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Research Article

# Apply Optimized Tensor Completion Method by Bayesian CP-Factorization for Image Recovery

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**Abstract.** In this paper, we are going to analyze big data (embedded in the digital images) with new methods of tensor completion (TC). The determination of tensor ranks and the type of decomposition are significant and essential matters. For defeating these problems, *Bayesian CP-Factorization (BCPF)* is applied to the tensor completion problem. The *BCPF* can optimize the type of ranks and decomposition for achieving the best results. In this paper, the hybrid method is proposed by integrating *BCPF* and general *TC*. The tensor completion problem was briefly introduced. Then, based on our implementations, and related sources, the proposed tensor-based completion methods emphasize their strengths and weaknesses. Theoretical, practical, and applied theories have been discussed and two of them for analyzing big data have been selected, and applied to several examples of selected images. The results are extracted and compared to determine the method's efficiency and importance compared to each other. Finally, the future ways and the field of future activity are also presented.

**Keywords.** Image recovery, Matrix completion, Optimization problems, Tensor completion, Variational Bayesian inference.

**MSC.** 65F99; 15A69; 58C05..

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## 1 Introduction

A TC problem is filling the missing or unobserved entries in the given partially observed tensors. Tensors are used for describing complex datasets, due to their multidimensional features. Because of this, TC algorithms have been widely developed for and applied in fields and domains, including data mining, computer vision, signal processing, and neuroscience. Different unpredictable or inevitable factors like mal-operation, limited permission, as well as random data missing lead to raw and incomplete multifunctional datasets [5]. Moreover, in practice, TC problems typically arise in data-driven applications, including image completion and processing, as well as video compression own to the multiway properties of modern datasets. TC is a generalized version of matrix completion (MC). TC has been emerged because of some constraints of MC in the analysis of big data. Intuitively, MC algorithms can solve TC problems via downgrading and decreasing them into matrix levels, where typically either slice a tensor into several small matrices or unfold it into a big matrix [9].

Incomplete data tensor factorization can be utilized as a potent procedure to impute missing entries (also referred to as TC) via explicitly obtaining latent multilinear structures [8]. In this paper, after introducing the TC problem, we state the famous method for image recovery: "Bayesian CP Factorization of Incomplete Tensors with Automatic Rank Determination (BCPF)" implement it on some examples. Tensors (that are multiway arrays) offer efficient and reliable representations and descriptions for data structural attributes data, particularly for multidimensional data or data collections that are in the influence of various factors and parameters. For example, third-order tensors can be used to express video sequences with the dimensions of $(Height \times Width \times Time)$; also, higher-order tensors having the dimensionality of $(Pixel \times Person \times Pose \times illumination)$ can be employed to represent image ensembles that are measured under various conditions [9]. The rest of this study is organized as follows. Section 2 reviews basic notations as well as preliminary results. Section 4 presents our main results. In section 5, conclusions and some future works are presented.

## 2 Notations and Preliminaries

Some preliminaries for tensor calculus as well as TC are shortly stated in this section. For more details, and information, please refer to [1], [5], and [8].

**Definition 1.** A tensor is a multidimensional array, with a dimensionality that is referred to as its "order". $X$ stands for a $N$th-order tensor (i.e., an $N$-way array) which is identified as $N$-dimensional or $N$-mode tensor, too. Here, the term ”order” is used for referring the dimensionality of a tensor (like $N$th-order tensor), and the word ”mode” is employed for describing operations on a particular dimension (like mode-$n$ product) [2]. We denote the set of all $n$-dimensional tensors of order $m$ by $T_{m,n}$. The tensor $A$ is called symmetric, if all $a_{i_1,\cdots,i_n}$ are invariant under any permutation of indices. The set of all real $n$-dimensional symmetric tensors of order $m$ is shown with $S_{m,n}$.

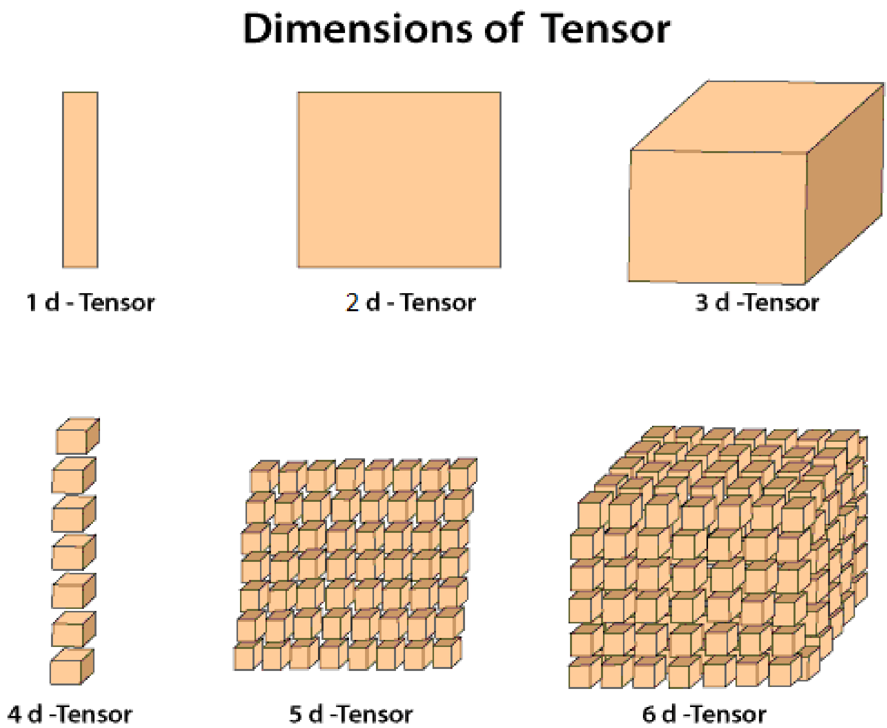


**Figure 1:** The representation of tensors as n-way array.

**Definition 2.** The notation $< X, Y >$ stands for the inner product of the two tensors $X$ and $Y$ with identical sizes. It is treated as a dot product with following defenition, unless otherwise stated [7]:

$$< X, Y >:= \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} X_{i_1,i_2,\cdots,i_N} y_{i_1,i_2,\cdots,i_N}. \tag{1}$$

**Definition 3.** The $F$-norm of a tensor $X$, as a generalization of Frobenius matrix norm, has the following definition [5]:

$$\|X\|_F := \sqrt{< X, X >} = \sqrt{\sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} X^2_{i_1,i_2,\cdots,i_N}}. \tag{2}$$

**Definition 4.** Suppose $X$ is a symmetric tensor of $S_{m,n}$, $r$ is a positive integer number, and $u^{(k)} \in \mathbb{R}^n$ for $k \in \{1, \cdots, r\}$ exist such that

$$X = \sum_{k=1}^{r} (u^{(k)})^m. \tag{3}$$

Therefore, $X$ is called a completely positive tensor (CP), and (3) is a CP-decomposition of $X$ (For example, see Figure 2). In the CP-decomposition of (3), the minimum of $r$ is called CP-rank of $X$ [5].

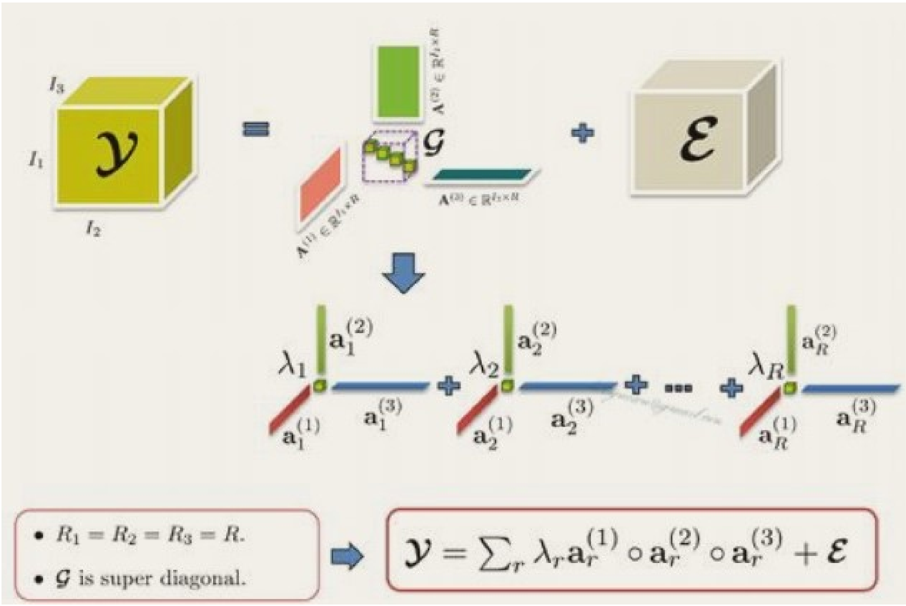


**Figure 2:** The representation of CP-Decomposition of 3-tensor.

**Definition 5.** The matrix completion (MC) problem is defined with the following optimization problem:

$$\min_{X} \quad \frac{1}{2}\|X - M\|_{\Omega}^2,$$
$$s.t. \quad rank(X) \leq r,$$

where $X, M \in \mathbb{R}^{p \times q}$, and the components of $M$ have been provided in the set $\Omega$, though the residual elements are needed. We use a low-rank matrix $X$ to approximate the missing components [1] and [3].

**Definition 6.** Tensors are generalizations of the matrix concept. Given a low-rank tensor $T$ with missing entries (either CP or another type of ranks), the following optimization problem can be used to describe the formulation of TC problem to complete $T$ [4]:

$$\min_{X} \quad \frac{1}{2}\|X - Y\|_F^2,$$
$$s.t. \quad \|X\|_{tr} \leq c,$$
$$Y_\Omega = T_\Omega.$$

Note that $X, Y, T$ are $n$-mode tensors having same sizes for every mode. Here, we define the tensor trace norm based on the completed positive (CP) rank as follows [5]:

$$\|X\|_{tr} = \frac{1}{n}\sum_{i=1}^{n} \|X_i\|_{tr}. \tag{4}$$

If the trace norms of all unfolded matrices along every mode are averaged, then the trace norm of tensor $T$ is obtained.

Figure 3 shows the comparison between matrix and tensor completion problems.

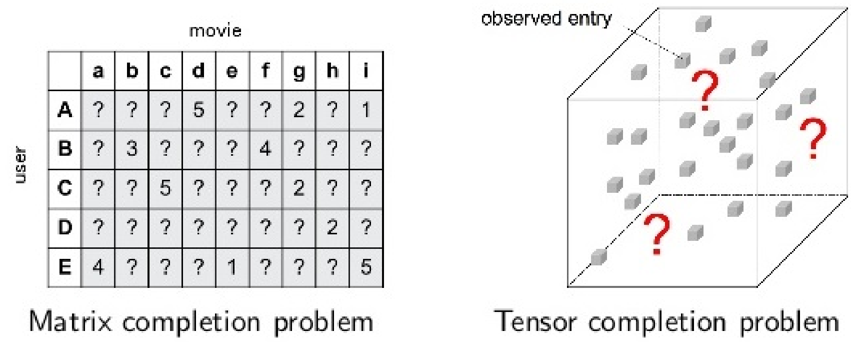


**Figure 3:** The comparison scheme of matrix completion, vs tensor completion.

## 3 Main Results

The method discussed in this paper is named "Bayesian CP Factorization of Incomplete Tensors with Automatic Rank Determination (BCPF)". Different from the matrix, it is a difficult challenge to evaluate the order of a tensor. In an actual implementation, the special measures including fitting or generalization errors are affected on the effective dimensionality of the latent space, which can be considered as a tunable parameter. As the complexity of the model can be controlled by this parameter, so the selection of this parameter is equivalent to a model selection problem. Anyway, in this work, the precision of every dimensionality of the latent space is controlled by a set of continual hyperparameters. The algorithm effectively avoids the overfitting of the problems by using this method.

In *BCPF*, a CP factorization was first formulated using a hierarchal probabilistic model and a Bayesian treatment over multiple latent factors were used for model learning via Bayesian inference. Then, the suitable hyperpriors over all hyper-parameters led to automatic model selecting (order determining) and noise detecting. Therefore, our model is a parameter-free procedure allowing the effective inference of the underlying multilinear factors related to the incomplete and noisy data of tensors having low-rank constraints, as well as predictable distributions over latent factors and missing entries estimation [5]. An exact Bayesian inference should be integrated over all latent variables and hyperparameters, leading to an analytical intractability. Hence, we must resort to the approximate inference. In this method, first, the incomplete tensor$X$ is inputted and then factorized by low-rank matrices via CP factorization, and the CP rank of each matrix is calculated. The optimization of this decomposition is obtained by Bayesian posterior parameters , and the full rank of tensor $X$ is calculated. Then, the TC problem is solved for $X$ subject to these resulting parameters. For the first time, the *BCPF* algorithm was proposed by Zhao, Zhang, and Cichocki in 2014 [9]. In the following section, we implemented this algorithm for the recovery of some images. Figure 4 shows the main scheme of the *BCPF* algorithm.

The hierarchical probabilistic model framework, and the Bayesian inference are the bases of the *BCPF* and various notable benefits including the following cases can be obtained:

1. The low-rank estimation of a tensor can be clearly achieved via the automatic measurement capability of tensor rank.

2. As this procedure is tunable and parameter-free, it is sufficient to rely on the observed data for inferring all model parameters, which bypasses the costly computational determination scheme of parameters.

3. Avoidance from the overfitting of the problem, because the posteriors are indicated by combining all extraneous variables.

4. *BCPF* method can be used to achieve the uncertainty information attributed to latent factors, and missing entries prediction.

5. The *BCPF* is a deterministic and effective algorithm, that is developed for Bayesian inference, leading to the quick empirical convergence, and there is a linear relationship between its computational complexity and data size.

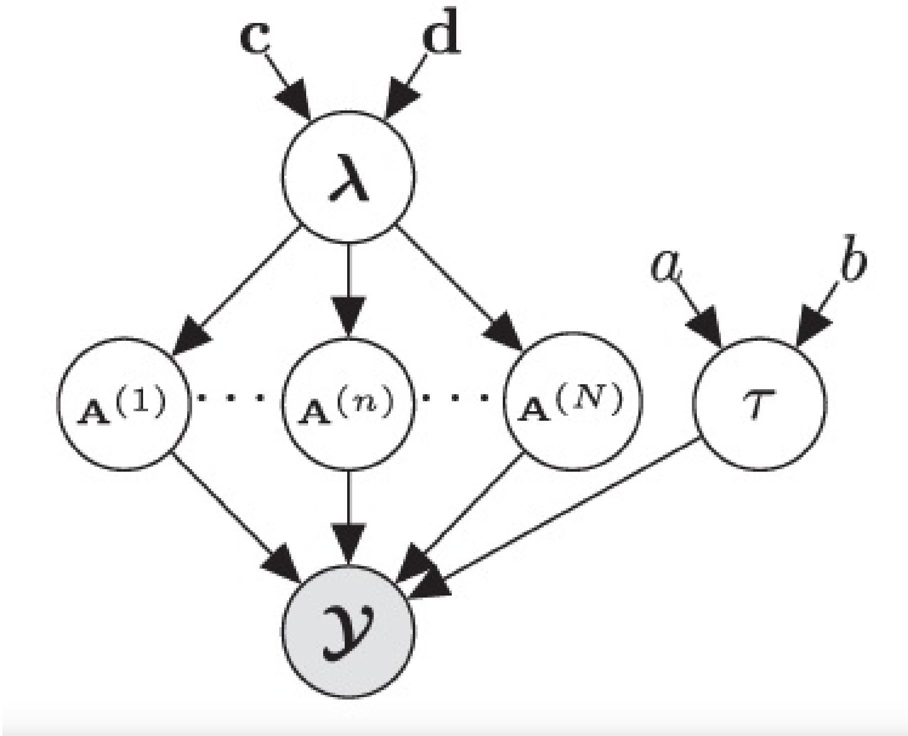


**Figure 4:** The BCPF algorithm: the main procedure demonstrated.

In *Matlab* R2020a, the BCPF codes use *tensor toolbox* 2.6 and *tensorlab*. By opening the file "DemoBayesCP-Images.m", in line 22, the "ObsRatio" is tunable. After determining that, by importing the image in lines 21 and 69, the *BCPF* is ready for implementation. The main results are as follows:

1. In "ObsRatio=50", after 72 seconds and 13 iterations, with a rank reduction from 100 to 41, the image restored up to 95.42 percent (see Figure 5).

2. In "ObsRatio=70", after 88 seconds and 14 iterations, with a rank reduction from 100 to 56, the image restored up to 96.51 percent (see Figure 6).

3. But in "ObsRatio=30", after 62 seconds and 13 iterations, with a rank reduction from 100 to 25, the image restored up to 93.47 percent (see Figure 7).

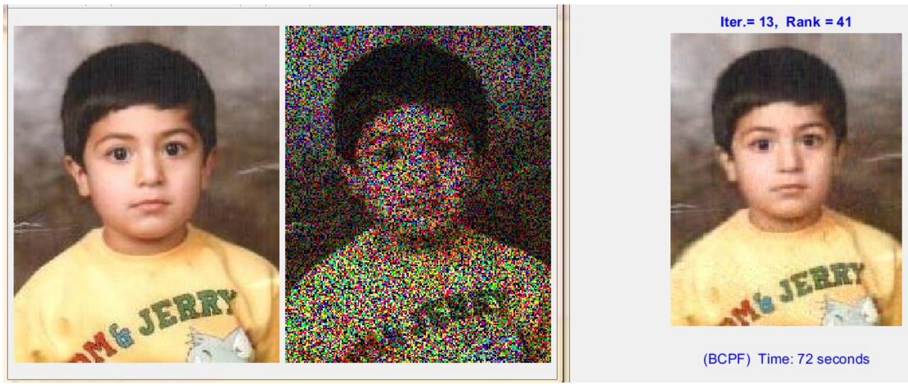


**Figure 5:** Expriment with ObsRatio=50.

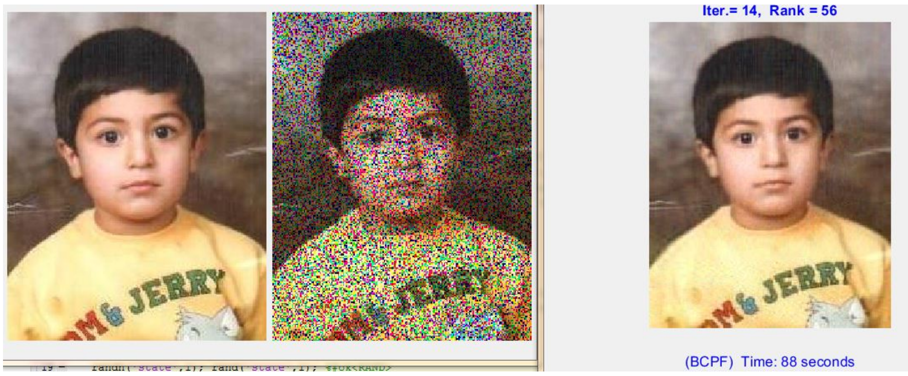


**Figure 6:** Expriment with ObsRatio=70.


**Figure 7:** Expriment with ObsRatio=30.

**Figure 8:** Visual effects of image inpainting. From top to bottom, two representative applications are shown. The first column shows the observed images, while from left to right, the images reconstructed by FBCP, CPWOPT, CPNLS, FaLRTC, FCSA, HardC, and STDC methods are shown.

## 4 Conclusions

Overall, *BCPF* is a great and robust method to recover noisy images. According to figure 8, the visual effects of image inpainting obtained by *BCPF* have a superior performance for getting the image inpainting visual effects in comparison with all other procedures (FBCP, CPWOPT, CPNLS, FaLRTC, FCSA, HardC, and STDC procedures) for two reasons. The superiority of *BCPF* can be confirmed in numerous real-world applications, including image completion, and image synthesis compared to the new methods of tensor factorization and tensor completion. Briefly, because of some impressive attributes, this procedure has been paid much attention to various potential applications [8]. Our experiments show that the *BCPF* method could restore the noisy and interrupted images with a minimum of 30 percent observed data. This method is not suitable for 90 percent noisy images, or large images . For future works, we propose to apply other tensor products, ranks, and decompositions for reaching better results based on the computational costs and efficiency.

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