Control and Optimization in Applied Mathematics (COAM)
Vol. 6, No. 2, Summer-Autumn 2021 (1-21), ©2016 Payame Noor University, Iran
DOI. 10.30473/coam.2022.59941.1169 (Cited this article)

Research Article

Chaotic Dynamics in a Fractional-Order Hopfield Neural Network and its Stabilization via an Adaptive Model-Free Control Method

Majid Roohi[1,*], Mohammad Pourmahmood Aghababa[2], Javid Ziaei[3], Chongqi Zhang[4,*]

[1,4]School of Economics and Statistics, Guangzhou University, P.O. Box. 510006, Guangzhou, China.
[2]Electrical and Computer Engineering Department, University of Windsor, Windsor, ON, Canada.
[3]Department of Automation, Biomechanics and Mechatronics, Lods University of Technology, Poland.

Received: July 14, 2021; **Accepted:** May 17, 2022.

Abstract. The present study introduces a kind of fractional-order Hopfield neural network (FOHNN), and its complex dynamic behavior is investigated through chaos analyses. With the use of phase space analysis and bifurcation diagrams and maximal Lyapunov exponent (MLE) it is demonstrated that for the values of $0.87 < \alpha < 1$, as the fractional-order (FO), the dynamical behavior of the mentioned FOHNN is chaotic. Then, the bounded trait of chaotic systems is utilized to derive an adaptive model-free control technique to suppress of complex dynamics of the FOHNN. Furthermore, according to the matrix analysis theorem of non-integer-order systems and the adaptive model-free control methodology, analytical consequences of the designed controller are evidenced. Eventually, two examples are reported to illustrate the applicability of the mentioned model-free control method.

Keywords. Fractional-order systems, Hopfield neural network, Bifurcation, Adaptive model-free controller, Stabilization.

MSC. 26A33; 92B20; 34H10.

* Corresponding author
majid.roohi67@gmail.com, m.p.aghababa@gmail.com, javid.ziaei@gmail.com, cqzhang@gzhu.edu.cn.
http://mathco.journals.pnu.ac.ir

1 Introduction

Recently, one of the essential challenges of sciences and engineering is the trustable modeling of natural phenomena by the lightest possible equations. In this regard, fractional differential equations provide new constructions to this field. In reality, fractional differential equations are a mix of arbitrary order differentiation and integration. The so-called "memory effect" is a distinguishing feature of a positive real-order derivative [46]. However, it is evident that many natural occurrences have long memory qualities. As a result, fractional order (FO) differential equations provide an effective tool for modeling such processes with certain unique qualities, such as extended memory and hereditary traits. [36, 38, 46]. There has recently been significant reporting in the literature on the application of the fractional calculus in a variety of sectors of science and engineering, including financial models [10], energy systems [2], optimization [15, 18], medical sciences [3, 12] and secure communications [8, 43].

Neural networks, introduced at the beginning of the twentieth century, have been used in many scientific fields and applications such as computer science [45], medical science [31], aerospace [22], optimization [14, 20], security [26] and so on. Hopfield neural networks (HNNs) [23, 48] abstracted from brain dynamics are some of the most important neural networks and are able of storing certain memories or patterns in a manner rather similar to the human brain. Owing to the wide applicability in pattern recognition [52, 53], medical science [7] and associative memories [6], there is nowadays much motivation on the study of Hopfield neural networks. Recent two decades have witnessed increasing attention to chaotic systems. High sensitivity to initial values, fractal properties of the motion in the phase space and broad Fourier transform spectra are some substantial properties of a chaotic system. Due to these significant features, chaos theory has been utilized in many practical research issues which include secure communications [54], information processing [39], and neural networks [47, 50]. Furthermore, chaotic and hyper-chaotic behaviors have been reported in many artificial neural networks. In [24], new classes of chaotic Hopfield neural networks have also been offered.

Based on this controversy, in the present study, a new type of three-dimensional fractional-order Hopfield neural network (FOHNN) is provided and its dynamical behaviors are investigated. To this end, we apply the bifurcation analysis and maximal Lyapunov exponent (MLE) criterion. The bifurcation diagram can identify different responses of the system under different situations derived by the change of some effective parameters on dynamics. To be sure that a dynamical system can show chaotic behavior, the positive value of the MLE criterion is sufficient.

It should be noted that the chaotic response is not always desirable. So, the control of FO chaotic systems has become an interesting subject for research. And, recently some control methods have been applied to control/stabilize chaotic treatment of FO dynamical systems [21]. Therefore, diverse approaches for stabilization of FO chaotic systems are applicable. In [29], to H_∞ synchronize uncertain FO complex systems, by using fuzzy logic, an adaptive controller has been suggested. But, as demonstrated in [1], the authors supposed that the procedure of non-integer differentials were the same as integer ones, hence the main results of their method were not correct. In [19, 41], the problem of stabilization of FO non-autonomous systems using active control methods was investigated. The authors of [5, 40] have introduced switching adaptive control methods to stabilize unknown FO complex systems. Recently, some sliding mode controller approaches have been designed in [42, 49, 51, 55] for the stabilization of FO chaotic systems. In [34], according to Laplace transformation, a robust control method is designed for chaos synchronization in a new fractional hybrid system. In [33], an adaptive sliding terminal is designed to synchronize FO quadratic complex flows. The authors of [25], to synchronize and stabilize the FO complex systems, have suggested an adaptive fuzzy type-2 control method by using a projection algorithm. In [30], using the FO version of Lyapunov-stability-theory, an FO adaptive SMC method is presented to synchronization FO neural networks. H_∞ performance analysis and stabilization for FO complex neural networks is reported in [35] with a non-fragile robust finite-time controller. In [16], based on a neural estimator, an adaptive FO SMC scheme is designed to control a class of complex systems with nonlinearities.

Although virtually all of the methods presented in the literature have a common weakness, almost all of them share a common weak point: they all employ all of the words associated with the FO systems in the control input. In contrast, there is no specific information regarding the linear and non-linear dynamic aspects of the FO-systems in real-world scenarios. As a result, the development of control mechanisms for the stabilization of FO chaotic systems with model-free structures is a critical subject in both theoretical and experimental studies. Nonetheless, as one knows, the model-free control approach has received little attention in the literature, and the control purpose of this study is to examine this topic further. An adaptive model-free control mechanism is presented in this study, which is intended to suppress the complicated behavior of the suggested FOHNN. The bounded property of the chaotic systems is used for introducing an efficient adaptive model-free control method, in which the linear-or-nonlinear elements in relations of the introduced FOHNN are not utilized. Two numerical simulations are provided to ensure the application and efficacy of the designed adaptive model-free stability strategy. For the purpose of developing an effective adaptive model-free control approach, the bounded property in chaotic systems is used, and the linear and nonlinear components of the dynamics of the newly presented FOHNN are not utilized. The applicability and effectiveness of the proposed adaptive model-free control approach are demonstrated by a number of simulations, which are also included.

The following is the structure of this work. It is discussed in Section 2 how to formulate preliminary thoughts concerning fractional differential equations. In Section 3, some essential details about bifurcation analysis and maximal Lyapunov exponent are given. Section 4 is devoted to introducing the fractional-order Hopfield neural network. Then, using the MLE criterion and bifurcation analysis, it is shown that the nonlinear behavior of the FOHNN is chaotic. In Section 5, according to the bounded property of the chaotic systems, an adaptive model-free control approach is designed and the theoretical results are proved. Two illustrated examples are presented in Section 6. Finally, a summery of the results of this work are presented in Section 7.

2 Preliminary Concepts

In this section, fundamental definitions and concepts about fractional calculus and an essential theorem for stability analysis of FO equations are presented.

Definition 1. [37] The Fractional integeral is called Riemann-Liouville FO integral for a fractional of FO α of a continuous function Ψ is introduced by

$$ {}_{t_0}I_t = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \Psi(\theta)(t-\theta)^{\alpha-1} d\theta, \tag{1}$$

where t_0 shows the initial amount of time and $\Gamma(.)$ shows the Gamma function

$$\Gamma(z) = \int_{t_0}^{\infty} t^{s-1} e^{-t} dt. \tag{2}$$

Definition 2. [37] Let $\alpha \in (r-1, r)$ and $r \in N$, then the Caputo FO derivative of order α for a function $\Psi : R^+ \to R$ is given as follows

$$ {}_{t_0}D_t^{\alpha}\Psi(t) = \frac{1}{\Gamma(r-\alpha)} \int_{t_0}^{t} \Psi^{(r)}(\theta)(t-\theta)^{r-\alpha-1} d\theta. \tag{3}$$

In the rest of the article, D^{α} demonstrates the Caputo derivative.

Lemma 1. [28] Let we have the following FO system:

$$D^{\alpha}X = G(X), \tag{4}$$

where $X(t) = [x_1(t), x_2(t), ..., x_n(t)]^T \in R^n$ and $\alpha \in (0,1)$, then equilibrium point in the FO system (4) reaches to asymptotical stability if there is a positive-definite symmetric, real matrix Ω such that the equation

$$\Delta = X^T \Omega D^\alpha X < 0.$$

3 Bifurcation Analysis and Maximal Lyapunov Exponent

In nonlinear dynamical systems, the variation of some parameters may cause structural deviations. If a dynamical system has instability in its nature, then the system can exhibit unexpected qualitative changes by enforcing it through an arbitrarily small variation in some parameter. Bifurcation theory is the nonlinear dynamic analysis of sudden changes in the qualitative behavior of a given nonlinear system. Bifurcation analysis can be applied for the study of continuous and discrete systems and can provide the possible long-term evolution of a system as a function of a bifurcation parameter [56, 13, 11].

The exponential divergence of initially nearby trajectories with time is known as sensitivity to initial conditions. This feature is a common property of chaotic systems. The Lyapunov exponent is a criterion for the identification of chaos through the quantification of the sensitivity on initial conditions. An m-dimensional dynamical system has m Lyapunov exponents defined as

$$\lambda_i = \lim_{n \to \infty} \frac{1}{t} ln \left\| \frac{d_i(t)}{d_i(0)} \right\|, \tag{5}$$

where $d_i(0)$ and $d_i(t)$ represent distance between two trajectories at times 0 and t in the ith direction, respectively and $\|.\|$ denotes the vector norm. However, just one positive Lyapunov exponent is sufficient for a nonlinear system to be chaotic. So, in many practical situations, the maximal Lyapunov exponent is computed. In the present study, the algorithm of Rosenstein [44] is applied to calculate the MLE of the system.

4 Model Description and Complex Dynamics

Here, a fractional-order description of the Hopfield neural network (HNN) in [23] is presented. Then, using maximal Lyapunov exponent and bifurcation analysis, the complex dynamics of the introduced fractional-order Hopfield neural network (FOHNN) are examined.

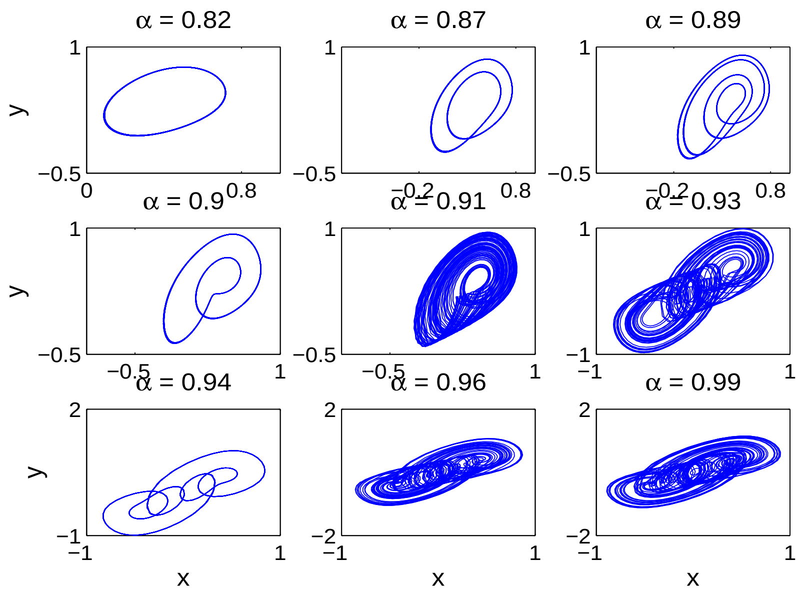

Figure 1: Phase space plot of FOHNN (9), for random values of the fractional order α.

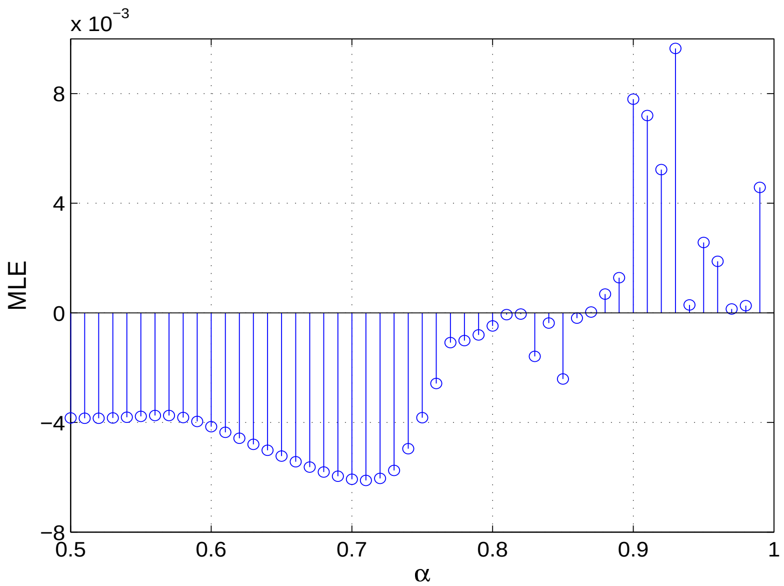

Figure 2: The MLEs of FOHNN (9) for different values of the fractional order α increasing from 0.5 to 1.

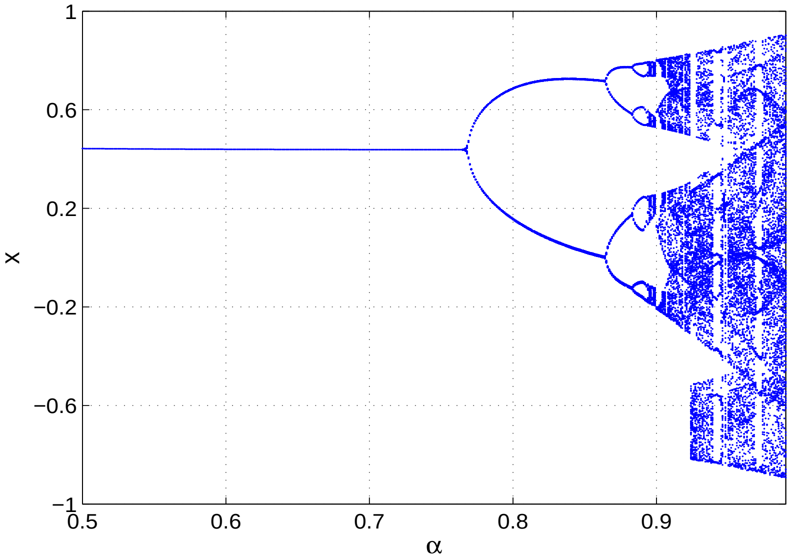

Figure 3: Bifurcation diagram of FOHNN (9) with control parameter α increasing from 0.5 to 1.

4.1 Fractional-order model of Hopfield neural network

An integer-order HNN has been given in [23] as

$$\dot{X} = -CX + Wf(X), \tag{6}$$

where $X = [x_1, x_2, , x_n]^t \in R^n$ is a vector of dynamical variables, C is the vector of constant parameters. The weight-matrix W, is an $n \times n$ matrix and connects neurons, and $f(X)$ is a bounded differentiable function defined by $f(X) = \tan(X)$.

In the three-dimensional case, the HNN (6) can be represented as

$$\dot{x}_i = -c_i x_i + \sum_{j=1}^{3} w_{ij} f(x_j), \qquad i = 1, 2, 3, \tag{7}$$

where $x_i \in R, i = 1, 2, 3$ is the variable of the neural network, c_i is a constant parameter, w_{ij} are the elements of W and $f(x_i) = tanh(x_i)$.

A new chaotic Hopfield neural network has been suggested in [27], which is presented as follows:

$$\begin{cases} \dot{x}_1 & = -x_1 + 2\tanh(x_1) - \tanh(x_2), \\ \dot{x}_2 & = -x_2 + 1.7\tanh(x_1) + 1.71\tanh(x_2) + 1.1\tanh(x_3), \\ \dot{x}_3 & = -2x_3 - 2.5\tanh(x_1) - 2.9\tanh(x_2) + (0.56 + p)\tanh(x_3). \end{cases} \tag{8}$$

The HNN (8) exhibits a chaotic and unpredictable behavior for $p = 0$. For more details, the Ref. [27] can be read.

Here, according to the definition of a fractional differential equation and using Equation (8), the following fractional order Hopfield neural network model is proposed.

$$\begin{cases} D^\alpha x_1 &= -x_1 + 2\tanh(x_1) - \tanh(x_2), \\ D^\alpha x_2 &= -x_2 + 1.7\tanh(x_1) + 1.71\tanh(x_2) + 1.1\tanh(x_3), \\ D^\alpha x_3 &= -2x_3 - 2.5\tanh(x_1) - 2.9\tanh(x_2) + (0.56 + p)\tanh(x_3), \end{cases} \tag{9}$$

where $\alpha \in (0, 1)$ is the fractional order of the FOHNN.

Remark 1. In (9), in the case of $p \neq 0$ have been considered in [43] a synchronization problem of a vast class of FO neural networks. It is demonstrated that considering $p = 0$ does not have any fundamental effect on the framework and behavior of the system.

Remark 2. There are different definitions of the dynamics of the Hopfield neural network. For instance, recently, a new class of the 3-dimensional FO Hopfield neural networks has been taken into account in [32]. Moreover, for chaos suppression of the system an adaptive SMC approach is generated and then the attractors are synchronized in a finite time.

4.2 Complex dynamical behavior

Now, it is illustrated that the proposed FOHNN can show the wide range of dynamical behaviors like stable behaviors, limit cycles and chaotic motions. These diverse behaviors can be seen in Figure 1, where the $2-d$ phase orbit of the system have been demonstrated for $\alpha = 0.82$ (period-2 orbit), $\alpha = 0.87$ (period-4 orbit), $\alpha = 0.89$ (period-8 orbit), $\alpha = 0.9$ (period-4 orbit), $\alpha = 0.91$ (single-scroll chaotic orbit), $\alpha = 0.93$ (double-scroll chaotic orbit), $\alpha = 0.94$ (period-10 orbit), $\alpha = 0.96$ (double-scroll chaotic orbit) and $\alpha = 0.99$ (double-scroll chaotic orbit).

As it is shown using maximal Lyapunov exponent criterion, dynamical behaviors of FOHNN (9) are chaotic. The MLE diagrams of FOHNN (9) for distinct values of the FO $\alpha \in [0.5, 0.99]$ are plotted in Figure 2. One can see that for the values of $0.87 \leq \alpha \leq 0.99$, the FOHNN (9) has positive value MLE, therefore it has chaotic dynamics in phase space.

Furthermore, the bifurcation diagram shown in Figure 3 aids us for having a pervasive view of the dynamics of the FOHNN. Clearly, the variation of α has divided dynamics of the system into three regions. The first region contains a single stable fixed point, i.e., system does not oscillate and the dynamics of system terminate in a single fixed point. This means that different trajectories in accordance with different values of α have the same qualitative structure and the small perturbation to the system due to the variation of α does not affect the long-time dynamic of system.

The second region contains consecutive period-doublings. The first period-doubling is the starting point of this region. The creation of new two stable fixed points and the fluctuation of system dynamics between them are distinctive points of the second region in comparison with the first region. Roughly speaking, the phase space of system is a periodic closed orbit. With the increase of α the generated fixed points are separated from one another. The creation of new fixed points and their divergence continues until the consecutive periods-doubling terminates in chaos where the dynamics possesses infinitely many equilibria and fixed points. The third region defines the chaotic zone, where the system fluctuates in an unsystematic way independent of the value of α.

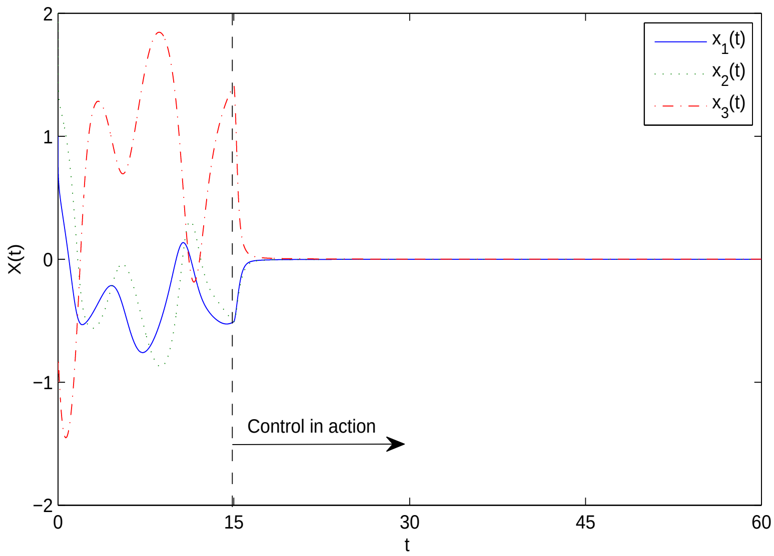

Figure 4: State trajectories of the controlled FOHNN system (10) in Example 1 (8).

Remark 3. Actually, the adaptive control approaches (15) and (16) are called model-free because the nonlinear/linear dynamical terms of the system have not been used in the design of the controller. This means that the formulation of the controller is only based on the states of the systems.

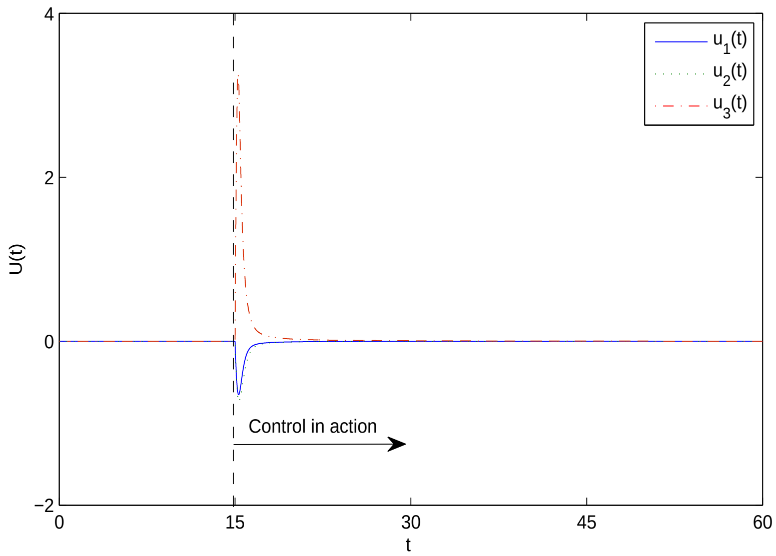

Figure 5: Time history of adaptive controller (15), applied on Example 1.

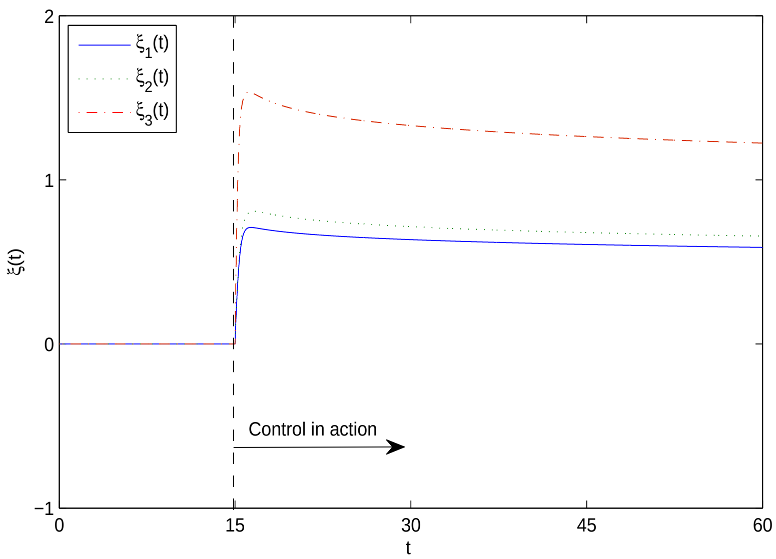

Figure 6: Time response of the updating laws (16), in Example 1.

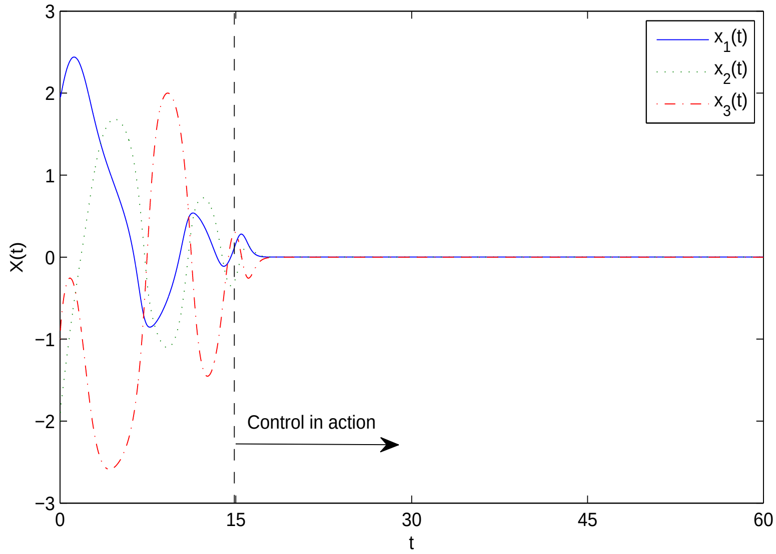

Figure 7: State trajectories of the controlled FOHNN system (10) in Example 2.

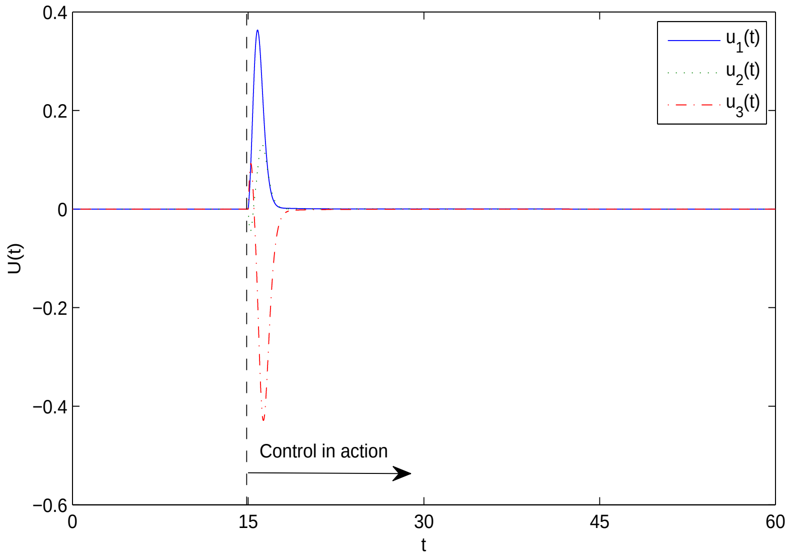

Figure 8: Time history of adaptive controller (15), applied on Example 2.

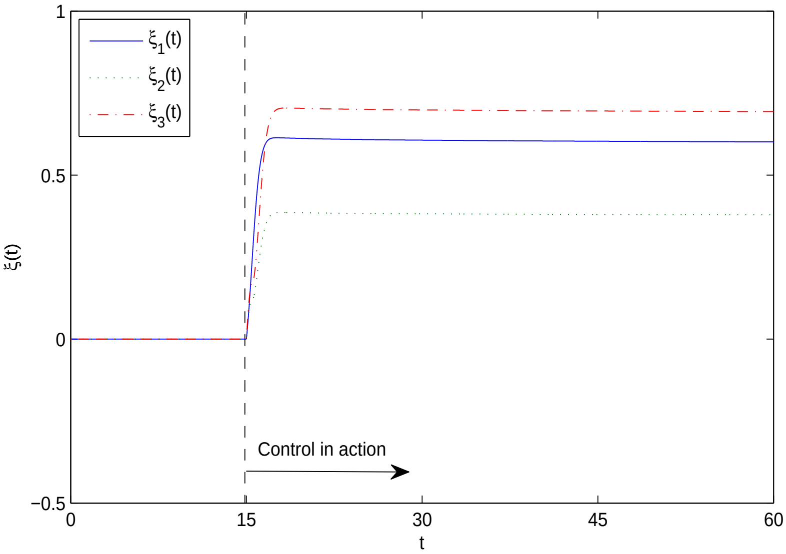

Figure 9: Time response of the updating laws (16), in example 2.

5 Control of the Proposed FOHNN

Here, we concern the chaos control problem in system (9) with the adaptive scheme. To suppress the chaotic behavior of system (9), we should add control signals to the system. Consider the following chaotic FOHNN with control input $u_i, \quad i = 1, 2, 3$:

$$\begin{cases} D^\alpha x_1 &= -x_1 + 2\tanh(x_1) - \tanh(x_2) - u_1, \\ D^\alpha x_2 &= -x_2 + 1.7\tanh(x_1) + 1.71\tanh(x_2) + 1.1\tanh(x_3) - u_2, \\ D^\alpha x_3 &= -2x_3 - 2.5\tanh(x_1) - 2.9\tanh(x_2) + 0.56\tanh(x_3) - u_3. \end{cases} \tag{10}$$

By converting Equation (10) to a matrix-form, one obtains

$$\begin{bmatrix} D^\alpha x_1 \\ D^\alpha x_2 \\ D^\alpha x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 1.7 & 1.71 & 1.1 \\ -2.5 & -2.9 & 0.56 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \tanh(x_1) \\ \tanh(x_2) \\ \tanh(x_3) \end{bmatrix}}_{Y} - \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{U}, \tag{11}$$

which is equivalent to

$$D^\alpha X = AX + BY - U. \tag{12}$$

If the rows of the matrices A and B are called a_i and b_i $(i = 1, 2, 3)$ respectively, then the Equation (12) is equivalent to

$$\begin{cases} D^\alpha x_1 &= g_1(X, Y) - u_1, \\ D^\alpha x_2 &= g_2(X, Y) - u_2, \\ D^\alpha x_3 &= g_3(X, Y) - u_3, \end{cases} \tag{13}$$

in which $g_i(X,Y) = a_i X + b_i Y, \quad i = 1,2,3.$

Assumption: On the basis of phase space, it has been proved that the state trajectories of chaotic systems have a global constraint in their state space [9, 17]. Actually, this is made by irregular attractors of the chaotic systems. As it turns out, this is a byproduct of chaotic systems' irregular attractors. It is therefore necessary to confine the right-hand portion of a complicated system. Because of this, there are finite-valued positive constants $\rho_i, \quad i = 1,2,3$ for the nonlinear components of the FOHNN (13), $g_i(X,Y) = a_i X + b_i Y, \quad i = 1,2,3$ such that

$$|g_i(X,Y)| < \rho_i < \infty, \qquad i = 1,2,3. \tag{14}$$

One of the well-known methods with too many desirable features is the adaptive control method. Here, an adaptive model-free controller is proposed to suppress the chaotic behaviors of the FOHNN (10). The controller is designed so that it does not require any information about dynamical terms of FOHNN (10). Thus on the basis of bounded feature in a chaotic system, the adaptive model-free controller is designed as

$$u_i(t) = \xi_i(t)\Big(k_i \Upsilon(x_i) + c_i \overline{\Upsilon}(x_i)\Big), \quad i = 1,2,3, \tag{15}$$

$$D^\alpha \xi_i(t) = \lambda_i |x_i|, \;\; \xi_i(0) = \xi_{i0}, \quad i = 1,2,3, \tag{16}$$

where

$$\Upsilon(x) = \begin{cases} sign(x), & if \quad x > 0, \\ 0, & if \quad x \le 0, \end{cases}$$

and

$$\overline{\Upsilon}(x) = \begin{cases} 0, & if \quad x > 0, \\ -tanh(x), & if \quad x \le 0, \end{cases}$$

where, $k_i > 1$, $c_i < -1$ and $\lambda_i > 0$ are constant values and $\xi_{i0} > 0$ shows the initial condition of the adaptive parameter $\xi_i(t)$.

It should be noted that, to avoid undesirable chattering phenomenon the Υ and the $\overline{\Upsilon}$ functions are joined in the control approach.

Theorem 1. Consider the fractional-order neural network (10) with control inputs. The trajectories of the neural network (10) will asymptotically converge to zero under the control scheme (15) and updating law (16).

Proof. Suppose that W is given by $W(t) = [X(t), \gamma(t)]^T$, where

$$X(t) = [x_1(t), x_2(t), x_3(t)]^T \quad \text{and} \quad \gamma(t) = \gamma_1(t), \gamma_2(t), \gamma_3(t)]^T.$$

Thus,

$$D^{\alpha}W(t) = [D^{\alpha}x_1(t), D^{\alpha}x_2(t), D^{\alpha}x_3(t), D^{\alpha}\gamma_1(t), D^{\alpha}\gamma_2(t), D^{\alpha}\gamma_3(t)]^T,$$

where

$$\gamma_i = (\rho_i - \xi_i) \quad i = 1,2,3.$$

Then,

$$D^{\alpha}\gamma_i = -D^{\alpha}\xi_i = -\lambda_i|x_i|, \quad i = 1,2,3.$$

Based on Theorem 1, the matrix P must be chosen in a way that the following relation holds.

$$\Delta = W^T P D^{\alpha} W < 0.$$

By selecting P as

$$P = diag(1,1,1,\frac{1}{\lambda_1},\frac{1}{\lambda_2},\frac{1}{\lambda_3}),$$

one has

$$\Delta = \sum_{i=1}^{3} x_i D^{\alpha}x_i + \sum_{i=1}^{3} \lambda_i^{-1}\gamma_i D^{\alpha}\gamma_i. \tag{17}$$

Inserting $D^{\alpha}x_i$ from Equation (13) into the above equation, one obtains

$$\begin{aligned}\Delta &= \sum_{i=1}^{3} x_i\Big(g_i(X,Y) - u_i\Big) + \sum_{i=1}^{3} \lambda_i^{-1}\gamma_i \underbrace{(-\lambda_i|x_i|)}_{D^{\alpha}\gamma_i} \\ &= \sum_{i=1}^{3} x_i\Big(g_i(X,Y) - u_i\Big) - \sum_{i=1}^{3} \underbrace{(\rho_i - \xi_i)}_{\gamma_i}|x_i|.\end{aligned}$$

According to the relation (14) and the design of $u_i(t)$ in relation (15), one gets

$$\begin{aligned}\Delta &< \sum_{i=1}^{3}\Big[\rho_i|x_i| - \xi_i(t)\Big(\xi(t)\Big(k_i\Upsilon(x_i) + c_i\overline{\Upsilon}(x_i)\Big)\Big)\Big] - \sum_{i=1}^{3}(\rho_i - \xi_i)|x_i| \\ &< \sum_{i=1}^{3} x_i\Big[\rho_i - \xi_i(t)\Big(k_i\Upsilon(x_i) + c_i\overline{\Upsilon}(x_i)\Big)\Big] - \sum_{i=1}^{3}(\rho_i - \xi_i)|x_i| \\ &= \sum_{i=1}^{3}\Big[\rho_i x_i - x_i\xi_i\Big(k_i\Upsilon(x_i) + c_i\overline{\Upsilon}(x_i)\Big)\Big] - \sum_{i=1}^{3}\Big[\rho_i|x_i| - \xi_i(t)|x_i|\Big] \\ &\le \sum_{i=1}^{3}\Big[\rho_i|x_i| - k_i x_i\xi_i\Upsilon(x_i) - c_i x_i\xi_i\overline{\Upsilon}(x_i) - \rho_i|x_i| + \xi_i(t)|x_i|\Big].\end{aligned}$$

Therefore,

$$\Delta < \sum_{i=1}^{3}\Big[\xi_i(t)|x_i| - k_i x_i\xi_i\Upsilon(x_i) - c_i x_i\xi_i\overline{\Upsilon}(x_i)\Big]. \tag{18}$$

- Case 1. If $x_i > 0$, in (18) we have

$$\begin{aligned}\Delta &< \sum_{i=1}^{3}\left[\xi_i(t)|x_i| - k_i x_i \xi_i sign(x_i)\right] \\ &= \sum_{i=1}^{3}\left(\xi_i(t)|x_i| - k_i \xi_i |x_i|\right) \\ &= \sum_{i=1}^{3}\left[\xi_i(t)|x_i|(1-k_i)\right]. \end{aligned} \tag{19}$$

Since $k_i > 1$ and the adaptive condition $\xi_i(t) > 0$ are ensured, we obtain

$$\Delta < \sum_{i=1}^{3}\left[\xi_i(t)|x_i|(1-k_i)\right] < 0. \tag{20}$$

- Case 2. If $x_i \leq 0$, in (18) one has

$$\Delta < \sum_{i=1}^{3}\left[\xi_i(t)|x_i| + c_i x_i \xi_i tanh(x_i)\right]. \tag{21}$$

Since $x_i tanh(x_i) \leq |x_i|$ is true for any $x_i \in R$, one obtains

$$\begin{aligned}\Delta &< \sum_{i=1}^{3}\left(\xi_i(t)|x_i| + c_i \xi_i |x_i|\right) \\ &= \sum_{i=1}^{3}\left[\xi_i(t)|x_i|(1+c_i)\right]. \end{aligned} \tag{22}$$

Since $c_i < -1$ and the adaptive condition $\xi_i(t) > 0$ are ensured,

$$\Delta < \sum_{i=1}^{3}\left[\xi_i(t)|x_i|(1+c_i)\right] < 0. \tag{23}$$

Thus, the stability condition in Lemma 1 is satisfied by relations (20) and (23). Therefore, the procedure of the proof is done. □

Remark 4. The parameters λ_i, k_i, c_i and ξ_i in (15) and (16) mean the switching control gain, which should be adjusted to some special values such that $\lambda_i > 0, k_i > 0, \xi_i > 0$ and $c_i < -1$ for $i = 1, 2, 3$, in order to assure the stability of system equilibrium point. According to the controller, it can be shown that the values of these parameters have an effect on the amount of control effort required in the suggested method. To put it another way, high values of the parameters result in a significant control energy.

6 Illustrative Examples

To validate the efficiency and applicability of the proposed adaptive control method (15) in suppressing the complex dynamics of the FOHNN in a given time interval, two illustrative examples are provided for different values of α. It should be indicated that a numerical algorithm, which is presented in [4], is utilized and the controller begins to operate after $t = 15$ sec.

6.1 Example 1

In this example we fix α, the order of derivative of the FOHNN (10), at 0.91 where the chaotic behavior of the complex network (10) is ensured. Moreover, the initial conditions of the FOHNN (10) are selected as $x_1(0) = 1$, $x_2(0) = 2$ and $x_3(0) = -1$. Subsequently, the parameters of the controller (15) and updating law (16) are chosen as $k_1 = k_2 = k_3 = 4.5$, $\lambda_1 = \lambda_2 = \lambda_3 = 3$ and $c_1 = c_2 = c_3 = -1.5$ respectively.

The state trajectories of the controlled FOHNN system (10) and the time history of adaptive controller (15) are plotted in Figures 4 and 5. It is obvious that the strange attractors of the chaotic FOHNN system are stabilized quickly. Furthermore, it is seen that the control input (15) converges to an equilibrium point. The time response of the updating law (16) is depicted in Figure 6. Obviously, all of the parameters of updating law (16) approach to some fixed value. This means that the designed adaptive controller can effectively suppress the chaotic behaviors of fractional-order HNN system and the controller is feasible in the real world.

6.2 Example 2

Here, we assume that the fractional order of the FOHNN (10) is equal to 0.99. With this choice, the behavior of (10) will be unstable and chaotic, surely. To overcome this unstability, the parameters of the control law (15) and updating law (16) are selected as $k_1 = k_2 = k_3 = 4$, $\lambda_1 = \lambda_2 = \lambda_3 = 2$ and $c_1 = c_2 = c_3 = -1$, respectively. Moreover, the initial values of the FOHNN (10) are chosen as $x_1(0) = 2$, $x_2(0) = -1$ and $x_3(0) = 1$. Figures 7-9 show the state trajectories of the FOHNN (10), the time response of the adaptive controller (15) and the time history of the updating law (16), respectively. It can be seen that the state trajectories and control signals converge to the equilibrium point. Furthermore, all of the parameters of the updating law (16), approach to the bounded values. This means that the proposed adaptive controller can effectively stabilize the chaotic behavior of the introduced FOHNN.

7 Conclusions

In this paper, a new fractional-order Hopfield neural network (FOHNN) was introduced and the existence of chaos was examined. Using the maximal Lyapunov exponent criterion and bifurcation analysis, it aws ensured that the FOHNN exhibits chaotic behavior. In this regard, an adaptive control scheme was designed for control and stabilization of the FOHNN. The designed control method aws independent of the linear/nonlinear terms of the complex system states. By applying the adaptive control method and the stability analysis theorem of the fractional-order systems, the analytical terms of this controller were proved. Simulation results showed that the introduced FOHNN exhibited chaotic attractors and the proposed adaptive controller could simply undertake the observed chaotic behaviors.

References

[1] Aghababa M.P. (2012). "Comments on H_∞ synchronization of uncertain fractional-order chaotic systems: Adaptive fuzzy approach [ISA Trans 50 (2011) 548–556]", ISA Transactions, 51, 11-12.

[2] Aghababa M.P. (2015). "Fractional modeling and control of a complex nonlinear energy supply-demand system", Complexity, 20, 74-86.

[3] Aghababa M.P., Borjkhani M. (2014). "Chaotic fractional-order model for muscular blood vessel and its control via fractional control scheme", Complexity, 20 (2), 37-46.

[4] Asl M.S., Javidi M. (2017). "An improved PC scheme for nonlinear fractional differential equations: Error and stability analysis", Journal of Computational and Applied Mathematics, 324, 101-117.

[5] Aghababa M.P., Haghighi A.R., Roohi M. (2015). "Stabilization of unknown fractional-order chaotic systems: an adaptive switching control strategy with application to power systems", IET Generation, Transmission & Distribution, 9, 1883-1893.

[6] Asakawa S., Kyoya I. (2013). "Hopfield neural network model for explaining double dissociation in semantic memory impairment", BMC Neuroscience, 14, 233.

[7] Banerjee S., Majumdar D. (2000). "Shape matching in multimodal medical images using point landmarks with Hopfield net", Neurocomputing, 30, 103-116.

[8] Chen Y., Tang C., Roohi M. (2021). "Design of a model-free adaptive sliding mode control to synchronize chaotic fractional-order systems with input saturation: An application in secure communications", Journal of the Franklin Institute, 8109-8137.

[9] Curran P.F., Chua L.O. (1997). "Absolute stability theory and the synchronization problem", International Journal of Bifurcation and Chaos, 7, 1375-1382.

[10] Danca M.F., Garrappa R., Tang K.S., Chen G. (2013). "Sustaining stable dynamics of a fractional-order chaotic financial system by parameter switching", Computers & Mathematics with Applications, 66, 702-716.

[11] Diks C., Wagener F.O.O. (2008). "A bifurcation theory for a class of discrete time Markovian stochastic systems", Physica D: Nonlinear Phenomena, 237, 3297-3306.

[12] Ding Y., Wang Z., Ye H. (2012). "Optimal control of a fractional-order HIV-immune system with memory", IEEE Transactions on Control Systems Technology, 20, 763-769.

[13] Du J., Sheng Z., Mei Q., Huang H. (2009). "Study on output dynamic competition model and its global bifurcation", International Journal of Nonlinear Sciences and Numerical Simulation, 10, 129-136.

[14] Effati S., Moghaddas M. (2016). "A novel neural network based on NCP function for solving constrained nonconvex optimization problems", Complexity, 21, 130-141.

[15] Esfahani Z., Roohi M., Gheisarnejad M., Dragicević T., Khooban M.H. (2019). "Optimal non-integer sliding mode control for frequency regulation in stand-alone modern power grids", Applied Sciences, 9, 3411.

[16] Fei J., Lu C. (2018). "Adaptive fractional-order sliding mode controller with neural estimator", Journal of the Franklin Institute, 355, 2369-2391.

[17] Fradkov A.L., Evans R. (2005). "Control of chaos: Methods and applications in engineering", Annual Reviews in Control, 29, 33-56.

[18] Gomaa Haroun A., Yin-Ya L. (2019). "A novel optimized fractional-order hybrid fuzzy intelligent PID controller for interconnected realistic power systems", Transactions of the Institute of Measurement and Control, 41, 3065-3080.

[19] Haghighi A., Aghababa M.P., Asghary N., Roohi M. (2020). "A nonlinear control scheme for stabilization of fractional-order dynamical chaotic systems", Journal of Advanced Mathematical Modeling, 10, 19-38.

[20] Hamza M.F., Yap H.J., Choudhury I.A. (2017). "Recent advances on the use of metaheuristic optimization algorithms to optimize the type-2 fuzzy logic systems in intelligent control", Neural Computing and Applications, 28, 979-999.

[21] He S., Sun K., Wang H. (2016). "Solution and dynamics analysis of a fractional-order hyperchaotic system", Mathematical Methods in the Applied Sciences, 39, 2965-2973.

[22] Hong C.H., Choi K.C., Kim B.S. (2009). "Applications of adaptive neural network control to an unmanned airship", International Journal of Control, Automation and Systems, 7, 911.

[23] Hopfield J.J. (1984). "Neurons with graded response have collective computational properties like those of two-state neurons", Proceedings of the national academy of sciences, 81, 3088-3092.

[24] Huang W.Z., Huang Y. (2008). "Chaos of a new class of Hopfield neural networks", 206, 1-11.

[25] Jafari P., Teshnehlab M., Tavakoli-Kakhki M. (2018). "Adaptive type-2 fuzzy system for synchronization and stabilization of chaotic non-linear fractional order systems", IET Control Theory & Applications, 12, 183-193.

[26] Javan D.S., Mashhadi H.R., Rouhani M. (2013). "A fast static security assessment method based on radial basis function neural networks using enhanced clustering", International Journal of Electrical Power and Energy Systems, 44, 988-996.

[27] Li J., Liu F., Guan Z., Li T. (2013). "A new chaotic Hopfield neural network and its synthesis via parameter switchings", Neurocomputing, 117, 33-39.

[28] Li C., Tong Y. (2013). "Adaptive control and synchronization of a fractional-order chaotic system", Pramana, 80, 583-592.

[29] Lin T.C., Kuo C.H. (2011). "H_∞ synchronization of uncertain fractional-order chaotic systems: Adaptive fuzzy approach", ISA Transactions, 50, 548-556.

[30] Liu H., Pan Y., Li S., Chen Y. (2018). "Synchronization for fractional-order neural networks with full/under-actuation using fractional-order sliding mode control", International Journal of Machine Learning and Cybernetics, 9, 1219-1232.

[31] Liu H., Wen Y., Luan F., Gao Y. (2009). "Application of experimental design and radial basis function neural network to the separation and determination of active components in traditional Chinese medicines by capillary electrophoresis", Analytica Chimica Acta, 88-93.

[32] Mahmoud E.E., Jahanzib L.S., Trikha P., Almaghrabi O.A. (2021). "Analysis and control of the fractional chaotic Hopfield neural network. Advances in Difference Equations", Advances in Difference Equations, 126.

[33] Mofid O., Mobayen S. (2018). "Adaptive synchronization of fractional-order quadratic chaotic flows with nonhyperbolic equilibrium", Journal of Vibration and Control, 24, 4971-4987.

[34] Ouannas A., Azar A.T., Vaidyanathan S. (2017). "A robust method for new fractional hybrid chaos synchronization", Mathematical Methods in the Applied Sciences, 40, 1804-1812.

[35] Peng X., Wu H. (2020). "Non-fragile robust finite-time stabilization and H_∞ performance analysis for fractional-order delayed neural networks with discontinuous activations under the asynchronous switching", Neural Computing and Applications, 32, 4045–4071.

[36] Petrás I. (2011). "An effective numerical method and its utilization to solution of fractional models used in bioengineering applications", Advances in Difference Equations, 652789.

[37] Podlubny I. (1998). “Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications”, Elsevier Science.

[38] Podlubny I. (2002). “Geometric and physical interpretation of fractional integration and fractional differentiation”, Fractional Calculus and Applied Analysis, 5, 367-386.

[39] Ramezani A., Safarinejadian B., Zarei J. (2019). “Fractional-order chaotic cryptography in colored noise environment by using fractional order interpolatory cubature Kalman filter”, Transactions of the Institute of Measurement and Control, 41, 3206-3222.

[40] Roohi M., Aghababa M.P., Haghighi A.R. (2015). “Switching adaptive controllers to control fractional-order complex systems with unknown structure and input nonlinearities”, Complexity, 21, 211-223.

[41] Roohi M., Hadian H., Aghababa M.P., Amiri S.M.M. (2017). “Design of an active control method for complete stabilization of unknown fractional-order non-autonomous systems”, International Workshop on Mathematics and Decision Science: Fuzzy Information and Engineering and Decision, 646, 131-142.

[42] Roohi M., Khooban M.H., Esfahani Z., Aghababa M.P., Dragicevic T. (2019). “A switching sliding mode control technique for chaos suppression of fractional-order complex systems”, Transactions of the Institute of Measurement and Control, 41, 2932-2946.

[43] Roohi M., Zhang C., Chen Y. (2020). “Adaptive model-free synchronization of different fractional-order neural networks with an application in cryptography”, Nonlinear Dynamics, 2020, 3979-4001.

[44] Rosenstein M.T., Collins J.J., De Luca C.J. (1993). “A practical method for calculating largest Lyapunov exponents from small data sets”, Physica D: Nonlinear Phenomena, 65, 117-134.

[45] Shrivastava Y., Singh B. (2019). “Stable cutting zone prediction in computer numerical control turning based on empirical mode decomposition and artificial neural network approach”, Transactions of the Institute of Measurement and Control, 41, 193-209.

[46] Sun H., Zhang Y., Baleanu D., Chen W., Chen Y.Q. (2018). “A new collection of real-world applications of fractional calculus in science and engineering”, Communications in Nonlinear Science and Numerical Simulation, 64, 213-231.

[47] Suykens J.A., Vandewalle J. (2005). “Cellular neural networks, multi-scroll chaos and synchronization”, World Scientific Series on Nonlinear Science Series A, 50, 248.

[48] Tank D., Hopfield J.J. (1986). “Simple neural optimization networks: An A/D converter, signal decision circuit, and a linear programming circuit”, IEEE Transactions on Circuits and Systems, 33, 533-541.

[49] Wang B., Liu Y. J., Yu F., Wang J., Yang J. (2017). “Robust finite-time control of fractional-order nonlinear systems via frequency distributed model”, Nonlinear Dynamics, 85, 2133-2142.

[50] Wang L., Li S., Tian F., Fu X. (2004). "A noisy chaotic neural network for solving combinatorial optimization problems: Stochastic chaotic simulated annealing", IEEE Transactions on Systems, Man, and Cybernetics, Part B, 34, 2119-2125.

[51] Wang, R, Liu, Y.J, Yu, F.S, Wang, J.Y, Yang, J.L. (2017). "Adaptive variable universe of discourse fuzzy control for a class of nonlinear systems with unknown dead zones", International Journal of Adaptive Control and Signal Processing, 31 (12), 1934-1951.

[52] Wang X.Y., Li Z.M. (2019). "A color image encryption algorithm based on Hopfield chaotic neural network. Optics and Lasers in Engineering", Optics and Lasers in Engineering, 115, 107-118.

[53] Wei J.D., Cheng H.J., Chang C.W. (2019). "Hopfield network-based approach to detect seam-carved images and identify tampered regions", Neural Computing and Applications, 31, 6479-6492.

[54] Xu G., Xu J., Liu F., Zang F. (2017). "Secure communication based on the synchronous control of hysteretic chaotic neuron", Neurocomputing, 227, 108-112.

[55] Yin C., Dadras S., Zhong S.M. (2012). "Design an adaptive sliding mode controller for drive-response synchronization of two different uncertain fractional-order chaotic systems with fully unknown parameters", Journal of the Franklin Institute, 349, 3078-3101.

[56] Yu J., Pan W.Z., Ng T.W. (2008). "Bifurcation and chaos in the generalized Korteweg-de Vries equation under a harmonic excitation", International Journal of Nonlinear Sciences and Numerical Simulation, 9, 37-40.

How to Cite this Article:

Roohi, M., Pourmahmood Aghababa, M., Ziaei, J., Zhang, C. (2022). "Chaotic dynamics in a fractional-order Hopfield neural network and its stabilization via an adaptive model-free control method", Control and Optimization in Applied Mathematics, 6(2): 1-21. doi: 10.30473/coam.2022.59941.1169