Estimation Framework

Before presenting the estimation procedures for the Weibull and Beta distributions, we first describe the general philosophy underlying the proposed methodology. The objective of this study is not to introduce a new estimation algorithm, but rather to provide a unified optimization-based inferential framework that brings together several well-established estimation paradigms within a common methodological perspective.

Although Bayesian estimation, hierarchical Bayesian estimation, E-Bayesian estimation, and the L2E method have been developed independently in prior studies, they all pursue the same statistical objective: the estimation of unknown model parameters. The fundamental difference between these approaches lies in the criterion adopted to obtain the estimator. Bayesian methods determine the estimator by minimizing posterior expected loss under an appropriate loss function; hierarchical Bayesian and E-Bayesian procedures further account for uncertainty in prior specifications through hyper-prior distributions; whereas the L2E method obtains the estimator by minimizing a divergence measure between the assumed probability model and the observed data. Consequently, despite their different inferential philosophies, all these approaches can be interpreted as optimization-based estimation procedures.

The proposed framework is intentionally formulated at a general level and is therefore not restricted to a particular probability distribution. To demonstrate its generality, it is implemented for two representative continuous parametric models with substantially different statistical characteristics. The two-parameter Weibull distribution represents lifetime data arising from reliability and survival studies under progressive Type-II censoring, whereas the Beta distribution represents bounded continuous data and proportion-type observations under complete sampling. These two settings illustrate that the proposed framework is applicable across different data structures while preserving a common inferential philosophy.

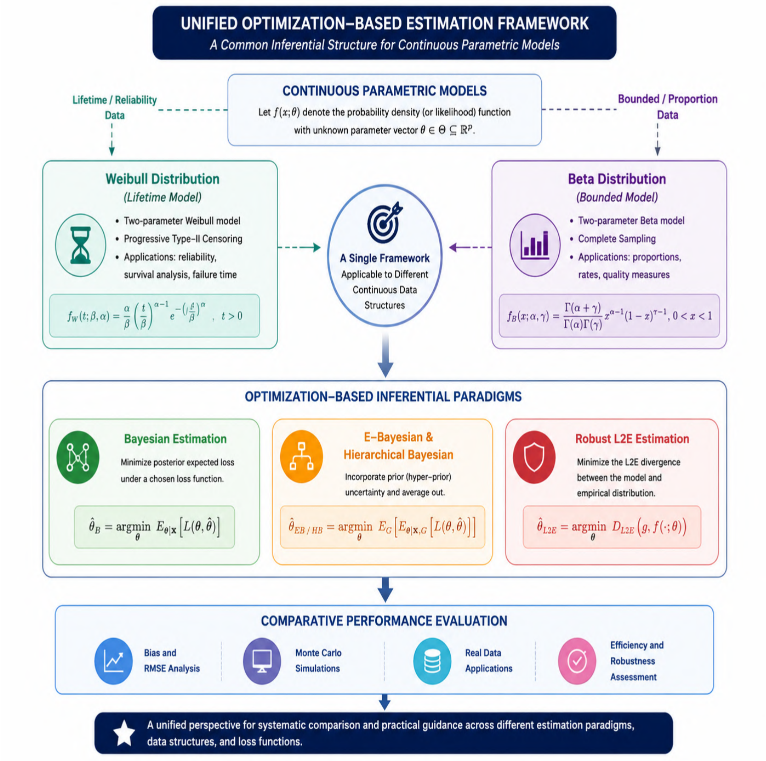

Figure 3: Conceptual illustration of the proposed unified optimization-based inferential framework. Bayesian, hierarchical Bayesian, E-Bayesian, and L2E estimation are integrated within a common optimization perspective and subsequently applied to two representative classes of continuous parametric models, namely lifetime data (Weibull distribution under progressive Type-II censoring) and bounded continuous data (Beta distribution under complete sampling).

As illustrated in Figure 3, rather than introducing a new estimation algorithm, the proposed framework provides a common inferential perspective for several well-established estimation paradigms. Although Bayesian, hierarchical Bayesian, E-Bayesian, and L2E estimation are based on different statistical principles, they all address the common objective of parameter estimation through optimization under different criteria. The framework is subsequently implemented for two representative continuous parametric models to demonstrate its flexibility across different data structures and inferential settings.

The following sections present the implementation of this general framework for each distribution and compare the resulting estimation procedures through simulation studies and real data analyses.

3 E-Bayesian Estimation of the β Parameter

In this section, the E-Bayesian estimators of β are derived. We assume that the prior distribution assigned to β follows a Gamma distribution with hyperparameters a and b:

$$\pi(\beta|a,b) = \frac{b^a}{\Gamma(a)}\beta^{a-1}e^{-b\beta}, \quad \beta > 0, \quad a > 0, \quad b > 0.$$

The first derivative of this prior with respect to β is:

$$\frac{d\pi(\beta|a,b)}{d\beta} = \frac{b^a}{\Gamma(a)}\beta^{a-2}e^{-b\beta}\left((a-1) - b\beta\right).$$

Following [21], the hyperparameters a and b are chosen such that $\pi(\beta|a,b)$ is a decreasing function of β, which requires $0 < a \leq 1$ and $b > 0$. As noted by [9], larger values of b reduce the efficiency of the Bayesian estimator for β, so b should be bounded above, i.e., $0 < b < c$. Furthermore, [23] suggested that a uniform distribution is the most suitable choice for b.

Accordingly, we set $b \sim \text{Unif}(0, c)$ and $a = 1$, which yields the following prior distribution for β:

$$\pi(\beta \mid b) = be^{-b\beta}, \quad \beta > 0, \quad b > 0. \tag{2}$$

Definition 1. Let $\hat{\beta}_B(b)$ denote the Bayesian estimator of β. The E-Bayesian estimator $\hat{\beta}_{EB}$, defined as the expectation of $\hat{\beta}_B(b)$ with respect to the prior distribution of b, is:

$$\hat{\beta}_{EB} = \int_0^c \hat{\beta}_B(b)\pi(b)\,db = E_\pi(\hat{\beta}_B(b)). \tag{3}$$

3.1 E-Bayesian Estimation of β under the Squared Error Loss Function

Under the squared error loss (SEL) function,

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2,$$

we derive the Bayesian estimator and then obtain the corresponding E-Bayesian estimator. Assume that the observed failure times are generated from the Weibull distribution with pdf (1), and consider a progressively Type-II censored sample $Y_{1:m:n}, Y_{2:m:n}, \ldots, Y_{m:m:n}$ with observed values y_i. According to [5], the likelihood function is:

$$L(\beta \mid \boldsymbol{Y}) = C \prod_{i=1}^{m} f(y_i \mid \beta) \left[1 - F(y_i \mid \beta)\right]^{R_i} = C\, \alpha^m \beta^m \left(\prod_{i=1}^{m} y_i\right)^{\alpha-1} e^{-\beta \sum_{i=1}^{m}(1+R_i) y_i^{\alpha}}, \tag{4}$$

where $\boldsymbol{Y} = (Y_{1:m:n}, Y_{2:m:n}, \ldots, Y_{m:m:n})$ and

$$C = n\,(n - R_1 - 1) \cdots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1).$$

Combining Equations (2) and (4), the posterior distribution of β is:

$$\pi_1^*(\beta \mid \boldsymbol{Y}) = \frac{\left(b + \sum_{i=1}^{m}(1+R_i) y_i^{\alpha}\right)^{m+1}}{m!} \beta^m e^{-\beta\left(b + \sum_{i=1}^{m}(1+R_i) y_i^{\alpha}\right)}, \quad \beta > 0. \tag{5}$$

Consequently, the Bayesian estimator of β under the SEL function is:

$$\hat{\beta}_{BS}(b) = E(\beta \mid \boldsymbol{Y}) = (m+1)\left(b + \sum_{i=1}^{m}(1+R_i) y_i^{\alpha}\right)^{-1}. \tag{6}$$

Applying Equations (3) and (5) with the uniform prior $\pi(b) = 1/c$, $0 < b < c$, the E-Bayesian estimator of β is:

$$\hat{\beta}_{EBS} = \frac{1}{c} \int_0^c \hat{\beta}_{BS}(b)\, db = \frac{(m+1)}{c} \left[\log\left(c + \sum_{i=1}^{m}(1+R_i) y_i^{\alpha}\right) - \log\left(\sum_{i=1}^{m}(1+R_i) y_i^{\alpha}\right)\right]. \tag{7}$$

3.2 E-Bayesian Estimation of β under the Entropy Loss Function

The entropy loss function is defined as:

$$L(\hat{\theta}, \theta) \propto \left(\frac{\hat{\theta}}{\theta}\right)^k - k \ln\left(\frac{\hat{\theta}}{\theta}\right) - 1, \quad k \neq 0.$$

Under this loss function, the Bayesian estimator of β is:

$$\hat{\beta}_{BG}(b) = \left(E(\beta^{-k} \mid \boldsymbol{Y})\right)^{-\frac{1}{k}} = \left[\frac{\Gamma(m+1-k)}{m!}\right]^{-\frac{1}{k}} \left(b + \sum_{i=1}^{m}(1+R_i) y_i^{\alpha}\right)^{-1}. \tag{8}$$

The E-Bayesian estimator, obtained from Equations (3) and (8), is:

$$\hat{\beta}_{EBG} = \frac{1}{c} \left[\frac{\Gamma(m+1-k)}{m!}\right]^{-\frac{1}{k}} \left[\log\left(c + \sum_{i=1}^{m}(1+R_i) y_i^{\alpha}\right) - \log\left(\sum_{i=1}^{m}(1+R_i) y_i^{\alpha}\right)\right]. \tag{9}$$

3.3 E-Bayesian Estimation of β under the Linex Loss Function

The Linex loss function is:

$$L(\hat{\theta}, \theta) = e^{h(\hat{\theta}-\theta)} - h(\hat{\theta}-\theta) - 1, \quad h \neq 0.$$

Computing the required posterior expectation:

$$\begin{aligned} E\left(e^{-h\beta} \mid \mathbf{Y}\right) &= \frac{1}{m!} \int_0^\infty \left(b + \sum_{i=1}^m (1+R_i) y_i^\alpha\right)^{m+1} \beta^m \exp\left[-\beta\left(h + b + \sum_{i=1}^m (1+R_i) y_i^\alpha\right)\right] d\beta \\ &= \left(\frac{b + \sum_{i=1}^m (1+R_i) y_i^\alpha}{h + b + \sum_{i=1}^m (1+R_i) y_i^\alpha}\right)^{m+1}. \end{aligned}$$

The Bayesian estimator of β under the Linex loss function is:

$$\hat{\beta}_{BL}(b) = -\frac{1}{h} \log E(e^{-h\beta} \mid \mathbf{Y}) = -\frac{m+1}{h} \log\left(\frac{b + \sum_{i=1}^m (1+R_i) y_i^\alpha}{h + b + \sum_{i=1}^m (1+R_i) y_i^\alpha}\right). \tag{10}$$

The E-Bayesian estimator is obtained by averaging $\hat{\beta}_{BL}(b)$ over the hyperprior distribution:

$$\begin{aligned} \hat{\beta}_{EBL} &= \int \hat{\beta}_{BL}(b) f(b)\, db = \frac{1}{c} \int_0^c \hat{\beta}_{BL}(b)\, db \qquad (11) \\ &= \frac{m+1}{hc}\left[-\int_0^c \log\left(b + \sum_{i=1}^m (1+R_i) y_i^\alpha\right) db + \int_0^c \log\left(h + b + \sum_{i=1}^m (1+R_i) y_i^\alpha\right) db\right]. \end{aligned}$$

Setting $u = \sum_{i=1}^m (1+R_i) y_i^\alpha$ and applying the standard integral $\int \log(x+u)\, dx = (x+u)\log(x+u) - (x+u) + C$, we obtain:

$$\begin{aligned} \int_0^c \log(b+u)\, db &= (c+u)\log(c+u) - c - u \log u, \\ \int_0^c \log(b+h+u)\, db &= (h+c+u)\log(h+c+u) - c - (h+u)\log(h+u). \end{aligned}$$

Substituting into Equation (11) gives:

$$\begin{aligned} \hat{\beta}_{EBL} = \frac{m+1}{hc}\Bigg[&\left(h + c + \sum_{i=1}^m (1+R_i) y_i^\alpha\right) \log\left(\frac{h + c + \sum_{i=1}^m (1+R_i) y_i^\alpha}{c + \sum_{i=1}^m (1+R_i) y_i^\alpha}\right) \\ &- \left(\sum_{i=1}^m (1+R_i) y_i^\alpha\right) \log\left(\frac{h + \sum_{i=1}^m (1+R_i) y_i^\alpha}{\sum_{i=1}^m (1+R_i) y_i^\alpha}\right)\Bigg]. \end{aligned} \tag{12}$$

4 Bayesian Estimation of β under a Hierarchical Framework

This section derives the hierarchical Bayesian estimators for β under the Squared Error, Entropy, and Linex loss functions.

Definition 2. Let λ be a hyperparameter associated with θ. If the prior density of θ is $\pi_2(\theta|\lambda)$ and the prior density of λ is $\pi_3(\lambda)$, the hierarchical prior density of θ is:

$$\pi_4(\theta) = \int_\Lambda \pi_2(\theta|\lambda)\pi_3(\lambda)\, d\lambda, \quad \lambda \in \Lambda. \tag{13}$$

Combining Equations (2) and (13) with a uniform prior for b over $(0, c)$, the hierarchical prior of β is:

$$\pi_4(\beta) = \int_0^c \pi(\beta|b)\pi_1(b)\, db = \frac{1}{c}\int_0^c b\, e^{-b\beta}\, db = \frac{1-(1+\beta c)e^{-c\beta}}{c\beta^2}, \quad \beta > 0.$$

The corresponding hierarchical posterior density of β is:

$$\pi_2^*(\beta \mid \boldsymbol{Y}) = \frac{(T+c)^m T^{m-1}}{(m-1)!\,\{(T+c)^m - [T+c(m+1)]T^{m-1}\}} \times \beta^{m-2} e^{-\beta\left(b+\sum_{i=1}^m (1+R_i)y_i^\alpha\right)} \left[1-(1+c\beta)e^{-c\beta}\right], \tag{14}$$

where $T = b + \sum_{i=1}^m (1+R_i)y_i^\alpha$.

The hierarchical Bayesian estimators under the three loss functions are defined as:

$$\hat\beta_{HBS} = E(\beta \mid \boldsymbol{Y}) = \int_0^\infty \beta\, \pi_2^*(\beta \mid \boldsymbol{Y})\, d\beta,$$

$$\hat\beta_{HBG} = \left[E(\beta^{-k} \mid \boldsymbol{Y})\right]^{-1/k} = \left[\int_0^\infty \beta^{-k}\pi_2^*(\beta \mid \boldsymbol{Y})\, d\beta\right]^{-1/k},$$

$$\hat\beta_{HBL} = -\frac{1}{h}\log E(e^{-h\beta} \mid \boldsymbol{Y}) = -\frac{1}{h}\log\left[\int_0^\infty e^{-h\beta}\pi_2^*(\beta \mid \boldsymbol{Y})\, d\beta\right].$$

Explicit closed-form expressions for these estimators are:

$$\hat\beta_{HBS} = \frac{m\{(T+c)^{m+1} - [T+c(m+2)]T^m\}}{(T+c)\,T\,\{(T+c)^m - [T+c(m+1)]T^{m-1}\}}, \tag{15}$$

$$\hat\beta_{HBG} = \left[\frac{(m-k-1)!\,[T(T+c)]^k\,\{(T+c)^{m-k} - [T+c(m-k+1)]T^{m-k-1}\}}{(m-1)!\,\{(T+c)^m - [T+c(m+1)]T^{m-1}\}}\right]^{-\frac{1}{k}}, \tag{16}$$

$$\hat\beta_{HBL} = -\frac{1}{h}\log f(T, c, h), \tag{17}$$

where

$$f(T,c,h) = \left(\frac{T}{T+c}\right)^{m-1}\left(\frac{T+c}{T+c+h}\right)^{m} \times \frac{(T+c+h)^m - [T+h+c(m+1)](T+h)^{m-1}}{(T+c)^m - [T+c(m+1)]T^{m-1}}. \tag{18}$$

5 L2E Estimation for the Simple Beta Model

The parameter θ is estimated using the L2E approach by minimizing the following objective function [32, 8]:

$$\hat{\boldsymbol{\theta}}_{L2E} = \mathrm{argmin}_{\theta}\left[\int_{-\infty}^{\infty} f(x \mid \theta)^2\,dx - \frac{2}{n}\sum_{i=1}^{n} f(x_i \mid \theta)\right], \tag{19}$$

where $f(x \mid \theta)$ is the probability density function of the Beta distribution with parameter vector $\theta = (\alpha, \beta)$.

Let $X_1, \ldots, X_n$ be a random sample from a Beta distribution with pdf

$$f_X(x \mid \alpha, \beta) = \frac{1}{B(\alpha,\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}, \quad x \in (0,1),\ \alpha > 0,\ \beta > 0.$$

After analytical simplification, Equation (19) reduces to:

$$\hat{\boldsymbol{\theta}}_{L2E} = \mathrm{argmin}_{\theta=(\alpha,\beta)}\left[\frac{B(2\alpha-1, 2\beta-1)}{(B(\alpha,\beta))^2} - \frac{2}{n}\sum_{i=1}^{n} f(x_i \mid \alpha, \beta)\right]. \tag{20}$$

To illustrate the procedure, we generate $n = 1000$ observations from the following Beta mixture distribution:

$$f_X(x \mid w, \alpha_0, \beta_0, \alpha_1, \beta_1) = w\,\mathrm{Beta}(\alpha_0, \beta_0) + (1-w)\,\mathrm{Beta}(\alpha_1, \beta_1),$$

with parameter values $w = 0.25$, $\alpha_0 = 3$, $\beta_0 = 8$, $\alpha_1 = 10$, and $\beta_1 = 4$. A kernel density estimate of the simulated data together with the true mixture density is shown in Figure 4.

The L2E objective function was minimized numerically using the `nlminb` algorithm in the R programming language. All R code is provided in the Appendix.

5.1 Estimation via the L2E Method with One Unknown Parameter

We first obtain the L2E estimate for the parameter of the simple Beta distribution when only α is unknown. Fitting the (misspecified) Beta$(\alpha, 4)$ model via Equation (20) yields $\hat{\alpha} = 8.891$.

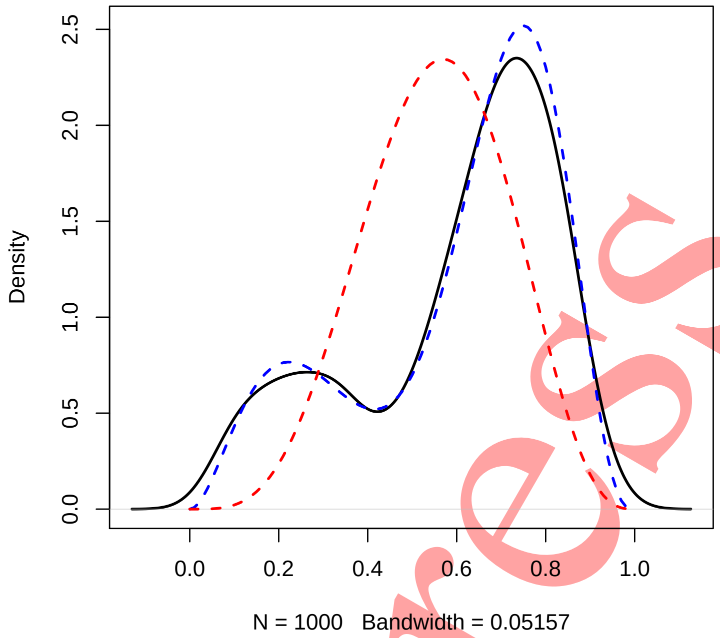

Figure 4: Kernel density estimate of the simulated data (black solid line), true mixture density (blue dotted line), and single MLE Beta fit (red dotted line).

Figure 5 plots the L2E criterion as a function of α; the relatively flat profile near the minimum confirms the robustness of the L2E estimator, which yields a stable solution even when the model is deliberately misspecified.

Figure 6 shows the L2E fit on the mixture data for the one-parameter case. The L2E fit is noticeably closer to the true mixture density than the MLE fit shown in Figure 4.

5.2 Parameter Estimation of the Beta Distribution with Two Unknown Parameters

When both parameters of the Beta distribution are unknown, the L2E surface over (α, β) is minimized jointly. Figure 7 displays this surface; the minimizing values are:

$$\hat{\alpha} = 5.509, \qquad \hat{\beta} = 2.635.$$

Figure 8 compares the two-parameter L2E fit with the one-parameter fit from Section 5.1. The two-parameter fit is closer to the true mixture density, which is expected since allowing β to vary increases the dispersion and provides additional flexibility.

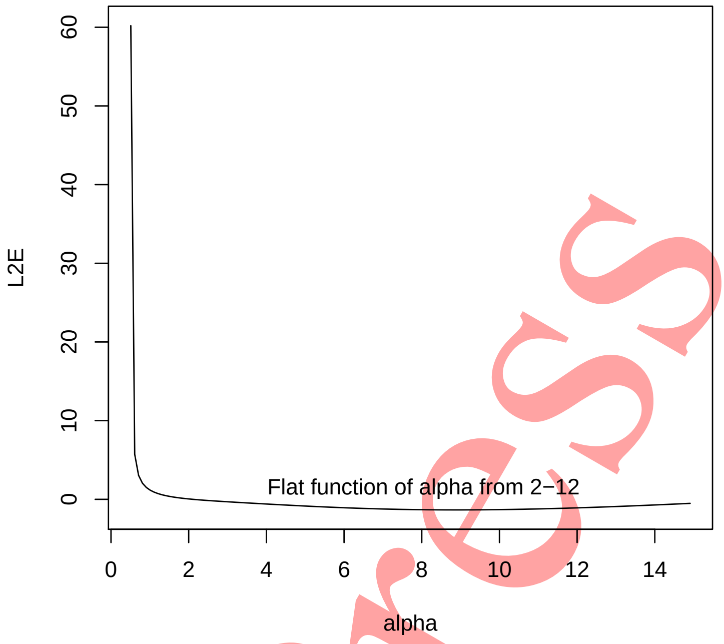

Figure 5: L2E criterion for the mixture-simulated data as a function of the parameter α.

6 L2E Estimation for the Beta Mixture Model

Consider the two-component Beta mixture model:

$$f(x \mid \theta) = w\,\text{Beta}(\alpha_1, \beta_1) + (1-w)\,\text{Beta}(\alpha_2, \beta_2),$$

which involves $q = 5$ unknown parameters. The integral term in the L2E criterion (19) evaluates to:

$$\int_0^1 f(x \mid \theta)^2\,dx = w^2\left[\frac{B(2\alpha_1 - 1, 2\beta_1 - 1)}{B^2(\alpha_1, \beta_1)}\right] + (1-w)^2\left[\frac{B(2\alpha_2 - 1, 2\beta_2 - 1)}{B^2(\alpha_2, \beta_2)}\right]$$
$$+\,2w(1-w)\left[\frac{B(\alpha_1 + \alpha_2 - 1, \beta_1 + \beta_2 - 1)}{B(\alpha_1, \beta_1)\,B(\alpha_2, \beta_2)}\right].$$

Minimizing the full L2E criterion (19) over the five-parameter space yields:

$$\hat{w} = 0.257, \quad \hat{\alpha}_1 = 2.681, \quad \hat{\alpha}_2 = 10.697, \quad \hat{\beta}_1 = 7.349, \quad \hat{\beta}_2 = 4.311.$$

The MLE values agree with these estimates to the number of decimal places shown. Figure 9 displays the five-parameter mixture fit superimposed on the true mixture density and the kernel density estimate.

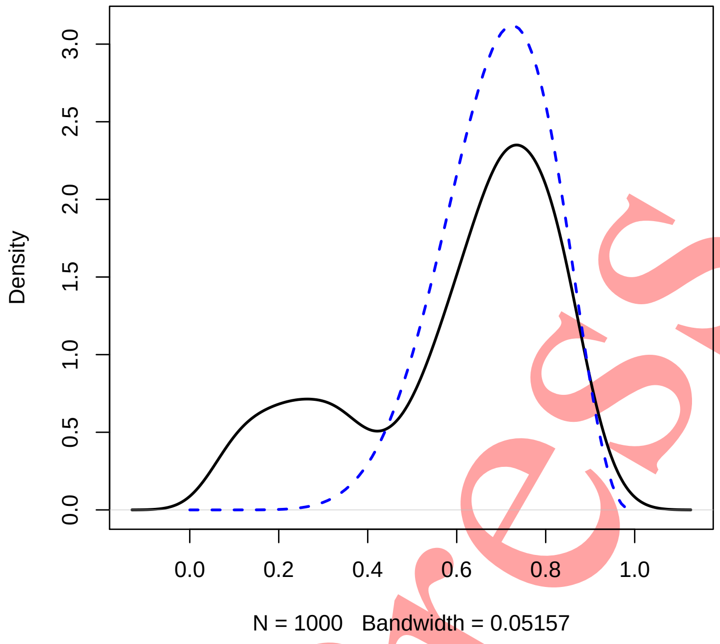

Figure 6: L2E fit (single one-parameter Beta) superimposed on the kernel density estimate of the simulated data.

7 E-Bayesian Estimation for the Beta Distribution

Following [18], [21], and [19], we now derive E-Bayesian estimators for the parameters of the Beta distribution. Let $X_1, \ldots, X_n$ be a random sample from a Beta distribution with parameter vector $\theta = (\alpha, \beta)$, whose joint density is:

$$f(x_1, \ldots, x_n \mid \alpha, \beta) = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \left(\prod_{i=1}^{n}(1-x_i)\right)^{\beta-1}.$$

For tractability, we set $\beta = 1$, so that the likelihood function reduces to:

$$L(x_1, \ldots, x_n \mid \alpha) = \alpha^n\, e^{\alpha \ln\left(\prod_{i=1}^{n} x_i\right)} \left(\prod_{i=1}^{n} x_i\right)^{-1}.$$

With a conjugate Gamma(a, b) prior on α, the posterior distribution of α is proportional to:

$$\Pi(\alpha \mid a, b, x_1, \ldots, x_n) \propto \alpha^{n+a-1}\, e^{-\alpha\left(b - \ln \prod_{i=1}^{n} x_i\right)},$$

which identifies as $\Gamma(n+a,\ b - \ln \prod_{i=1}^{n} x_i)$. The Bayes estimator under squared error loss is therefore:

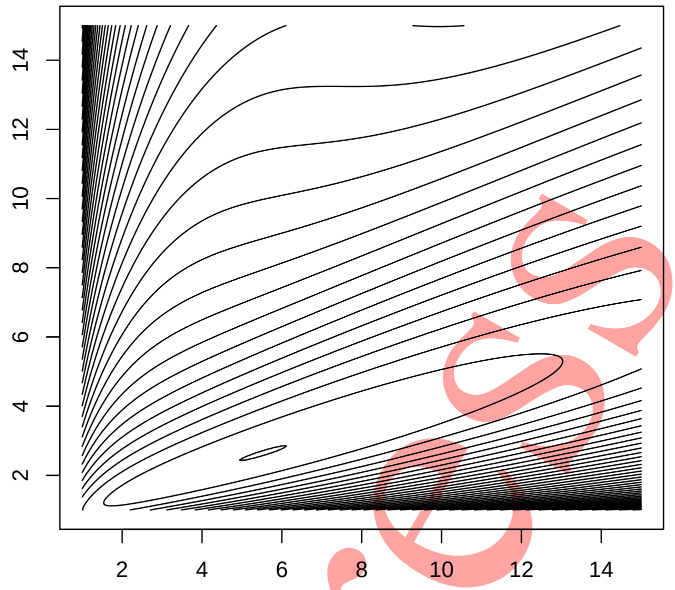

Figure 7: L2E criterion surface for the two-parameter Beta distribution, with minimum at $(\hat{\alpha}, \hat{\beta}) = (5.509,\ 2.635)$.

$$\delta_B = E(\alpha \mid x_1, \dots, x_n, a, b) = \frac{n+a}{b - \ln\prod_{i=1}^{n} x_i}. \tag{21}$$

Following [21], the E-Bayesian estimator of α is:

$$\hat{\alpha}_{EB} = \iint \frac{n+a}{b - \ln\prod_{i=1}^{n} x_i}\, \Pi(a,b)\, da\, db. \tag{22}$$

We consider three prior densities for (a, b) over the region $0 < a < c$, $0 < b < c$:

$$\begin{aligned} \Pi_1(a,b) &= \frac{2(c-b)}{c^2} \cdot \frac{1}{c}, && 0 < b < c,\ 0 < a < c, \\ \Pi_2(a,b) &= \frac{1}{c} \cdot \frac{1}{c}, && 0 < b < c,\ 0 < a < c, \\ \Pi_3(a,b) &= \frac{2b}{c^2} \cdot \frac{1}{c}, && 0 < b < c,\ 0 < a < c, \end{aligned}$$

where $a \sim \text{Unif}(0, c)$ and the priors of b are, respectively, a decreasing, uniform, and increasing function of b. The corresponding E-Bayesian estimators of α are:

$$\hat{\alpha}_{EB1} = \frac{2\left(\frac{c}{2}+n\right)}{c^2} \left[\left(-\ln\prod_{i=1}^{n} x_i + c \right) \ln\left(\frac{-\ln\prod_{i=1}^{n} x_i + c}{-\ln\prod_{i=1}^{n} x_i} \right) - c \right], \tag{23}$$

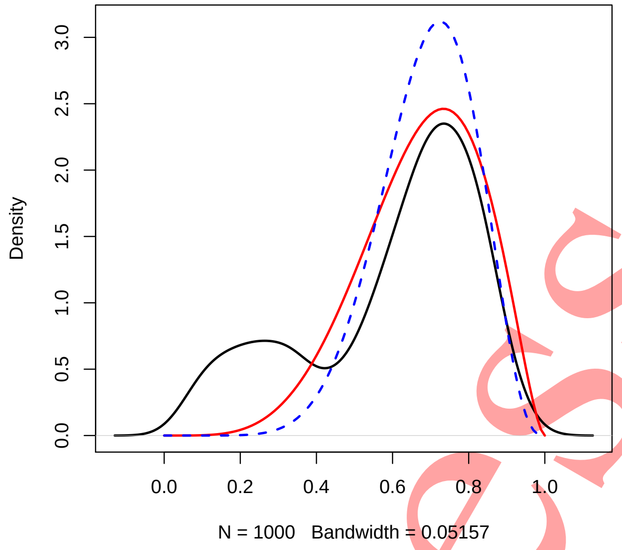

Figure 8: Comparison of the two-parameter L2E fit (red solid line) and the one-parameter L2E fit (blue dotted line).

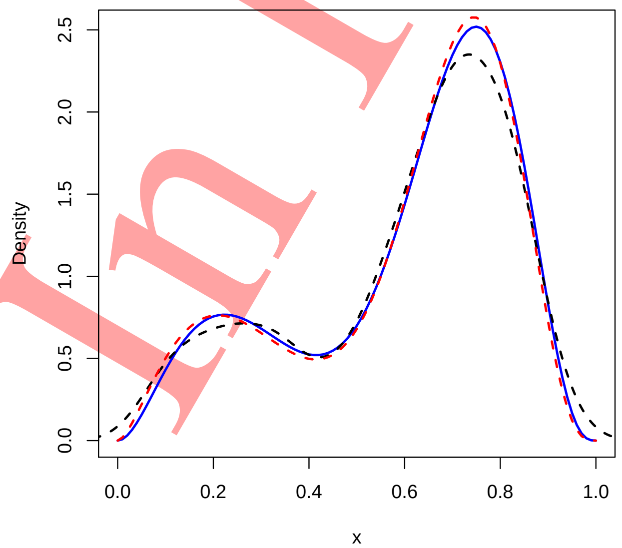

Figure 9: Comparison of the five-parameter L2E mixture fit (red dotted line), the true mixture density (blue solid line), and the kernel density estimate (black dotted line).

$$\hat{\alpha}_{EB2} = \frac{\frac{c}{2}+n}{c}\ln\left(\frac{-\ln\prod_{i=1}^{n} x_i + c}{-\ln\prod_{i=1}^{n} x_i}\right), \tag{24}$$

$$\hat{\alpha}_{EB3} = \frac{2\left(\frac{c}{2}+n\right)}{c^2}\left[c+\ln\prod_{i=1}^{n} x_i \cdot \ln\left(\frac{-\ln\prod_{i=1}^{n} x_i + c}{-\ln\prod_{i=1}^{n} x_i}\right)\right]. \tag{25}$$

7.1 Remarks on the General Two-Parameter Beta Model

The E-Bayesian estimation developed in this section has been illustrated under the simplified setting in which the second shape parameter is assumed known (specifically, $\beta = 1$). This assumption was adopted to obtain explicit analytical expressions for the posterior distribution and the corresponding E-Bayesian estimators, thereby allowing a transparent comparison with the L2E estimation procedure.

It should be emphasized that the proposed optimization-based inferential framework is not inherently restricted to this special case. In principle, the same methodology can be extended to the general Beta(α, β) model by assigning suitable prior and hyper-prior distributions to both shape parameters. However, such an extension leads to a substantially more complicated joint posterior distribution together with higher-dimensional hyperparameter integration, making analytical derivations considerably more involved and computationally demanding.

Since the primary objective of this paper is to demonstrate the proposed unified optimization-based inferential framework across different classes of continuous parametric models, the simplified setting was adopted as an illustrative example. Extending the proposed methodology to the general two-parameter Beta distribution constitutes an interesting direction for future research.

8 Simulation Study

The purpose of the Monte Carlo study is to investigate the finite-sample performance of the competing estimators under the proposed inferential framework. Because no general theoretical result establishes the superiority of Bayesian, E-Bayesian, or hierarchical Bayesian estimators across all statistical models, loss functions, and censoring schemes, numerical experiments are necessary to evaluate their relative performance under the specific settings considered in this study.

The comparative study is intentionally restricted to estimation procedures formulated within a common Bayesian decision-theoretic framework. Since the Bayesian, E-Bayesian, and hierarchical Bayesian estimators are all derived under explicit loss functions, their comparison provides meaningful insight into the effect of different loss structures on estimation performance.

Conventional methods such as Maximum Likelihood Estimation and the Method of Moments were therefore not included, as they are not directly associated with posterior expected loss minimization and pursue a fundamentally different inferential objective.

This section consists of two subsections: the first focuses on the Weibull model under progressive Type-II censoring (Section 8.1), and the second examines the Beta model under complete sampling (Section 8.2).

8.1 Simulation Study 1: Weibull Distribution

To assess and compare the finite-sample performance of the Bayesian, E-Bayesian, and hierarchical Bayesian estimators of β, progressively Type-II censored samples are generated from the Weibull distribution with pdf (1), with true parameter values $\alpha = 2$ and $\beta = 1.5$. Sample generation follows the algorithm of Balakrishnan and Sandhu [4], which proceeds through the four steps described below.

Step 1. Generate m independent uniform random numbers $W_1, W_2, \ldots, W_m \sim \text{Unif}(0,1)$.

Step 2. For each $i = 1, 2, \ldots, m$, compute

$$V_i = W_i^{1 \Big/ \left(i + \sum_{j=m+1-i}^{m} R_j \right)}.$$

Step 3. For each $i = 1, 2, \ldots, m$, set

$$U_{i:m:n} = 1 - V_m V_{m-1} \cdots V_{m+1-i},$$

so that $U_{1:m:n}, \ldots, U_{m:m:n}$ are progressively Type-II censored order statistics from the standard uniform distribution.

Step 4. For each $i = 1, 2, \ldots, m$, apply the quantile transformation

$$y_{i:m:n} = F^{-1}(U_{i:m:n}),$$

where $F^{-1}(\cdot)$ denotes the quantile function of the Weibull distribution with pdf (1). The resulting values $y_{1:m:n}, \ldots, y_{m:m:n}$ constitute progressively Type-II censored order statistics from the target Weibull distribution.

Using the generated samples, all Bayesian, E-Bayesian, and hierarchical Bayesian estimators given in Equations (6)–(17) are evaluated at $b = 0.4$ under two sets of loss-function parameter values: $(k, h) = (0.5, 0.5)$ and $(k, h) = (-1, 2)$. The entire procedure is replicated 1 000

Table 2: Mean estimates with $\sqrt{\text{MSE}}$ and $Mean\ Bias$ under the squared error loss function for $k = h = 0.5$.

n	m	**Design**	$\hat{\beta}_{BS}$	$\hat{\beta}_{EBS}$	$\hat{\beta}_{HBS}$
10	5	$(5,0,0,0,0)$	{1.178} (1.587) 5.198	{0.967} (1.143) 4.793	{0.208} (0.128) 1.791
10	5	$(2,3,0,0,0)$	{2.078} (3.651) 10.21	{0.899} (0.815) 2.818	{0.547} (0.201) 1.452
20	10	$(10,0,\dots,0)$	{2.514} (1.571) 7.723	{1.251} (1.576) 10.17	{0.558} (0.138) 3.441
20	10	$(3,4,3,0,\dots,0)$	{13.40} (3.469) 16.40	{2.415} (0.923) 5.415	{0.261} (0.214) 2.773
50	30	$(20,0,\dots,0)$	{7.828} (1.591) 12.17	{3.869} (2.987) 23.35	{1.172} (0.162) 6.823
50	40	$(1,4,5,0,\dots,0)$	{40.71} (14.71) 41.71	{1.649} (0.752) 2.649	{0.818} (0.351) 1.818
50	25	$(1,1,\dots,1)$	{153.6} (34.84) 154.6	{3.035} (0.835) 4.035	{2.156} (0.499) 3.156

Table 3: Mean estimates with $\sqrt{\text{MSE}}$ and $Mean\ Bias$ under the entropy loss function for $k = h = 0.5$.

n	m	**Design**	$\hat{\beta}_{BG}$	$\hat{\beta}_{EBG}$	$\hat{\beta}_{HBG}$
10	5	$(5,0,0,0,0)$	{1.841} (1.706) 6.553	{1.485} (1.002) 4.199	{1.431} (0.113) 1.568
10	5	$(2,3,0,0,0)$	{1.551} (1.255) 3.504	{0.646} (0.714) 2.468	{0.630} (0.177) 1.269
20	10	$(10,0,\dots,0)$	{5.153} (2.408) 15.06	{1.245} (1.469) 9.486	{6.792} (0.129) 3.207
20	10	$(3,4,3,0,\dots,0)$	{4.193} (1.532) 7.193	{2.047} (0.861) 5.047	{0.415} (0.199) 2.584
50	30	$(20,0,\dots,0)$	{15.47} (4.082) 35.47	{3.270} (2.881) 22.52	{2.742} (0.156) 26.35
50	40	$(1,4,5,0,\dots,0)$	{2.033} (1.023) 3.033	{1.470} (0.701) 2.470	{0.693} (0.328) 1.649
50	25	$(1,1,\dots,1)$	{3.474} (1.049) 4.474	{2.919} (0.811) 3.919	{2.065} (0.484) 3.065

Table 4: Mean estimates with $\sqrt{\text{MSE}}$ and $Mean\ Bias$ under the Linex loss function for $k = h = 0.5$.

n	m	**Design**	$\hat{\beta}_{BL}$	$\hat{\beta}_{EBL}$	$\hat{\beta}_{HBL}$
10	5	$(5,0,0,0,0)$	{3.158} (1.201) 5.753	{2.378} (0.692) 2.621	{1.333} (0.114) 1.666
10	5	$(2,3,0,0,0)$	{1.435} (1.046) 3.405	{0.641} (0.465) 1.452	{0.638} (0.183) 1.361
20	10	$(10,0,\dots,0)$	{2.243} (1.522) 12.06	{1.340} (0.955) 5.659	{1.802} (0.121) 3.197
20	10	$(3,4,3,0,\dots,0)$	{3.581} (1.203) 6.581	{0.483} (0.528) 2.786	{0.389} (0.190) 2.610
50	30	$(20,0,\dots,0)$	{6.961} (2.310) 26.35	{6.871} (1.727) 13.12	{3.672} (0.140) 6.322
50	40	$(1,4,5,0,\dots,0)$	{2.015} (0.939) 3.014	{0.385} (0.382) 1.308	{0.741} (0.326) 1.741
50	25	$(1,1,\dots,1)$	{3.404} (0.985) 4.404	{0.979} (0.409) 1.979	{2.061} (0.469) 3.061

Table 5: Mean estimates with $\sqrt{\text{MSE}}$ and $Mean\ Bias$ under the squared error loss function for $k = -1$ and $h = 2$.

n	m	**Design**	$\hat{\beta}_{BS}$	$\hat{\beta}_{EBS}$	$\hat{\beta}_{HBS}$
10	5	$(5,0,0,0,0)$	{1.147} (1.428) 5.398	{0.910} (0.973) 4.566	{0.227} (0.118) 1.772
10	5	$(2,3,0,0,0)$	{7.993} (3.805) 9.993	{0.974} (0.864) 2.893	{0.532} (0.209) 1.467
20	10	$(10,0,\dots,0)$	{2.463} (1.142) 7.622	{1.102} (1.395) 10.19	{0.551} (0.109) 3.488
20	10	$(3,4,3,0,\dots,0)$	{14.56} (4.440) 17.56	{2.189} (1.029) 5.189	{0.312} (0.257) 2.710
50	30	$(20,0,\dots,0)$	{7.702} (3.387) 12.29	{3.339} (2.632) 22.93	{1.196} (0.142) 6.808
50	40	$(1,4,5,0,\dots,0)$	{46.69} (14.67) 49.64	{1.155} (0.768) 3.155	{1.122} (0.365) 2.172
50	25	$(1,1,\dots,1)$	{155.1} (33.98) 156.1	{2.997} (0.845) 3.997	{2.132} (0.497) 3.133

Table 6: Mean estimates with $\sqrt{\text{MSE}}$ and $Mean\ Bias$ under the entropy loss function for $k = -1$ and $h = 2$.

n	m	**Design**	$\hat{\beta}_{BG}$	$\hat{\beta}_{EBG}$	$\hat{\beta}_{HBG}$
10	5	$(5, 0, 0, 0, 0)$	$\{2.244\}$ (1.719) 7.108	$\{0.910\}$ (0.973) 4.566	$\{0.227\}$ (0.118) 1.772
10	5	$(2, 3, 0, 0, 0)$	$\{2.153\}$ (1.523) 4.136	$\{0.974\}$ (0.864) 2.893	$\{0.532\}$ (0.209) 1.467
20	10	$(10, 0, \ldots, 0)$	$\{6.209\}$ (2.266) 16.21	$\{1.102\}$ (1.395) 10.19	$\{0.550\}$ (0.109) 3.440
20	10	$(3, 4, 3, 0, \ldots, 0)$	$\{4.333\}$ (1.805) 7.333	$\{2.189\}$ (1.029) 5.189	$\{0.312\}$ (0.257) 2.710
50	30	$(20, 0, \ldots, 0)$	$\{16.28\}$ (3.767) 36.28	$\{3.339\}$ (2.632) 22.93	$\{1.145\}$ (0.142) 6.808
50	40	$(1, 4, 5, 0, \ldots, 0)$	$\{2.867\}$ (1.104) 3.867	$\{1.155\}$ (0.768) 3.155	$\{1.172\}$ (0.365) 2.172
50	25	$(1, 1, \ldots, 1)$	$\{3.456\}$ (1.102) 4.559	$\{2.997\}$ (0.845) 3.997	$\{2.132\}$ (0.497) 3.132

Table 7: Mean estimates with $\sqrt{\text{MSE}}$ and $Mean\ Bias$ under the Linex loss function for $k = -1$ and $h = 2$.

n	m	**Design**	$\hat{\beta}_{BL}$	$\hat{\beta}_{EBL}$	$\hat{\beta}_{HBL}$
10	5	$(5, 0, 0, 0, 0)$	$\{1.400\}$ (1.517) 3.599	$\{0.612\}$ (0.657) 5.359	$\{0.609\}$ (0.077) 1.380
10	5	$(2, 3, 0, 0, 0)$	$\{0.648\}$ (0.622) 2.533	$\{1.971\}$ (0.828) 3.971	$\{0.425\}$ (0.145) 1.174
20	10	$(10, 0, \ldots, 0)$	$\{2.479\}$ (0.582) 7.520	$\{1.083\}$ (0.746) 10.99	$\{0.323\}$ (0.067) 2.676
20	10	$(3, 4, 3, 0, \ldots, 0)$	$\{1.605\}$ (0.768) 4.605	$\{4.249\}$ (1.023) 7.249	$\{0.800\}$ (0.173) 2.199
50	30	$(20, 0, \ldots, 0)$	$\{4.340\}$ (1.869) 15.65	$\{2.694\}$ (1.144) 22.67	$\{1.476\}$ (0.086) 5.247
50	40	$(1, 4, 5, 0, \ldots, 0)$	$\{1.977\}$ (2.686) 2.977	$\{1.121\}$ (1.037) 5.120	$\{0.865\}$ (0.276) 1.865
50	25	$(1, 1, \ldots, 1)$	$\{2.883\}$ (0.787) 3.883	$\{1.956\}$ (0.268) 6.956	$\{1.800\}$ (0.396) 2.800

times, and the mean estimate, mean bias, and root mean squared error ($\sqrt{\text{MSE}}$) are recorded for each estimator across all simulation configurations. The results are presented in Tables 2–7.

The results presented in Tables 2–7 show that, across all three loss functions—Squared Error, Entropy, and Linex—the hierarchical Bayesian estimator of the scale parameter of the Weibull distribution consistently outperforms both the Bayesian and E-Bayesian estimators in terms of bias and $\sqrt{\text{MSE}}$. Moreover, the E-Bayesian estimator shows better performance compared to the standard Bayesian estimator in most settings.

8.2 Simulation Study 2: Beta Distribution

To evaluate the accuracy of the E-Bayesian estimators $\hat{\alpha}_{EB1}$, $\hat{\alpha}_{EB2}$, and $\hat{\alpha}_{EB3}$ defined in Equations (23)–(25), we simulate $N = 100\,000$ samples from the Beta$(3, 1)$ distribution with sample sizes $n = 25, 50, 75$, and 100. The results, obtained with β assumed known, are presented in Table 8 and Figure 10. As the sample size increases, both the bias and the standard deviation (SD) of the E-Bayes estimators decrease. Increasing the hyperparameter c tends to increase the bias while reducing the SD. Among the three estimators, $\hat{\alpha}_{EB1}$ achieves the smallest bias but exhibits the largest SD.

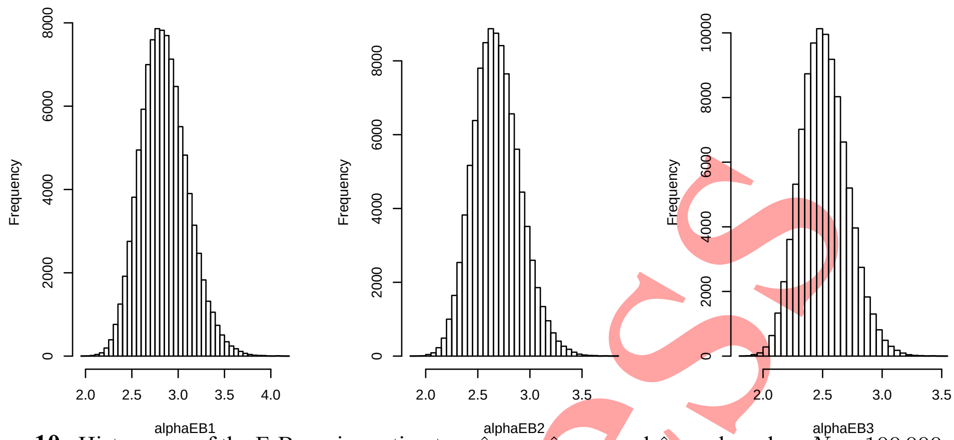

Figure 10: Histograms of the E-Bayesian estimators $\hat{\alpha}_{EB1}$, $\hat{\alpha}_{EB2}$, and $\hat{\alpha}_{EB3}$ based on $N = 100\,000$ simulation replications with $n = 100$ and $c = 15$.

Table 8: Simulation results for the E-Bayes estimators of α. Values in parentheses are standard deviations.

n	**Estimator**	$c = 5$	$c = 10$	$c = 15$
25	$\hat{\alpha}_{EB1}$	2.8734 (0.5028)	2.7415 (0.4289)	2.6653 (0.3840)
	$\hat{\alpha}_{EB2}$	2.6551 (0.4308)	2.4150 (0.3375)	2.2703 (0.2856)
	$\hat{\alpha}_{EB3}$	2.4368 (0.3590)	2.0885 (0.2463)	1.8753 (0.1874)
50	$\hat{\alpha}_{EB1}$	2.9264 (0.3852)	2.8303 (0.3465)	2.7602 (0.3185)
	$\hat{\alpha}_{EB2}$	2.8001 (0.3529)	2.6204 (0.2984)	2.4897 (0.2616)
	$\hat{\alpha}_{EB3}$	2.6738 (0.3207)	2.4105 (0.2503)	2.2193 (0.2047)
75	$\hat{\alpha}_{EB1}$	2.9472 (0.3233)	2.8737 (0.2989)	2.8150 (0.2797)
	$\hat{\alpha}_{EB2}$	2.8584 (0.3042)	2.7186 (0.2681)	2.6079 (0.2413)
	$\hat{\alpha}_{EB3}$	2.7696 (0.2851)	2.5634 (0.2373)	2.4009 (0.2029)
100	$\hat{\alpha}_{EB1}$	2.9582 (0.2855)	2.8990 (0.2683)	2.8493 (0.2540)
	$\hat{\alpha}_{EB2}$	2.8898 (0.2725)	2.7759 (0.2463)	2.6813 (0.2256)
	$\hat{\alpha}_{EB3}$	2.8213 (0.2595)	2.6528 (0.2243)	2.5133 (0.1973)

8.3 Sensitivity Analysis of the E-Bayesian Estimator

To further investigate the robustness of the proposed E-Bayesian estimator, a sensitivity analysis was carried out with respect to the hyperparameters c_0 and c_1. Nine different combinations of (c_0, c_1) were considered while keeping all remaining simulation settings unchanged. For each configuration, the mean estimate, bias, standard deviation (SD), and root mean squared error (RMSE) were computed from Monte Carlo simulations.

The numerical results are reported in Table 9. The proposed estimator exhibits reasonably stable behavior across the considered range of hyperparameter values. Although the choice of (c_0, c_1) slightly affects the bias and RMSE, the overall estimation performance remains satisfactory. Among the combinations investigated, $(c_0, c_1) = (20, 10)$ produces the smallest bias and RMSE, whereas increasing c_1 generally leads to a gradual increase in both quantities. These results indicate that the proposed E-Bayesian estimator is not excessively sensitive to moderate changes in hyperparameter specification, thereby supporting the robustness of the proposed methodology.

Table 9: Sensitivity analysis of the E-Bayesian estimator with respect to the hyperparameters c_0 and c_1.

c_0	c_1	**Mean**	**Bias**	**SD**	**RMSE**
10	10	2.7753	−0.2247	0.2500	0.3361
10	15	2.6223	−0.3777	0.2208	0.4375
10	20	2.4898	−0.5102	0.2007	0.5482
15	10	2.8428	−0.1572	0.2443	0.2905
15	15	2.6817	−0.3183	0.2243	0.3894
15	20	2.5445	−0.4555	0.2069	0.5003
20	10	2.9070	−0.0930	0.2573	0.2735
20	15	2.7456	−0.2544	0.2282	0.3417
20	20	2.6051	−0.3949	0.2133	0.4488

9 Application: Two Real Data Examples

The real data analysis is not merely illustrative. It is included to demonstrate the practical applicability of the proposed unified framework and to examine how different estimation methods behave under actual observed datasets. In particular, the analysis highlights the sensitivity of the estimators to the underlying data structure and provides additional insight beyond the simulation study regarding the comparative performance of the proposed inferential procedures.

We analyze the carbon fiber strength data reported by [2], which measure the strength (in GPa) of single carbon fibers and impregnated 1000-carbon fiber tows. The single fibers were subjected to tensile testing at gauge lengths of 20 mm (Data Set 1) and 10 mm (Data Set 2). The data are displayed in Tables 10 and 11.

Table 10: Data Set 1: Tensile strength of single carbon fibers tested at a gauge length of 20 mm (in GPa).

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585	

Table 11: Data Set 2: Tensile strength of single carbon fibers tested at a gauge length of 10 mm (in GPa).

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397
2.445	2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614
2.616	2.618	2.624	2.659	2.675	2.738	2.740	2.856	2.917
2.928	2.937	2.937	2.977	2.996	3.030	3.125	3.139	3.145
3.220	3.223	3.235	3.243	3.264	3.272	3.294	3.332	3.346
3.377	3.408	3.435	3.493	3.501	3.537	3.554	3.562	3.628
3.852	3.871	3.886	3.971	4.024	4.027	4.225	4.395	5.020

The goodness-of-fit of the Weibull model to both datasets was assessed using the Kolmogorov–Smirnov (K-S) and Anderson–Darling (AD) statistics. The results are summarized in Table 12. The large p-values and small AD statistics confirm that the Weibull distribution provides an adequate fit to both datasets.

Table 12: Goodness-of-fit statistics (Kolmogorov–Smirnov and Anderson–Darling) for the two carbon fiber datasets.

Data Set	K–S Statistic	p-value	A–D Statistic
1	0.0412	0.8592	0.0284
2	0.0508	0.8145	0.0421

For the censoring scheme $(R_1, R_2, R_3, R_4, R_5) = (2, 3, 3, 2, 4)$, the maximum likelihood estimate of α was computed and treated as a fixed known value. The Bayesian, E-Bayesian, and hierarchical Bayesian estimates of β were then obtained under each of the three loss functions for both datasets. The results are presented in Table 13. Specifically, the MLE of α for Data Set 1 under the squared-error, Linex, and generalized-entropy loss functions is 1.879, 1.349, and 1.921, respectively. For Data Set 2, the corresponding estimates are 2.182, 1.993, and 2.152.

Table 13: Bayesian estimates of β under different loss functions for the two carbon fiber datasets. B: Bayesian; HB: Hierarchical Bayesian; EB: E-Bayesian.

Dataset	Squared-error loss	Linex loss	Generalized-entropy loss
1	2.562 (B)	2.782 (HB)	2.696 (EB)
2	2.125 (B)	2.535 (HB)	2.328 (EB)

The fitted Weibull probability density functions for various estimates of β under the three loss functions are plotted in Figure 11 for Data Set 1 and in Figure 12 for Data Set 2. These plots clearly indicate the superiority of the hierarchical Bayesian estimate over the E-Bayesian and Bayesian estimates of β.

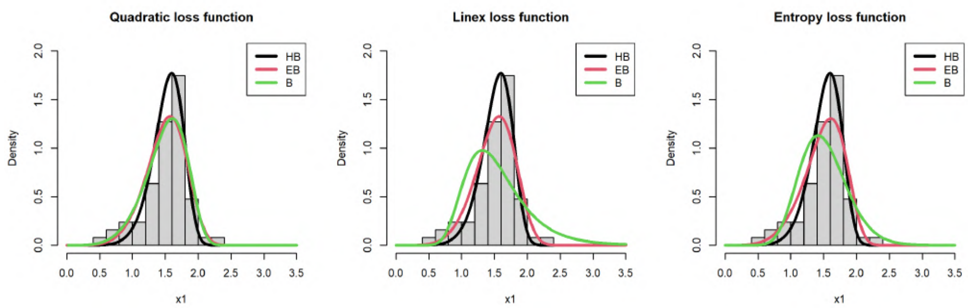

Figure 11: Histogram of Data Set 1 together with the fitted Weibull probability density functions obtained using Bayesian (B), E-Bayesian (EB), and Hierarchical Bayesian (HB) estimation under the squared error, entropy, and Linex loss functions.

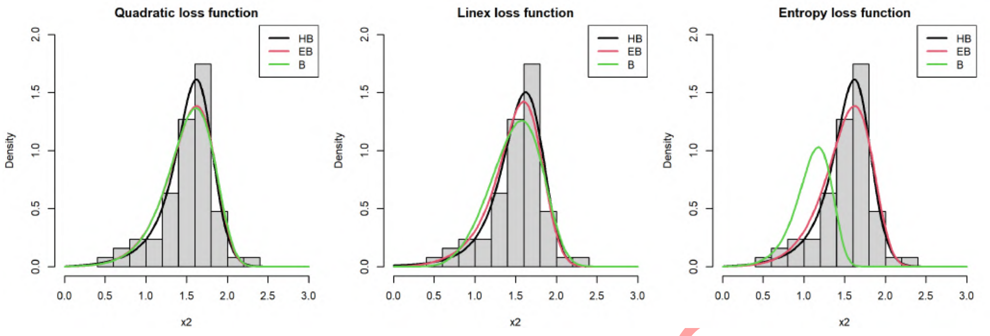

Figure 12: Histogram of Data Set 2 together with the fitted Weibull probability density functions obtained using Bayesian (B), E-Bayesian (EB), and Hierarchical Bayesian (HB) estimation under the squared error, entropy, and Linex loss functions.

10 Conclusion

This paper developed a unified inferential framework for continuous parametric models by integrating Bayesian, E-Bayesian, hierarchical Bayesian, and L2E estimation methods. The study considered two important statistical models: the two-parameter Weibull distribution under progressive Type-II censoring and the Beta distribution under complete sampling. For the Weibull model, E-Bayesian and hierarchical Bayesian estimators of the scale parameter were derived under Squared Error, Entropy, and Linex loss functions. Monte Carlo simulation results—evaluated in terms of bias and $\sqrt{\text{MSE}}$—indicated that the hierarchical Bayesian estimator consistently outperforms both the E-Bayesian and classical Bayesian estimators (see Tables 2–7). Moreover, the E-Bayesian estimator demonstrated improved performance relative to the standard Bayesian estimator. Real data analyses using the carbon fiber strength datasets (Tables 10–13 and Figures 11–12) further confirmed these findings, as the hierarchical Bayesian estimates provided better fitting performance across all considered loss functions. For the Beta distribution, robust parameter estimation was investigated using the L2E and E-Bayesian methods. Simulation studies showed that the L2E estimator performs competitively with, and in some scenarios better than, the MLE, particularly in the presence of outliers, which highlights its robustness. The E-Bayesian estimator was also derived and evaluated; although bias reduction was achieved in many cases, it was sometimes accompanied by a moderate increase in variability, which reflects the inherent trade-off in the Bayesian framework. Beyond the numerical results, the primary contribution of this work lies in the development of a unified optimization-based inferential framework, as illustrated in Figure 3, rather than in proposing a new estimation algorithm. The present work demonstrates that Bayesian, hierarchical Bayesian, E-Bayesian, and L2E estimation can be formulated and analyzed within a common optimization structure. This unified perspective enables a systematic comparison of Bayesian and robust divergence-based estimation strategies across fundamentally different classes of continuous parametric models—namely lifetime data under progressive Type-II censoring and bounded continuous data under complete sampling—and provides a general methodological basis for selecting appropriate estimation strategies according to the underlying data structure and inferential objective.

Limitations. One limitation of the present study concerns the E-Bayesian analysis for the Beta distribution, which was developed under the simplified assumption that the second shape parameter is known (specifically, $\beta = 1$). Although this assumption facilitates analytical derivations and allows a clear illustration of the proposed framework, it restricts the practical scope of the Beta-distribution component of the study. Additionally, the comparative analysis in Section 8 was intentionally limited to estimators formulated within a common Bayesian decision-theoretic framework, and conventional methods such as MLE and the method of moments were not included. While this restriction supports a focused comparison, it means that the proposed estimators have not been benchmarked against the widest possible range of competing methods.

Future Directions. Several natural extensions of this work are worth pursuing. The most immediate is to extend the proposed E-Bayesian and hierarchical Bayesian methodology to the general two-parameter Beta distribution, which would require handling a more complex joint posterior and higher-dimensional hyperparameter integration. More broadly, the proposed unified optimization-based framework can be applied to other continuous distributions, including inverse Gaussian, log-normal, and generalized gamma models, as well as to more complex censoring mechanisms such as adaptive progressive and hybrid censoring schemes. Incorporating MCMC-based posterior sampling within the proposed framework would further extend its applicability to settings where closed-form estimators are unavailable. Finally, developing theoretical guarantees—such as consistency and asymptotic normality—for the E-Bayesian and hierarchical Bayesian estimators within the proposed optimization structure constitutes an important direction for future research.

Declarations

Availability of Supporting Data

The data that support the findings of this study are available from the corresponding author upon reasonable request. Due to organizational confidentiality, the data are not publicly available.

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Conflict of Interest

The authors declare no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Authors Contributions

Iman Makhdoom: Conceptualization; Methodology; Software and Validation; Formal Analysis; Investigation; Resources; Data Curation; Writing – Original Draft; Writing – Review & Editing; Visualization; Supervision; Project Administration. **Shahram Yaghoobzadeh Shahrastani**: Conceptualization; Software and Validation; Formal Analysis; Resources; Data Curation; Writing – Original Draft; Project Administration.

Artificial Intelligence Statement

Artificial intelligence (AI) tools, including large language models, were used solely for language editing and improving readability. AI tools were not used for generating ideas, performing analyses, interpreting results, or writing the scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

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Appendix: Computational Algorithm and Representative R Code

This appendix presents the computational implementation of the proposed unified optimization-based estimation framework. Algorithm 1 outlines the general inferential procedure, followed by representative R code that illustrates its application to the Beta mixture model via the L2E method.

Algorithm 1 Unified Optimization-Based Estimation Framework

Input: Observed sample $\mathbf{x} = (x_1, \ldots, x_n)$ and a continuous parametric model $f(x \mid \theta)$

Output: Estimated parameter vector $\hat{\theta}$

1. Specify the probability density function $f(x \mid \theta)$.

2. Construct the likelihood function $L(\theta \mid \mathbf{x})$.

3. Select the estimation paradigm:

- ▷ Bayesian
- ▷ Hierarchical Bayesian
- ▷ E-Bayesian
- ▷ L2E (divergence-based)

4. Assign prior and hyperprior distributions (for Bayesian paradigms).

5. Derive or approximate the posterior distribution.

6. Choose a loss function: squared error (SELF), entropy (ELF), or Linex (LLF).

7. Solve the resulting optimization problem (analytically or numerically).

8. Extract the point estimator $\hat{\theta}$.

9. Assess performance through Monte Carlo simulation and real data analysis.

Representative R Code

```
##########################################################
## Simulation from a mixture Beta distribution
##########################################################
```

```
set.seed(153)

alpha0 <- 3
beta0  <- 8
alpha1 <- 10
beta1  <- 4
w0     <- 0.25
n0     <- 1000

x22 <- numeric(n0)

for (i in 1:n0) {
    x22[i] <- ifelse(runif(1) < w0,
                     rbeta(1, alpha0, beta0),
                     rbeta(1, alpha1, beta1))
}

#########################################################
## L2E objective function
#########################################################

L2E1 <- function(par, x) {
    beta(2*par[1]-1, 2*par[2]-1) /
    (beta(par[1], par[2]))^2 -
    (2/n0) * sum(dbeta(x, par[1], par[2]))
}

#########################################################
## Numerical optimization
#########################################################

fit <- nlminb(c(3, 3),
              L2E1,
              lower = 0.51,
              upper = 100,
              x = x22)

fit$par
```

Authors' Bio-sketches

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