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Research Article

Extraction of Approximate Solution for a Class of Nonlinear Optimal Control Problems Using $\frac{1}{G'}$-Expansion Technique

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Abstract. In this paper, the benefits of $\dfrac{1}{G'}$-expansion technique are utilized to create a direct scheme for extracting approximate solutions for a class of optimal control problems. In the given approach, first state and control functions have been parameterized as a power series, which is constructed according to the solutions of a Bernoulli differential equation, where the number of terms in produced power series is determined by the balance method. A proportionate replacement and solving the created optimization problem lead to suitable solutions close to the analytical ones for the main problem. Numerical experiments are given to evaluate the quality of the proposed method.

Keywords. Optimal control problem, $\frac{1}{G'}$-Expansion method, Parametrization.

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1 Introduction

Today, modeling in the form of optimal control problem (OCP) is a tool to achieve optimal parameters in a wide range of various problems in sciences such as economics, biology, physics, and engineering. Since the nature of optimal control problems is to search in infinite-dimensional spaces to find analytic functions that give rise the optimality of targets and finding of such functions are not easily possible except for some limited class, inevitably, numerical methods have been considered to find the approximate of such analytical functions. The parameterization of state and control functions and then converting OCP to nonlinear programming are one of the most common methods in the category of methods for finding approximate solutions for OCPs, called direct methods. In applying this strategy, in some cases, parameterization of control, for example, Spangelo [19], and in other cases, parameterization of states, Kafash et al. [12], and sometimes parameterization of control and states at the same time, [5, 6, 9], have been considered. Also, numerous polynomials, such as Chebyshev, Legendre, B-spline, and even Fourier series and Bezier curve's have been used to represent the parameterized form of functions [3, 6, 9, 11, 12, 14]. Undoubtedly, the choice of the basic functions will play an important role in the accuracy of the approximation and ultimately how the solutions are obtained from this class of direct methods. Obviously, in this regard, a kind of parameterization of control and state functions that leads to more accurate solutions with fewer calculations for the dynamic system governing the problem is more important. Also, strategies for finding analytical and approximate solutions of ordinary and partial differential equations governing the problem of optimal control have a significant impact on how to parameterize control and status functions.

Among the approaches that try to find solutions close to the analytical solution, schemes that use the solutions of simple and well-known ODE's such as Riccati and Bernoulli equations, as auxiliary differential equations, have suitable capabilities to track approximate solutions by parameterizing the state and control functions according to the functions in terms of the solutions of these equations. Some of these approaches are Sub-ODE method [1, 24, 25], $(\frac{1}{G'})$-expansion method [8, 21, 23, 22], $(\frac{G'}{G})$-expansion method [20, 26], and $(\frac{G'}{G}, \frac{1}{G})$-expansion method [2, 13].

In this paper, the components of $(\frac{1}{G'})$-expansion approach are manifested in the way of parameterizing control and state functions to start a direct method in order to extract acceptable approximate solutions for a class of nonlinear OCPs as follows:

$$
\begin{aligned}
&\text{Min} \quad J = g(x(t_0)) + \int_{t_0}^{t_f} F(t, x, u, x', u', x'', u'', \ldots)dt \\
&\text{s.t:} \\
&H(x, u, x', u', x'', x'u', u'', \ldots) = 0, \\
&\alpha_{i,0}x(t_0) + \alpha_{i,1}x(t_f) = c_i, \quad i = 0, 1, 2, \ldots, \\
&\beta_{j,0}x^{(k)}(t_0) + \beta_{j,1}x^{(l)}(t_f) = d_{j,k,l},
\end{aligned}
$$

where F and H are polynomials of their arguments, $\alpha_{i,0}, \alpha_{i,1}, \beta_{j,0}, \beta_{j,1}, c_i$, and $d_{j,k,l}$ for $i, j, k, l = 0, 1, 2, \cdots$, are proportional known constants, and $x(\cdot), u(\cdot), g(\cdot)$ are real-valued functions with continuous derivatives of any order on the time interval $I_0 = [t_0, t_f]$. Of course, it is assumed that the above optimal control problem is a controllable problem.

The advantage of this method over a number of existing methods is that if the solution of the algebraic equation system is accurate, after optimizing the cost function, then the exact solution to the optimal control problem will be obtained. On the other hand, in particular, the solutions obtained from this method are in the form of logistic functions that these types of solutions are widely used in population growth models in life sciences and medicine [15]. Therefore, the solutions obtained from this method are of great practical importance.

In Section 2, a summary of $(\frac{1}{G'})$-expansion approach is provided and then, using the benefits of the method introduced in this section, an approach based on parameterization of control and state functions with an algorithm and along with convergence analysis is presented in Section 3. In Section 4, experimental results of utilizing the given approach in some optimal control problems are described. Finally, the conclusions are explained in Section 5.

2 An Overview of $(\frac{1}{G'})$-Expansion Approach

In the $\frac{1}{G'}$-expansion method, the solutions of the ordinary differential equation

$$G'' + \lambda G' + \mu = 0, \quad \lambda \neq 0, \tag{1}$$

are used to find exact traveling wave solutions of nonlinear partial differential equations (NPDE) [8, 19, 21].

The solutions of equation (1) are of the form $G(t) = -\frac{\mu t}{\lambda} + ce^{-\lambda t} + d$, where λ, μ, c, and d are constant parameters. So

$$\frac{1}{G'} = -\frac{\lambda}{\mu + c\lambda^2[\cosh(\lambda t) + \sinh(\lambda t)]}. \tag{2}$$

By applying the rules of derivation and using equation (1), the different order derivatives of expression $\frac{1}{G'}$ can be written as a power series of $\frac{1}{G'}$.

Consider the two-variables nonlinear partial differential equation

$$E(v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial t}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial t^2}, \frac{\partial^2 v}{\partial x \partial t}, \ldots) = 0, \tag{3}$$

where E is a polynomial of $v(x,t)$ and its partial derivatives. With transform $v(\xi) = v(x,t)$, $\xi = x + kt$, where k is a nonzero constant, equation (3) may be converted to an ordinary differential equation as follows:

$$P(v, v', v'', v''', \ldots) = 0. \tag{4}$$

Assume that v is an extension to the power series of $\frac{1}{G'}$ as follows:

$$v(\xi) = \sum_{i=0}^{n} a_i (\frac{1}{G'(\xi)})^i, \tag{5}$$

in which G is the solution of equation (1) and n will be obtained by balancing the expression with the highest order derivative and the highest nonlinear expression in (4). By substituting (5) in (4) and after rearranging the obtained polynomials of different powers of $\frac{1}{G'}$, we set the coefficients equal to zero. Thus, a system of algebraic equations is obtained. By solving the system of algebraic equations, the parameters of a_i, $i = 0, 1, 2, \cdots, n$, may be determined. With these parameters, exact traveling wave solutions of the form (5) can be found. This process can be applied to solve ordinary differential equations, partial differential equations, and/or a system of these equations.

3 An Approach Via a Different Parametrization

Suppose that state and control functions x and u can be represented as a finite-series of $\frac{1}{G'}$, in the form

$$x(t) = \sum_{i=0}^{n} a_i \left(\frac{1}{G'(t)}\right)^i, \tag{6}$$

$$u(t) = \sum_{j=0}^{m} b_j \left(\frac{1}{G'(t)}\right)^j, \tag{7}$$

where a_i, $i = 0, 1, 2, \ldots, n$, and $b_j, j = 0, 1, 2, \ldots, m$, are constants, m and n are positive integers that are described below, and $G = G(t)$ is the solution of Bernoulli ordinary differential equation (1).

In order to determine the quantities m and n in

$$H(x, u, x^{'}, u^{'}, x^{''}, x^{'}u^{'}, u^{''}, \ldots) = 0, \tag{8}$$

we choose two terms with the highest power of $\frac{1}{G'}$ in (8), such that

i. at least one of these two terms contains x^i or $x^{(i)}$ for $i = 1, 2, 3, \ldots$.

ii. at least one of these two terms contains u^j or $u^{(j)}$, $j = 1, 2, \ldots$, (if there is no u^j or $u^{(j)}$ in (8), then equilibrium is done between the terms $x^{(i)}$ and u^j for $i, j = 1, 2, 3, \ldots$, with the highest order $\frac{1}{G'}$).

Suppose that in the balancing step above, $m = h(n)$, where h is an increasing and integer value function of n. For each fixed natural number n, in order to find the parameters a_i, $i = 0, 1, 2, \ldots, n$ and b_j, $j = 0, 1, 2, \ldots, h(n)$, we substitute the power series (6) and (7) in (8), which after simplifying the left-hand side of (8), a polynomial of $\frac{1}{G'}$ will be obtained. The coefficients of this polynomial are equal to zero that gives rise to a system of algebraic equations. By solving this system, we find parameters a_i and b_j.

By substituting a_i and b_j in (6) and (7), state and control variables may be obtained and then the expansion form of state and control functions x and u may be extracted by solving a nonlinear programming problem. In what follows, an algorithm based on the scheme described above is presented.

3.1 The Algorithm

Given $\epsilon_1 > 0$, $\epsilon_2 > 0$ and k, put $n = 0$.
step 1 Compute n and m using (i), (ii), in the form $m = h(n)$.
step 2 Add a unit to n.
step 3 Consider x_n and $u_{h(n)}$ in the form (6) and (7)
Step 4 Substitute x_n and $u_{h(n)}$ in (8) and set the obtained polynomial coefficients to zero.
Step 5 Find a_i and b_j by solving the system of algebraic equations obtained in the previous step.
Step 6 Put a_i and b_j in (6) and (7) and compute x_n and $u_{h(n)}$.
Step 7 Compute the value cost function j_n using of x_n and $u_{h(n)}$.

Step 8 If $n = 1$, then go to Step 2.
Step 9 If one of the following conditions is satisfied, then go to Step 2 else end.
(i) If the optimal value of the cost function J^* is known and $|J_n - J^*| > \epsilon_1$,
(ii) If $n < k$, where k is the maximum number of iterations of the algorithm,
(iii) For $n \geq 3$ if $|e_n - e_{n-1}| > \epsilon_2$, where $e_n = |j_n - j_{n-1}| + |j_n|$, and for $n < 3$, if $|j_n - j_{n-1}| > \epsilon_1$.

3.2 Convergence analysis

Suppose that x and u have expansions as (6) and (7), respectively. By substituting in the dynamics of the problem and the initial conditions, we obtain

$$H(\sum_{i=0}^{n} a_i(\frac{1}{G'(t)})^i, \sum_{j=0}^{m} b_j(\frac{1}{G'(t)})^j, \sum_{i=1}^{n} ia_i(\frac{1}{G'(t)})^i(\lambda(\frac{1}{G'(t)}) + \mu)$$
$$+ \sum_{j=1}^{m} ib_j(\frac{1}{G'(t)})^j(\lambda(\frac{1}{G'(t)}) + \mu), \dots) = 0.$$
$$\alpha_{i,0} \sum_{i=0}^{n} a_i(\frac{1}{G'(t_0)})^i + \alpha_{i,1} \sum_{i=0}^{n} a_i(\frac{1}{G'(t_1)})^i = c_i, \quad i = 0, 1, 2, \dots$$
$$\beta_{j,0}(\sum_{i=0}^{n} a_i(\frac{1}{G'(t)})^i)^{(k)}|_{t_0} + \beta_{j,1}(\sum_{i=0}^{n} a_i(\frac{1}{G'(t)})^i)^{(l)}|_{t_1} = d_{j,k,l}.$$

Consider $\alpha = (a_0, a_1, \dots, a_n)$ and $\beta = (b_0, b_1, \dots, b_m)$. Thus, dynamic and initial conditions of problem can be written of the form
$$M[\alpha, \beta] = 0,$$
where M is a coefficients matrix and the cost function J is as follows:
$$J = J(\alpha, \beta).$$

So the optimal control problem coverts to

$$\begin{aligned}&\text{Min} \quad J(\alpha, \beta)\\ &\text{s.t:}\\ &M[\alpha, \beta] = 0.\end{aligned}$$

If Z is the set of all vector functions of the form $(\sum_{i=0}^{\infty} a_i(\frac{1}{G'(t)})^i, \sum_{j=0}^{\infty} b_j(\frac{1}{G'(t)})^j)$ that are defined on subinterval $[t_0, t_1]$ and satisfy in dynamic and initial conditions optimal control problem, then consider $Z_{n,m}$ set of all pair regularly $(x(t), u(t))$ such that $x(t)$ and $u(t)$ are polynomials at most of degree n and m -respectively- of $\frac{1}{G'(t)}$, which satisfy in subjects and initial conditions of optimal control.

Suppose that the minimum of J on $Z_{n,m}$ is $\rho_{n,m}$. In the balancing step of Algorithm 3.1, if $m = h(n)$ in which h is an increase and integer value function of n, then $Z_{n,m}$ can be written of the form $Z_{n,h(n)}$.

If optimal solutions of state and control can be written in the form of power series $\frac{1}{G'(t)}$), then, the following proposition may show the convergence of the scheme.

Proposition 1. If $\rho_{n,h(n)} = \inf_{Z_{n,h(n)}} J$ for $n = 1, 2, 3, \ldots$, then $\lim_{n\to+\infty} \rho_{n.h(n)} = \rho$, where $\rho = \inf_Z J$.

Proof. Consider

$$\rho_{n,h(n)} = \min_{(\alpha_n,\beta_{h(n)})\in\mathbb{R}^{n+1}\times\mathbb{R}^{h(n)+1}} J(\alpha_n, \beta_{h(n)}).$$

Then

$$\rho_{n,h(n)} = J(\alpha_n^*, \beta_{h(n)}^*),$$

where

$$(\alpha_n^*(t), \beta_{h(n)}^*) \in arg\min\{J(\alpha_n, \beta_{h(n)}) : (\alpha_n, \beta_{h(n)}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{h(n)+1}\}.$$

So,

$$(x_n^*(t), u_{h(n)}^*(t)) \in arg\min\{J(x(t), u(t)) : (x(t), u(t)) \in Z_{n,h(n)}\},$$

we obtain

$$J(x_n^*(t), u_{h(n)}^*(t)) = \min_{(x(t),u(t))\in Z_{n,h(n)}} J((x(t), u(t)).$$

Therefore

$$\rho_{n,h(n)} = J(\alpha_n^*(t), \beta_{h(n)}^*(t)).$$

Since $h(n)$ is an increasing function, we obtain $Z_{n,h(n)} \subset Z_{n+1,h(n+1)}$. Hence

$$\min_{(x(t),u(t))\in Z_{n+1,h(n+1)}} J(x(t), u(t)) \leq \min_{(x(t),u(t))\in Z_{n,h(n)}} J(x(t), u(t)).$$

Sequence $\{\rho_{n,h(n)}\}_{n=1}^{+\infty}$ is a numeric non-increasing sequence. Because that sequence is bounded from the below, so $\{\rho_{n,h(n)}\}_{n=1}^{+\infty}$ is convergent.

If

$$\lim_{n\longrightarrow+\infty} \rho_{n,h(n)} = \rho^*$$

that $\rho^* > \rho$, then for $\varepsilon = \dfrac{\rho^* - \rho}{2} > 0$, exists $(x(\cdot), u(\cdot)) \in Z$ such that

$$J((x(\cdot), u(\cdot)) < \rho + \varepsilon = \rho + \frac{\rho^* - \rho}{2} = \frac{\rho^* + \rho}{2} < \rho^*,$$

and this is a contradiction with the continuity of J.

Hence,

$$\lim_{n\longrightarrow+\infty} \rho_{n,h(n)} = \min_{(x,u)\in Z} J(x(\cdot), u(\cdot)) = \rho.$$

□

4 Numerical Experiments

In this section, some examples for checking the ability of Algorithm 3.1 are presented.

Example 1. ([14, 16]) Consider the optimal control problem of the following form:

$$
\begin{aligned}
&\text{Min} \quad \frac{1}{2}\int_0^1 (3x^2(t)+u^2(t))dt\\
&\text{s.t:}\\
&u(t)=x^{'}(t)+x(t),\\
&x(0)=0, x(1)=2.
\end{aligned}
$$

The exact state and control functions of the above problem are

$$
\begin{aligned}
x(t)&=\frac{2}{\sinh(2)}\sinh(2t),\\
u(t)&=\frac{2}{\sinh(2)}(2\cosh(2t)+\sinh(2t)),
\end{aligned}
$$

with cost function $J^*=6.149258885$.

By applying Algorithm 3.1 proposed in the previous section, we consider

$$
\begin{aligned}
x^{'}&=n\mu a_n(\frac{1}{G'})^{n+1}+\cdots,\\
u&=b_m(\frac{1}{G'})^{m}+\cdots,
\end{aligned}
$$

where $m=n+1$.

We put $n=3$. By using algorithm 3.1, the expansion $x(t)$ and $u(t)$ in forms (6) and (7) are considered. Consider boundary conditions, and substituting in constrains, we have an algebraic differential system:

$$
\begin{aligned}
&(\frac{1}{G'})^0: a_0-b_0=0,\\
&(\frac{1}{G'})^1: a_1(1+\lambda)-b_1=0,\\
&(\frac{1}{G'})^2: \frac{-1}{\lambda^4c(e^{\lambda}-1)}(2((a_0c^3\lambda^7+(\frac{1}{2}a_0c^3-a_1c^2)\lambda^6+3(a_0\mu-\frac{1}{6}a_1)c^2\lambda^5\\
&\quad+\frac{3}{2}(a_0c\mu-\frac{5}{3}a_1\mu+\frac{1}{3}b_2)c\lambda^4+3(a_0\mu-\frac{1}{3}a_1)c\mu\lambda^3+\frac{3}{2}a_0c\lambda^2\mu^2\\
&\quad+2\lambda\mu^3+\mu^3)e^{3\lambda}-\lambda^2(((-\frac{5}{2}a_1\mu+\frac{1}{2}b_2)\lambda^2+((3a_0-6)\mu^2-a_1\mu)\lambda\\
&\quad+\frac{3}{2}\mu^2(a_0-2))e^{2\lambda}+((-a_1\lambda+3(a_0-2)\mu)e^{\lambda}+c\lambda^2(a_0-2))\lambda^2(\lambda\\
&\quad+\frac{1}{2})c)c)e^{-2\lambda})=0,\\
&(\frac{1}{G'})^3: \frac{-1}{c\lambda^5(e^{\lambda}-1)}(3((a_0c^3\lambda^7+(-a_1c^2+\frac{1}{3}a_0c^3)\lambda^6+3(a_0\mu-\frac{1}{9}a_1)c^2\lambda^5\\
&\quad+c^2\lambda^4a_0\mu+(6c\mu^2-\frac{1}{3}b_3)\lambda^3+2(\frac{7}{6}a_1+c)\mu^2\lambda^2+((-4a_0+10)\mu^3\\
&\quad+\frac{1}{3}a_1\mu^2)\lambda-\frac{2}{3}\mu^3(a_0-3))\lambda^2ce^{2\lambda}+(\frac{5}{3}a_0c^3\lambda^7\mu+\frac{1}{3}c^2\mu(a_0c-5a_1)\lambda^6\\
&\quad+(5(\mu(a_0\mu-\frac{1}{15}a_1)c+\frac{1}{15}b_3)c\lambda^5+c\mu^2(-\frac{7}{3}a_1+a_0c)\lambda^4+4((a_0+\frac{1}{2})\mu\\
&\quad-\frac{1}{12}a_1)c\mu^2\lambda^3+\frac{2}{3}c\mu^3(a_0+1)\lambda^2+\frac{10}{3}\lambda\mu^4+\frac{2}{3}\mu^4)e^{3\lambda}-((-a_1\lambda+3(a_0
\end{aligned}
$$

$$
\begin{aligned}
&-2)\mu)e^{\lambda}+c\lambda^2(a_0-2))\lambda^4(c\lambda^3+\frac{1}{3}c\lambda^2+\frac{5}{3}\mu\lambda+\frac{1}{3}\mu)c^2)e^{-2\lambda})=0,\\
(\frac{1}{G'})^4 : &\frac{-1}{c\lambda^5(e^{\lambda}-1)}(3(\lambda^2c(a_0c^3\lambda^6\mu-a_1c^2\lambda^5\mu+3a_0c^2\lambda^4\mu^2-\frac{1}{3}b_4\lambda^3\\
&+6c\lambda^2\mu^3+a_1\lambda\mu^3-2\mu^4(a_0-3))e^{2\lambda}+(a_0c^3\lambda^6\mu^2-c(a_1c\mu^2-\frac{1}{3}b_4)\lambda^5\\
&+3a_0c^2\lambda^4\mu^3-a_1c\lambda^3\mu^3+2c\mu^4(a_0+1)\lambda^2+2\mu^5)e^{3\lambda}\\
&-((-a_1\lambda+3(a_0-2)\mu)e^{\lambda}+c\lambda^2(a_0-2))\lambda^4(c\lambda^2+\mu)\mu c^2)e^{-2\lambda})=0.
\end{aligned}
$$

By solving algebraic differential system, parameters a_1, b_0, b_1, b_2, and b_3 in terms of other parameters will obtain by solving nonlinear optimization problem as

$$
\begin{aligned}
&a_0=-3.06470738157878, a_1=-30.09601805, a_2=-110.0396068,\\
&a_3=-198.3459486, b_0=-3.06470738157878, b_1=-47.38560230,\\
&b_2=-236.6285454, b_3=-541.3366443, b_4=-3.11709818882398,\\
&\lambda=0.574480788556751, \mu=0.00523848726282568, c=9.35591990512390.
\end{aligned}
$$

Optimization cost function obtained by Algorithm 3.1 is $J = 6.14944148506950938$ for $\epsilon_1 = 0.0018$.

In Table 1, the cost function obtained by proposed method for different values of n and with $\epsilon_1 = 1.9\times10^{-4}$ are given.

Table 1: Cost function of $\frac{1}{G'}$-expansion method for different values of n

n	Cost funtion J_n	$\lvert J_n - J^*\rvert$
1	6.15038925186366026	0.001130367
2	6.15033693261559478	0.001078048
3	6.14944148506950938	0.000182600

In Table 2, comparing the cost function extracted by $\frac{1}{G'}$-expansion approach with several methods has been shown. The results show that the cost function of the given approach is acceptable.

Table 2: Cost function of $\frac{1}{G'}$-expansion method and other methods

Methods	Cost funtion J	Absolate error
$\frac{1}{G'}$-expansion method (n=3)	6.14944148506950938	0.182600e-3
Mehne [14]	6.1748	0.25541115e-1
Rafiei [17] (n=3)	6.14989648031908	0.637595e-3

In Figures 1 and 2, the diagram of approximate optimal state and control, which is resulted from the given approach, have been compared with analytical ones. In Figure 3, the error constraint has been plotted.

It is obvious that the proposed solution is more accurate than the one presented in [8].

Example 2. ([4]) Consider optimization problem in form

$$\text{Min } J = -x(2)$$

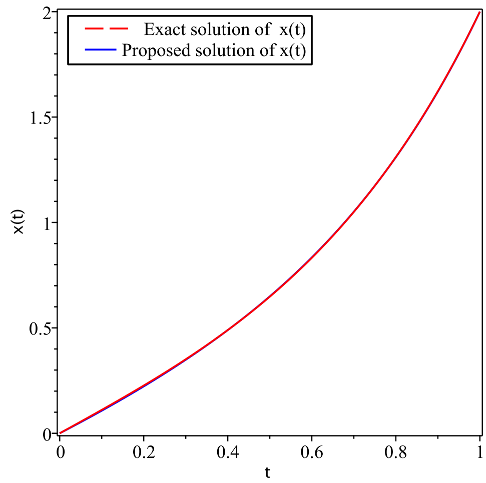

Figure 1: Approximate state function for Example 1.

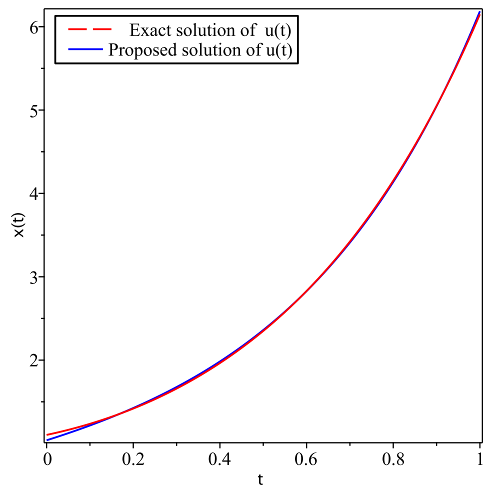

Figure 2: Approximate control function for Example 1.

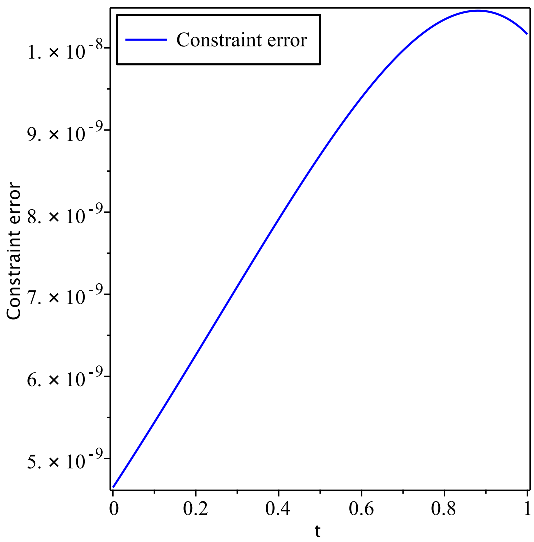

Figure 3: Constraint error $|x'(t) + x(t) - u(t)|$ for Example 1.

$$
\begin{aligned}
&\text{s.t:}\\
&x' = \frac{5}{2}(-x + ux - u^2),\\
&x(0) = 1.
\end{aligned}
$$

The exact solutions of this problem are in the following form:

$$
x(t) = \frac{4}{\left(1 + 3e^{\frac{5}{2}t}\right)},
$$

$$u(t) = \frac{1}{2}x(t),$$

with cost function $J^* = -0.0089637968$.

For finding m and n by using the balance terms u^2 and xu, where

$$u^2 = {b_m}^2(\frac{1}{G'})^{2m} + \cdots,$$
$$xu = a_n b_m(\frac{1}{G'})^{n+m} + \cdots,$$

we have $m+n = 2m$ that produce $m = n$. We put $n = 1$, after parameterization state and control functions and by using of the algorithm and solving the following algebraic differential system:

$$(\frac{1}{G'})^0 : 2.5 + \frac{5a_1\lambda}{2c\lambda^2 + 2\mu} - 2.5b_0(1 + \frac{a_1\lambda}{c\lambda^2 + \mu}) + 2.5b_0^2 = 0,$$
$$(\frac{1}{G'})^1 : a_1\lambda + 2.5a_1 - 2.5b_0a_1 - 2.5b_1(1 + \frac{a_1\lambda}{c\lambda^2 + \mu}) + 5b_0b_1 = 0,$$
$$(\frac{1}{G'})^2 : a_1\mu - 2.5a_1b_1 + 2.5b_1^2 = 0.$$

The parameters b_0, λ, μ, and c are obtained in terms of the other ones. After optimizing the cost function, we obtain

$$a_0 = 0, \quad a_1 = 1.33190388351046,$$
$$b_0 = 0, \quad b_1 = 0.665951941779148,$$
$$\lambda = -2.5, \quad \mu = 0.8324399275, \quad c = 0.3995711652,$$

and the value of cost function is $J = -0.008963796804$ that absolute error is 2.e-12($\epsilon_1 = 10^{-11}$). In Figures 4 and 5, state and control functions along with exact solution, are drawn.

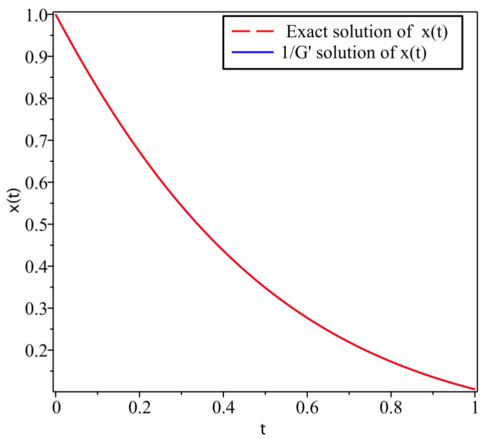

Figure 4: Approximate state function for Example 2.

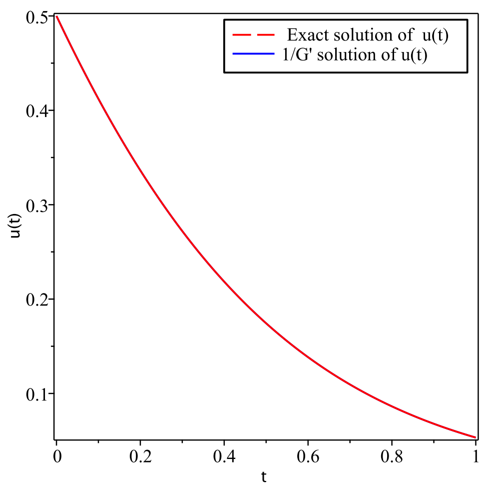

Figure 5: Approximate control function for Example 2.

It is obvious that the solution of proposed method is very accurate.

Example 3. ([10]) Consider the optimal control problem

$$\begin{aligned}
&\text{Min} \quad J = \int_0^1 (x^{'}(t) + u^2(t))dt \\
&\text{s.t:} \\
&x^{''}(t) = u(t), \\
&x(0) = 0, x(1) = 1, x^{'}(0) = 0.
\end{aligned}$$

The exact solutions are

$$x(t) = \frac{3t^2}{2} - \frac{t^3}{2}, \; u(t) = 3 - 3t.$$

We use expansion state and control functions in form series of (6) and (7). By balancing $x^{''}$ and u statements, that is,

$$\begin{aligned}
&x^{''} = n(n+1)\mu^2 a_n(\frac{1}{G'})^{n+2} + \cdots, \\
&u = b_m(\frac{1}{G'})^m + \cdots,
\end{aligned}$$

we obtain $m = n+2$. For $n = 2$ and $\epsilon_1 = 10^{-3}$, by solving the algebraic differential system

$$\begin{aligned}
&(\frac{1}{G'})^0 : \frac{1}{c^2\lambda^6(e^\lambda - 1)^2}(-e^{2\lambda}b_0c^2\lambda^6 + 2c^2\lambda^6(-\frac{1}{2} + e^\lambda)b_0) = 0, \\
&(\frac{1}{G'})^1 : \frac{1}{c^2\lambda^6(e^\lambda - 1)^2}((-b_1c^2\lambda^6 + 2c\lambda^5\mu^2 + 2\lambda^3\mu^3)e^{2\lambda} \\
&\quad + 2c\lambda^5((2c\lambda^2\mu + b_1c\lambda + 2\mu^2)e^\lambda + c\lambda(c\lambda^3 + \mu\lambda - \frac{1}{2}b_1))) = 0, \\
&(\frac{1}{G'})^2 : \frac{1}{c^2\lambda^6(e^\lambda - 1)^2}(-\lambda^2(c^2(-4\mu^2 + b_2)\lambda^4 - 14c\mu^3\lambda^2 - 10\mu^4)e^{2\lambda} \\
&\quad + 4c\lambda^4((2c^2\mu\lambda^4 + \frac{1}{2}c(14\mu^2 + b_2)\lambda^2 + 5\mu^3)e^\lambda + c\lambda^2(c^2\lambda^4 + \frac{7}{2}c\lambda^2\mu \\
&\quad + \frac{5}{2}\mu^2 - \frac{1}{4}b_2))) = 0,
\end{aligned}$$

$$\begin{aligned}
&(\frac{1}{G'})^3 : \frac{1}{c^2\lambda^6(e^\lambda - 1)^2}(-\lambda(-10c^2\lambda^4\mu^3 + b_3c^2\lambda^5 - 24c\lambda^2\mu^4 - 14\mu^5)e^{2\lambda} \\
&\quad + 10c\lambda^3((2c^2\lambda^4\mu^2 + \frac{24}{5}c\mu^3\lambda^2 + \frac{1}{5}cb_3\lambda^3 + \frac{14}{5}\mu^4)e^\lambda + c\lambda^2(c^2\mu\lambda^4 \\
&\quad + \frac{12}{5}c\lambda^2\mu^2 + \frac{7}{5}\mu^3 - \frac{1}{10}b_3\lambda))) = 0, \\
&(\frac{1}{G'})^4 : \frac{1}{c^2\lambda^6(e^\lambda - 1)^2}((6c^2\lambda^4\mu^4 - b_4c^2\lambda^6 + 12c\lambda^2\mu^5 + 6\mu^6)e^{2\lambda} \\
&\quad + 6c\lambda^2(((2c^2\mu^3 + \frac{1}{3}cb_4)\lambda^4 + 4c\lambda^2\mu^4 + 2\mu^5)e^\lambda + \lambda^2(c^2\lambda^4\mu^2 \\
&\quad + (2c\mu^3 - \frac{1}{6}b_4)\lambda^2 + \mu^4)c)) = 0,
\end{aligned}$$

and after optimization of cost function, parameters are obtained as follows:

$$\begin{aligned}
a_0 =& 8.201904799, a_1 = 39.57122592, a_2 = 47.72921532, \\
b_0 =& 0, b_1 = 45.87396616, b_2 = 422.9886746, b_3 = 1007.798143, \\
b_4 =& 712.8551340, c = 0.879519392578573, \lambda = 1.07669672598249,
\end{aligned}$$

μ =1.57773067177685.

The value of the cost function is $J = 4.00097044460308471$. Therefore, $|J_2 - J^*| = 0.00097044460308471$. In Figures 6 and 7, we plot real and proposed method state and control functions, and in Figure 8, error of x and in Figure 9, error of u are drawn. In Figure 10, we plot the error constraint.

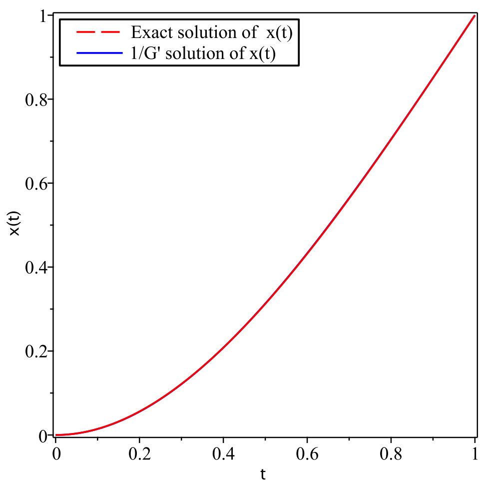

Figure 6: Approximate state function for Example 3.

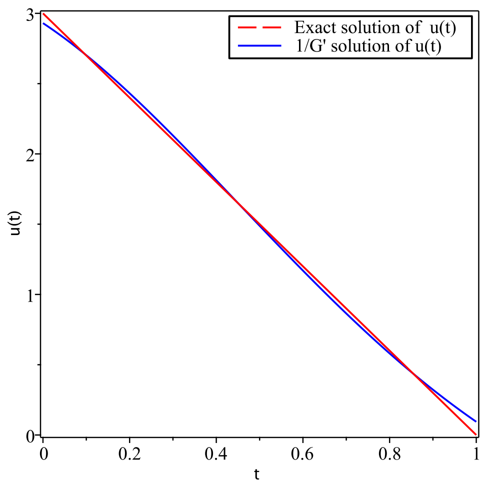

Figure 7: Approximate control function for Example 3.

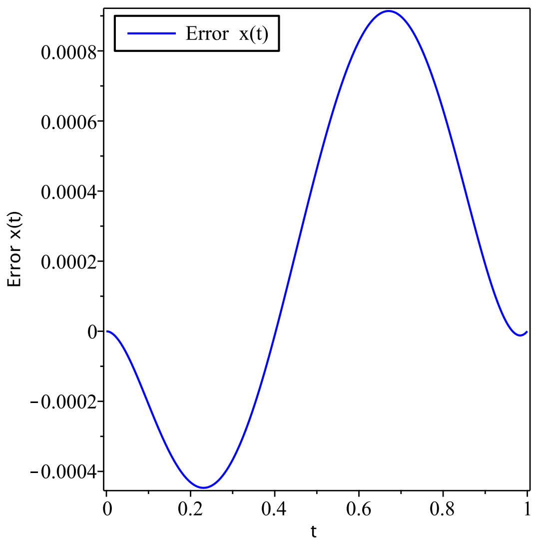

Figure 8: Error of $x(t)$ for Example 3.

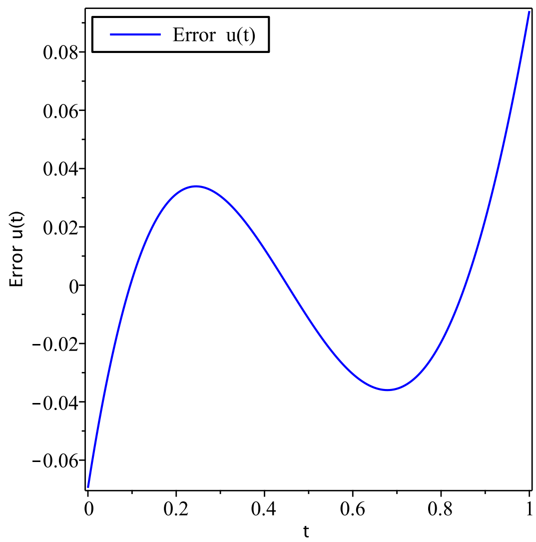

Figure 9: Error of $u(t)$ for Example 3.

Example 4. ([7]) Consider the optimal control problem

$$\text{Min} \quad J = \int_0^1 u^2(t)dt$$

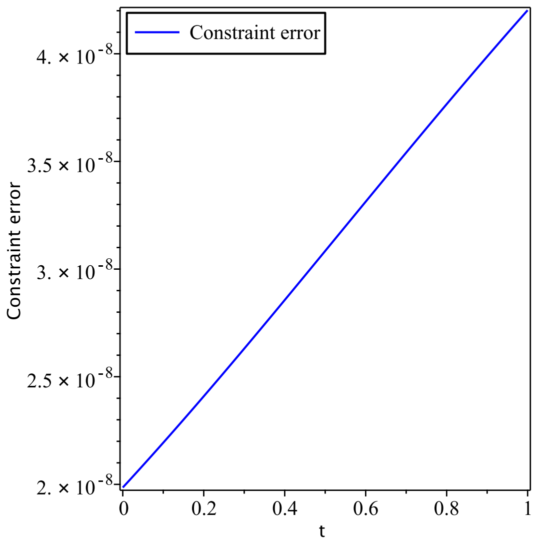

Figure 10: Constraint error $|x'(t) + x(t) - u(t)|$ for Example 3.

$$\text{s.t:}$$
$$x'(t) = x^2(t) + u(t),$$
$$x(0) = 0, x(1) = 0.5.$$

Using (6) and (7), in balancing x^2 and u terms, in the forms

$$x^2 = a_n^2(\frac{1}{G'})^{2n} + \cdots,$$
$$u = b_m(\frac{1}{G'})^m + \cdots,$$

we can obtain $m = 2n$. Using Algorithm 3.1, the cost function is 0.100618665278466746 for $n = 2$. In Table 3, the cost function obtained by the $\frac{1}{G'}$-expansion method with $n = 1, 2$ and $\epsilon_1 = 10^{-6}$ is given.

Table 3: Cost function of $\frac{1}{G'}$-expansion method for different values of n

n	Cost funtion J_n	$\lvert J_n - J_{n-1}\rvert$
1	0.178964705119591200	
2	0.178963753364739853	9.517×10^{-7}

In Table 4, the comparison of the cost function of $\frac{1}{G'}$-expansion method with several methods has been shown.

In Figures 11 and 12, we plot the proposed method state and control functions. In Figure 13, we plot the error constraint.

Example 5. ([18]) Consider the following optimal control problem:

$$\text{Min} \quad J = \int_0^{0.5} (x^2(t) + u^2(t))dt$$

Table 4: Cost function of $\frac{1}{G'}$-expansion method and other methods

Methods	Cost funtion J
$\frac{1}{G'}$-expansion method (n=2)	0.178963753364739853
Fard [7]	0.4447
Rafiei [17] (n=3)	0.1791666668

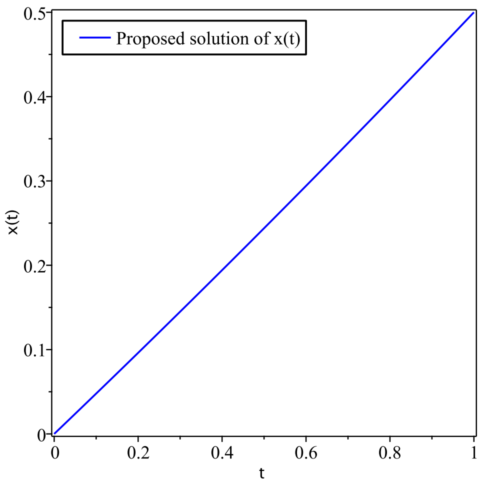

Figure 11: Approximate state function for Example 4.

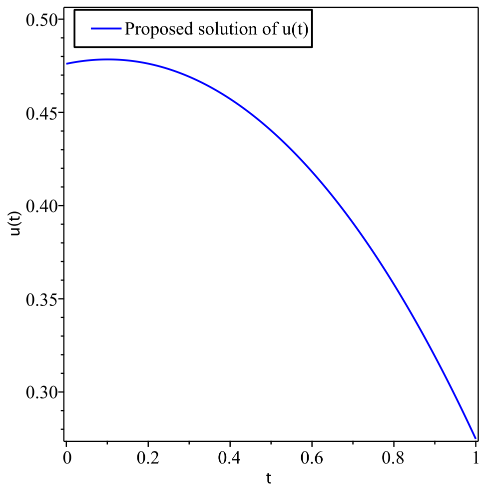

Figure 12: Approximate control function for Example 4.

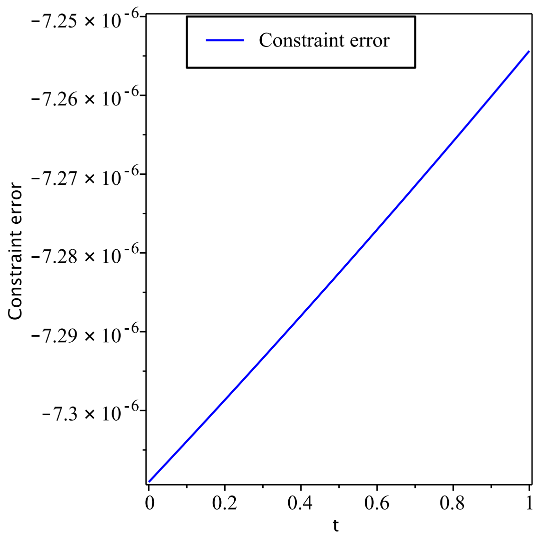

Figure 13: Constraint error $|x'(t) - x^2(t) - u(t)|$ for Example 4.

s.t:

$$x'(t) = -x(t) - 2x^2(t) - 0.5x^3(t) + u(t),$$

$$x(0) = 0.5.$$

In the first step of Algorithm 3.1, two terms $x^3(t)$ and $u(t)$ are selected for balance. In this case, we obtain

$$x^3 = a_n^3(\frac{1}{G'})^{3n} + \cdots,$$
$$u = b_m(\frac{1}{G'})^m + \cdots.$$

Therefore, $m = 3n$. In the mentioned algorithm, by setting $n = 3$, the cost function will be equal to 0.05528629215776097\87. Table 5 compares the results obtained from the proposed method with other method.

Table 5: Cost function of $\frac{1}{G'}$-expansion method and other method

Methods	Cost funtion J
$\frac{1}{G'}$-expansion method (n=3)	0.0552880257005159623
Rafiei [17] (n=3)	0.0899784856743520989

In Figures 14 and 15, we plot the proposed method state and control functions. In Figure 16, we plot the error constraint.

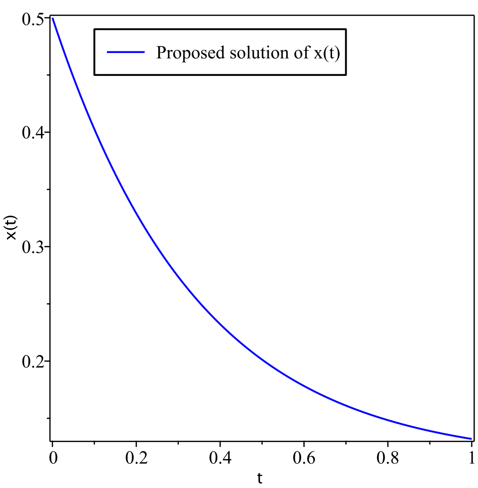

Figure 14: Approximate state function for Example 5.

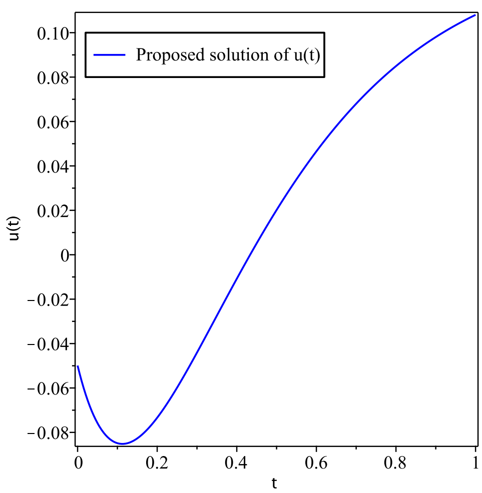

Figure 15: Approximate control function for Example 5.

In Table 6, the cost function obtained by the proposed method for different values of $n = 1, 2, 3$ and with $\epsilon_2 = 10^{-5}$ are given.

5 Conclusion

In this paper, using the brilliant properties of expansion methods in extracting near-analyticall solution of ODEs and PDEs have been utilized to build notable approximate solutions for a

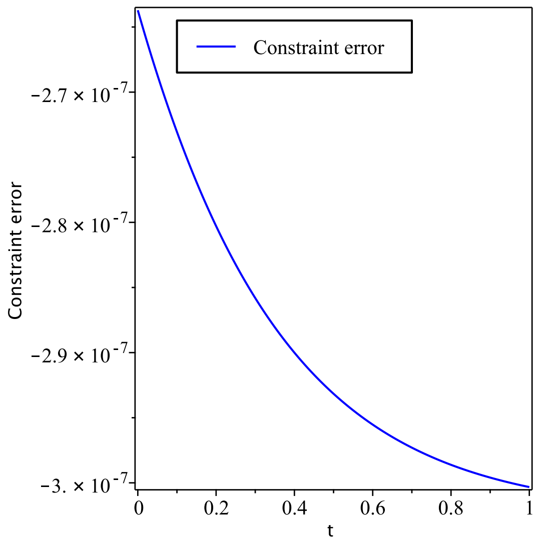

Figure 16: Constraint error $|x'(t) - (-x(t) - 2x^2(t) - 0.5x^3(t) + u(t))|$ for Example 5.

Table 6: Cost function of $\frac{1}{G'}$-expansion method

n	Cost funtion J_n	$e_n = \lvert J_n - J_{n-1}\rvert + \lvert J_n\rvert$	$\lvert e_n - e_{n-1}\rvert$
1	0.0553081902502270545		
2	0.0553013615950241760	0.05530819025	
3	0.0552880257005159623	0.05530136160	0.00000682865

class of OCP's. Of course, this is the first step in using similar methods to find solutions to such problems, and we hope to find near-analytical solutions to a variety of optimal control problems by combining these schemes with successful nonlinear optimization methods as well as valuable methods for finding the solutions of algebraic equations. There is no claim to express the preference of this method over similar methods because comparing the practice of this approach with other methods requires a serious study.

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