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Research Article

Solving a Class of Nonlinear Optimal Control Problems Using Haar Wavelets and Hybrid GA

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Abstract. In this paper, we solve a class of nonlinear optimal control problems using a hybrid genetic algorithm (HGA) and a direct method based on the Haar wavelets where the performance index is Bolza-form and the dynamic system is linear. First, we change the problem by using HWs to a static optimization problem in which the decision variables are the unknown coefficients of the state and control variables in the Haar series. Next, we apply HGA with a local search for higher power of GA in investigating the search space for solving optimization problems. Finally, we give some examples to illustrate the high accuracy of the proposed method.

Keywords. Optimal control problem, Haar wavelet, Hybrid genetic algorithm.

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1 Introduction

Nonlinear optimal control problems (NOCPs) are dynamic optimization problems for many applications in industrial processes such as aircraft, robotic arms, bioprocess systems, biomedical, power systems, and plasma physics, etc., [5].

The numerical strategies for solving NOCPs are separated into two groups, direct [22] or indirect [34]. For the indirect approach, the problem is transformed into a two-boundary value problem (TBVP) using Pontryagin's Maximum Principle (PMP), which can be solved by numerical strategies such as the shooting strategy [22]. These strategies require a large initial guess that lies within the convergence space. For direct strategies, control and/or state parameterizations are utilized to convert a continuous problem into a discrete problem. Utilizing an orthogonal set of functions as the base is a common approach to direct strategies. In these strategies, obscure capacities within the issue are approximated as a series of orthogonal capacities with obscure coefficients. Therefore, the dynamic conditions in NOCPs are changed over to arithmetical conditions. There are three classes of orthogonal functions: piecewise continuous functions such as Walsh [7], Block pulse [36], Haar wavelet [18] etc., orthogonal polynomials such as Legendre [11], Chebyshev [20], Lagrange [23], sine and cosine functions as Fourier series [29].

Metaheuristics as worldwide optimization are utilized for solving NOCPs. To discover the optimal solution, these algorithms, employing an arrangement of developmental administrators, overview the search space. They don't truly require great beginning guesses and deterministic rules. These algorithms broadly utilized for solving NOCPs as illustrations Genetic Algorithm (GA) [1, 32, 33], Genetic Programming (GP) [35], Particle Swarm Optimization (PSO) [25, 30, 31], Ant Colony Optimization (ACO) [2] and Differential Evolution (DE) [9, 24, 37].

Rastegar et al. [28], illuminated consolidation equation utilizing the matrix-based rationalized Haar wavelet change strategy. Two ordinary and most commonly utilized cases were fathomed and compared with the classical arrangement strategy which was based on Taylor arrangement extension. The proposed strategy was quick, helpful, and requires less computational effort due to its lattice calculation usage and it yields exactly what comes about when compared with the classical solution. Moreover, this strategy may be utilized for executing explicit functions instead of a steady value for the union coefficient c_v. Erfanian et al. [16], introduced a computational method for a class of Darboux problems that transform into two-dimensional nonlinear Volterra integral equations, based on expanding the solution as a set of Haar functions. Also, by applying the Banach fixed point theorem, they obtain an upper bound on the error of our method. The examples in this article are taken from a variety of references, so the numerical results obtained here can be compared with other numerical methods.

Erfanian and Mansoori [13], investigated mixed nonlinear integral-differential equations (MNIDE) using the rationalized Haar (RH) wavelet concept. The complexity of the MNIDE solution was known to everyone. For this purpose, they presented a numerical method by applying RH wavelets to approximate solutions of MNIDE of the second kind in the complex plane. First, they discussed the continuous integral operator. Erfanian et al. [14], approximated the solution of the nonlinear Fredholm integral equation of the second kind with a method based on the properties of RH wavelets and the use of matrix operators. The Banach fixed point theorem also guarantees the convergence of the method and also gets the error bound. In addition, Erfanian et al. [14] analyzed the order of convergence. Algorithms and some numerical examples for computing the solution are also given. Erfanian et al. [15], presented a method of numerically approximating the fixed-point operator, especially for the Volterra-Fredholm mixed integral-differential equations. The main tool for error analysis was the Banach fixed point theorem. The advantage of this method is that it does not use numerical integration. We used properties of the rationalized Haar wavelet to approximate the integral. Algorithm cost improves accuracy and greatly reduces computation. See other works in [17, 26, 27].

In this paper, a new method based on a combination of direct numerical methods and metaheuristic algorithms is proposed to solve a class of NOCPs. The direct methods are based on HW and meta-heuristics are based on HGA. HWs as a special class of wavelets is applied to various technical problems such as [3, 21, 38]. Furthermore, the HW is used to resolve NOCP. Hsiao and Wang [19] applied HWs to solve the optimal time-varying control system. Dai and Cochran [10] applied the wavelet collocation method to solve NOCPs. In this regard, firstly, we use HWs as the direct method, similar to [18], the NOCP is changed to an NLP. Then, we apply the hybrid genetic algorithm (HGA) to solve the sequential quadratic programming, [6] (SQP) as a local search is run to solve the NLP.

The structure of paper is organized as follows. In Section 2, the formulation of the problem is introduced. In Section 3, preliminaries are presented. At first, HWs are described and then the NOCP is converted to an NLP. Finally, the new problem is solved using HGA. In Section 4, we introduce the algorithm for solving NOCP. In Section 5, we implement our proposed algorithm on several test problems. We conclude in Section 6.

2 Problem Formulation

We consider a bounded continuous-time NOCP in which a control function, u, is exerted over the planning horizon $[t_0, t_f]$. A particular problem considered is finding the control input $u(\cdot) \in \mathbb{R}$ that minimizes the cost functional with Bolza form and linear state equation:

$$\min \quad J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t)dt \tag{1}$$

s.t.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in [t_0, t_f] \tag{2}$$

$$x(t_0) = x_0, \tag{3}$$

where, $x(\cdot) \in \mathbb{R}^n$ denotes the state variables for the system, g and ϕ are continuous scaler functions, $A(t)$ is an $n \times n$ matrix and $B(t)$ is an $n-$dimensional vector. The cost function (1) must be minimized subject to linear dynamic (2) with initial condition (3).

3 Preliminaries

In this section preliminaries, which are needed for the main algorithm, are presented. In Section 3.1, HWs are described. In Section 3.2, the problem is changed to NLP. In Section 3.3, The operators of the genetic algorithm (GA) are introduced. Finally, in Section 3.4 the HGA is introduced.

3.1 Haar wavelets [8]

The HWs are the simplest class of wavelets, which are square waves with the magnitude of ± 1 in some intervals and zeros elsewhere. The scale function for these wavelets is $h_0(t) = 1, 0 \leq t < 1$, which is applied to construct the mother wavelet as $h_1(t) = h_0(2t) - h_0(2t-1),\ 0 \leq t < 1$. The other HWs are made by the dilations and translations of $h_1(t)$ as follows.

$$h_n(t) = h_1(2^j t - k), \quad j \geq 0, \quad k = 0, \dots,\ 2^j - 1, \quad n = 2^j + k, \tag{4}$$

which construct a sequence of orthonormal functions. Each twice integrable function $y(t),\ 0 \leq t < 1$, can be expanded by the Haar series as follows.

$$y(t) = \sum_{i=0}^{\infty} \alpha_i h_i(t), \tag{5}$$

where $\alpha_i = \int_0^1 y(t)h_i(t)dt$ is Haar coefficient. For a piecewise constant approximation, (5) can be as a finite series as follow.

$$y(t) \simeq \sum_{i=0}^{m-1} \alpha_i h_i(t) = \alpha^T h_{(m)}(t), \tag{6}$$

where $h_{(m)}(t) = [h_0(t), h_1(t), \dots, h_{m-1}(t)]^T$ and $\alpha = [\alpha_0, \alpha_1, \dots, \alpha_{m-1}]^T$, m is called the level of wavelet and equals to 2^j, j is the level resolution.

3.2 Changing NOCP to NLP

Consider a continuous-time NOCP in Equations (1)-(3). Without loss the generality, let $[t_0, t_f] = [0, 1]$, otherwise by using a simple transformation $\tau = (t - t_0)/(t_f - t_0)$ the time interval, $[t_0, t_f]$, can be embedded in $[0, 1]$. For two separate cases, linear time-invariant, LTI, and linear time-variant, LTV, the problem can be converted to NLP, which is done by following.

3.2.1 Special LTI systems

Let an LTI dynamic system of order n as follows.

$$\dot{x}_1 = \gamma_1 x_2, \quad \dot{x}_2 = \gamma_2 x_3, \quad \dots \quad , \dot{x}_{n-1} = \gamma_{n-1} x_n, \quad , \dot{x}_n = \gamma_n x_n + \beta u \tag{7}$$

where $\gamma_i,\ i = 1, 2, \dots, n$ and $\beta \neq 0$ are constant real numbers. Similar to Equation (6), the unknown function $\dot{x}_n(t)$, in last equation in (7), can be approximated by a HW series as following

$$\dot{x}_n(t) = \alpha_0 h_0(t) + \alpha_1 h_1(t) + \dots + \alpha_{m-1} h_{m-1}(t) = \alpha^T h_{(m)}(t), \quad 0 \leq t < 1 \tag{8}$$

where $\alpha = [\alpha_0, \alpha_1, \dots, \alpha_{m-1}]^T$ is called Haar coefficients. Using Haar operational matrix of HW, P, [18], and integrate of Equation (8) we have

$$\begin{aligned} x_n(t) &= x_n(0) + \int_0^t \dot{x}_n(\tau) d\tau \\ &= x_n(0) + \int_0^t \alpha^T h_{(m)}(\tau) d\tau = x_n(0) + \alpha^T P h_{(m)}(t) \\ &= ([x_n(0), \underbrace{0, \dots, 0}_{m-1}] I_{m\times m} + \alpha^T P) h_{(m)}(t) = \beta_{\alpha,n}^T h_{(m)}(t), \end{aligned} \tag{9}$$

where, $\beta_{\alpha,n} = ([x_n(0), 0, \dots, 0] I_{m\times m} + \alpha^T P)^T$. By replacing (8) and (9) in last Equation (7), the control signal can be represent by HWs as

$$u(t) = \frac{1}{\beta}(\dot{x}_n(t) - \gamma_n x_n(t)) = \frac{1}{\beta}(\alpha^T - \gamma_n \beta_{\alpha,n}^T)h_{(m)}(t) = \eta_{\alpha,n}^T h_{(m)}(t), \tag{10}$$

where, $\eta_{\alpha,n} = \frac{1}{\beta}(\alpha - \gamma_n \beta_{\alpha,n})$. Now by replacing Equation (9) in Equation $\dot{x}_{n-1} = \gamma_{n-1}x_n$, from dynamic system in (8), and repeat the above procedure, we obtain

$$\begin{aligned} x_{n-1}(t) &= x_{n-1}(0) + \int_0^t \dot{x}_{n-1}(\tau)d\tau = x_{n-1}(0) + \int_0^t \gamma_{n-1}x_n(\tau)d\tau \\ &= x_{n-1}(0) + \gamma_{n-1}\int_0^t \beta_{\alpha,n}^T h_{(m)}(\tau)d\tau = ([x_{n-1}(0), 0, \dots, 0]I_{m\times m} \\ &\quad + \gamma_{n-1}\beta_{\alpha,n}^T P)h_{(m)}(t) = \beta_{\alpha,n-1}^T h_{(m)}(t), \end{aligned} \tag{11}$$

where, $\beta_{\alpha,n-1} = ([x_{n-1}(0), 0, \dots, 0]I_{m\times m} + \gamma_{n-1}\beta_{\alpha,n}^T P)^T$. By following this procedure, we obtain $x_1(t) = \beta_{\alpha,1}^T h_{(m)}(t)$, and so all state variables can be approximated by HWs with unknown coefficients, α, which can be denoted as

$$x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T = [\beta_{\alpha,1}, \beta_{\alpha,2}, \dots, \beta_{\alpha,n}]^T h_{(m)}(t) = \beta_\alpha^T h_{(m)}(t). \tag{12}$$

The final condition can be calculated. For $k \neq n$,

$$\begin{aligned} x_k(1) &= x_k(0) + \int_0^1 \dot{x}_k(\tau)d\tau = x_k(0) + \int_0^1 \gamma_k x_{k+1}(\tau)d\tau \\ &= x_k(0) + \gamma_k \int_0^1 \beta_{\alpha,k+1}^T h_m(\tau)d\tau = x_k(0) + \gamma_k \beta_{\alpha,k+1}^T \int_0^1 h_{(m)}(\tau)d\tau \\ &= x_k(0) + \gamma_k \beta_{\alpha,k+1}^T [1, 0, \dots, 0]^T = c_{\alpha,k}, \end{aligned} \tag{13}$$

and for $k = n$

$$\begin{aligned} x_n(1) &= x_n(0) + \int_0^1 \dot{x}_n(\tau)d\tau = x_n(0) + \int_0^1 \alpha^T h_m(\tau)d\tau \\ &= x_n(0) + \alpha^T [1, 0, \dots, 0]^T = c_{\alpha,n}. \end{aligned} \tag{14}$$

Now, from Equations (13) and (14)

$$x(1) = [x_1(1), x_2(1), \dots, x_n(1)]^T = [c_{\alpha,1}, c_{\alpha,2}, \dots, c_{\alpha,n}]^T = c_\alpha^T. \tag{15}$$

By replacing Equations (10), (12) and (15) in (1), the performance index can be given by

$$\begin{aligned} J(\alpha) &= \phi(x(1), 1) + \int_0^1 f(x(\tau), u(\tau))d\tau \\ &= \phi(c_\alpha^T, 1) + \int_0^1 f(\beta_\alpha^T h_{(m)}(\tau), \eta_{\alpha,n}^T h_{(m)}(\tau))d\tau. \end{aligned} \tag{16}$$

Hence, the NOCP in Equations (1)-(3) can be converted to (17) as an unconstrained NLP.

$$\begin{aligned} &\min \quad J(\alpha), \\ &\alpha \in \mathbb{R}. \end{aligned} \tag{17}$$

3.2.2 General LTI systems

For general LTI dynamic systems, Equation (2) is $\dot{x} = Ax(t) + Bu(u)$. The approximations (18) can be applied.

$$\dot{x} = Fh_{(m)}, \quad u = \alpha^T h_{(m)}, \tag{18}$$

where F is an $n \times m$ coefficient matrix and α is a $m \times 1$ vector. So, similar to Equation (9), we obtain

$$\begin{aligned} x(t) &= x(t_0) + \int_0^t \dot{x}(\tau)d\tau = x(t_0) + F\int_0^t h_{(m)}(\tau)d\tau \\ &= ([x(t_0), 0, \ldots, 0] + FP)h_{(m)}(t) = \bar{F}h_{(m)}(t), \end{aligned} \tag{19}$$

where $\bar{F} = \bar{X}_0 + FP$, $\bar{X}_0 = [x(t_0), 0, \ldots, 0]$. Replacing (18) and (19) in (2), we have

$$Fh_{(m)} = A\bar{F}h_{(m)} + B\alpha^T h_{(m)}, \tag{20}$$

Hence,

$$F - A\bar{F} - B\alpha^T = F - A(\bar{X}_0 + FP) - B\alpha^T = 0. \tag{21}$$

Therefore, $F - AFP = A\bar{X}_0 + B\alpha^T$. So,

$$F = (I - A \otimes P^T)G_\alpha, \tag{22}$$

where, $\otimes$ is Kronecker product of matrices and $G_\alpha = A\bar{X}_0 + B\alpha^T$. So, F is a function of α and it can be represented by F_α. Furthermore, for the final state, similar to (13), we have

$$x(1) = x(0) + \int_0^1 \dot{x}(\tau)d(\tau) = \bar{C}_{F_\alpha}. \tag{23}$$

By replacing Equations (18) and (22) in (1), the performance index is given by (24).

$$\begin{aligned} J(\alpha) &= \phi(x(1), 1) + \int_0^1 f(x(\tau), u(\tau))d\tau = \phi(\bar{C}_{F_\alpha}, 1) \\ &+ \int_0^1 f(\bar{F}_\alpha h_{(m)}(\tau), \alpha^T h_{(m)}(\tau))d\tau, \end{aligned} \tag{24}$$

Hence, the NOCP problem in Equations (1)-(3) can be converted to an NLP, similar to (17).

For the LTV dynamic systems, the above procedure can be applied (see [19] for more details). In the next Section, a new hybrid metaheuristic algorithm, called HGA, is proposed for solving the new problem in (17).

3.3 The GA operators

The GA introduced by Holland in 1975, is a class of heuristics and probabilistic methods. These algorithms start with an initial population of solutions. This population is evaluated by using genetic operators that include selection, crossover, and mutation. Here, in HGA, the underlying GA has the following steps.
Initialization. We consider a random $N_p \times m$ matrix as $A = (\alpha_1, \alpha_2, \ldots, \alpha_{N_p})^T$. This matrix is the coefficient matrix, in which each row of it is as the α in (8) or (18), as Haar coefficients vector, which is

$$\alpha_i = \alpha_{left} + (\alpha_{right} - \alpha_{left}).r_i, \quad i = 1, 2, \ldots, m, \tag{25}$$

where, r_i is a random number in $[0, 1]$ with a uniform distribution and $\alpha_{lelt}, \alpha_{right} \in \mathbb{R}$ are the lower and the upper bounds of coefficients of control input values, which can be given by the user.
Evaluation. Using $i-$th individual in population, $i-$th row in matrix A or α_i, the corresponding fitness can be calculate. By Replacing α_i with α in (16) or (24), the fitness of the $i-$th individual can be calculated.
Selection. To select two parents, we use a tournament operator with size 8 (see [12]).
Crossover. When two parents α_1 and α_2 are selected, we use the following stages to construct an offspring:

1. Select random numbers as below:

$$\lambda_1 \in [0, 1], \lambda_2 \in [-\lambda_{max}, 0], \lambda_3 \in [1, 1 + \lambda_{max}], \tag{26}$$

   where λ_{max} is a random number in $[0, 1]$.

2. Let

$$of^k = \lambda_k \alpha_1 + (1 - \lambda_k)\alpha_2, \quad k = 1, 2, 3, \tag{27}$$

   where $\lambda_k, k = 1, 2, 3$ is defined in (26). For $i = 1 \ldots m$, if $(of^k)_i > \alpha_{right}$, then let

$$(of^k)_i = \alpha_{right},$$

   and if $(of^k)_i < \alpha_{left}$, then let $(of^k)_i = \alpha_{left}$.

3. Let $of = of^*$, where of^* is the best $of^i, i = 1, 2, 3$ constructed by (27).

Mutation. We apply a perturbation on each component of the offspring. To do that, we use the relation (28).

$$(of)_i = (of)_i + r_i.\gamma, \quad i = 1, 2, \ldots, m, \tag{28}$$

where, r_i is selected randomly in $\{-1, 1\}$ and γ is a random number in $[0, 1]$. If $(of)_i > \alpha_{right}$, then let $(of)_i = \alpha_{right}$ and if $(of)_i < \alpha_{left}$, then let $(of)_i = \alpha_{left}$.
Replacement. Here, underling the GA uses a traditional replacement strategy. A replacement occurs if the new descendant has two characteristics as below.

1. Better than the worst people in the population.

2. Less like people in the population

Termination conditions. Underlying GA is terminated when at least one of the following conditions occurs.

1. The maximum number of generations, N_g, is reached.

2. Over a specified number of generations, N_i, we do not have any improvement (the best individual is not changed), or the two-norm of final state constraints, or equality constraints, will be reached to a prescribed number, ε.

3.4 Hybrid GA for solving SQP

In HGA, GA uses a local search method to improve solutions. Here, we use SQP as a local search. Using SQP as a local search in the hybrid meta-heuristic is common, see for example [25]. SQP is an iterative algorithm for solving constrained NLP, in which the problem is converted to sequential quadratic programming, QP, using a quadratic approximation of the Lagrangian function as an objective function and linearising the nonlinear constraints. The derivation of the function is approximated by the finite differences method. The Hessian matrix of the Lagrangian function, which is a positive definite, updated by any of the quasi-Newton methods, such as the BFGS method. The solution of QP, in each iterative, is applied to a search direction of the line search procedure. The step length parameter is determined by an appropriate line search procedure. Here, the maximum number of iterations allowed is a parameter, named $sqpmaxiter$. A complete overview of SQP is found in [6]. Using this approach, we may decrease the needed running time (the reason for using this approach is discussed in [4]).

4 The Proposed Algorithm

Here, we give the proposed algorithm, which is a direct approach, using HGA and HWs, for solving NOCPs. Using an initial population of random coefficients, for HWs series, Equations (8) or (18), a finite sequence of approximation solution signals is constructed. So, the continuous NOCP is changed to the finite-dimensional NLP. Next, the HGA is applied for updating these coefficients. The proposed algorithm is given in Algorithm 1.

Algorithm 1 The Structure of Proposed Algorithm

- **Initialization.** Input the accuracy number $m = 2^j$, the size of population N_p, the maximum number of generations without improvement N_i, the mutation implementation probability P_m, the initial value of the maximum number of iterations in SQP, $sqpmaxiter$, the lower and upper bounds of coefficients, i.e., α_{left} and α_{right}, and the random matrix $A = (\alpha_i)$ as the matrix of coefficients.
- **Evaluation.** Evaluate the fitness of each individual by (16) or (24).
- **Local Search.** Run SQP for each individual in the population if the maximum number of iterations is $sqpmaxiter$.
  While stopping conditions are not satisfied do
  - **Selection.** Using a tournament of eight players from the population, select two parents α_1 and α_2.
  - **Crossover.** Construct a new offspring, of, from α_1 and α_2 by using (26) and (27).
  - MutationApply (28) on of with probability P_m.
  - **Local Search.** Perform SQP on of when the maximum number of iteration is $sqpmaxiter$.
  - **Replacement.**
    If replacement conditions are satisfied (see Section 3.3) **then** replace of with the worst individual of the population. Let $sqpmaxiter := sqpmaxiter + 1$.
- **Return** the best individual in the final population as an approximate solution of NOCP.

5 Numerical Examples

In this section, to investigate the efficiency of the proposed algorithm, two numerical examples are considered. The numerical results in both cases are compared with the exact solutions, which allows validation of the proposed method by comparison with the result of exact solutions. The error of approximation for the control signal and the state signals are calculated by following

$$E_{x_i} = \sqrt{\int_0^1 (\tilde{x}_i(t) - x_i^*(t))^2 dt}, \quad E_u = \sqrt{\int_0^1 (\tilde{u}(t) - u^*(t))^2 dt}, \tag{29}$$

where, $x_i^*,\ i = 1, 2, \ldots, n$, and u^* are exact signals, and $\tilde{u}$ and $\tilde{x}_i,\ i = 1, 2, \ldots, n$ are the approximate signals, which is achieved by the proposed algorithm. The results are achieved with the level of wavelets $m = 2^j,\ j = 0, 1, 2, 3, 4$ and the parameters $P_m = 0.8$. Furthermore, we use $E_J = |J - J^*|$ as the absolute error of the performance index.

5.1 One dimensional problem

Consider the following linear-quadratic optimal control problem, LQOCP, with performance index as

$$J = \int_0^5 (x^2 + u^2) dt$$

$$s.t.$$

$$\dot{x} = x + u, \quad x(0) = 1.$$

The exact analytical solution to this problem, using necessary optimal conditions or PMP, is

$$u^*(t) = c_1 e^{5\sqrt{2}t} + c_2 e^{-5\sqrt{2}t},$$
$$x^*(t) = c_1(\sqrt{2} - 1)e^{5\sqrt{2}t} - c_2(\sqrt{2} + 1)c_2 e^{-5\sqrt{2}t},$$

where,

$$c_1 = \frac{\sqrt{2} + 1}{\sqrt{2} + 1 + (\sqrt{2} - 1)e^{10\sqrt{2}}}, \quad c_2 = 1 - c_1.$$

To solve the problem with the proposed algorithm, at first, the time interval is converted to $[0, 1]$ by variable change $\tau = t/5$. Then Algorithm 1 is applied with $N_p = 100,\ N_i = 60,\ u_{left} = -3$ and $u_{right} = 3$. The numerical results of the proposed algorithm are reported in Table 1. The graphical comparison between the exact and the approximate solution is shown in Figure 1, with $m = 16$.

Table 1: Numerical results of the proposed algorithm for Example 5.1, $m = 2,\ 4,\ 8,\ 16$

m	E_J	E_u	E_x
2	8.19×10^{-2}	1.1194	4.18×10^{-1}
4	1.17×10^{-2}	6.73×10^{-1}	2.91×10^{-1}
8	1.81×10^{-1}	5.46×10^{-1}	1.99×10^{-1}
16	9.86×10^{-2}	3.59×10^{-1}	8.89×10^{-2}

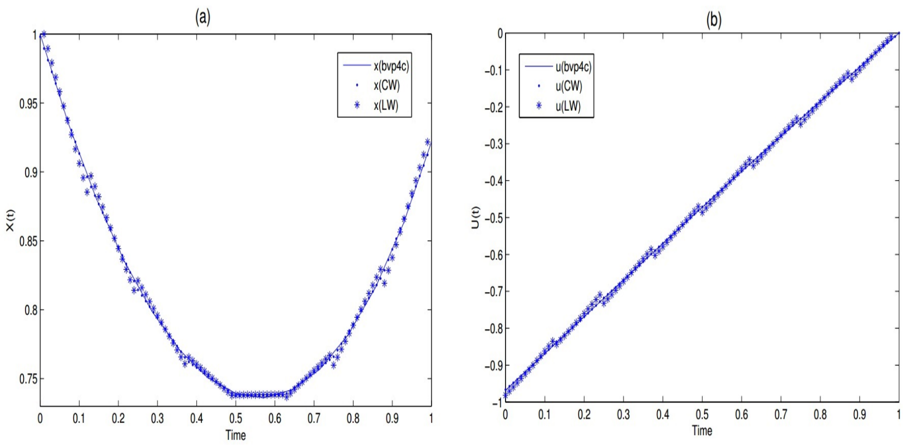

Figure 1: Comparison of exact and approximate solutions for the control (a) state (b) signals to Example 5.1, $m = 16$.

5.2 Two dimensional problem

Consider the following two-dimensional LQOCP:

$$\begin{aligned} &\min \quad J = \tfrac{1}{2}x_1^2(1) + \tfrac{1}{2}\int_0^1 u^2(t)dt,\\ &s.t.\\ &\dot{x}_1 = x_2,\\ &\dot{x}_2 = u,\\ &x(0) = [-1, 0]^T. \end{aligned}$$

The exact state signals of the problem are $x_1^*(t) = 3/8t^2 - 1/8t^3 - 1$ and $x_2^*(t) = 3/4t - 3/8t^2$ and the exact optimal control signal is $u^*(t) = 3/4(1-t)$. The numerical results of the proposed algorithm with $N_p = 100,\ N_i = 60,\ u_{left} = -5$ and $u_{right} = 5$ is summarized in Table 2. The graphical comparison of the exact and approximate solutions for states and control signals are shown in Figures 2 and 3, respectively.

Table 2: Numerical results of the proposed algorithm for Example 5.2, $m = 2, 4, 8, 16$

m	E_J	E_u	E_{x_1}	E_{x_2}
2	5.94×10^{-3}	1.08×10^{-1}	4.0×10^{-2}	6.21×10^{-2}
4	1.45×10^{-3}	5.39×10^{-2}	2.02×10^{-2}	3.09×10^{-2}
8	3.67×10^{-4}	2.69×10^{-2}	1.02×10^{-2}	1.85×10^{-2}
16	1.57×10^{-3}	1.76×10^{-2}	6.28×10^{-3}	9.74×10^{-3}

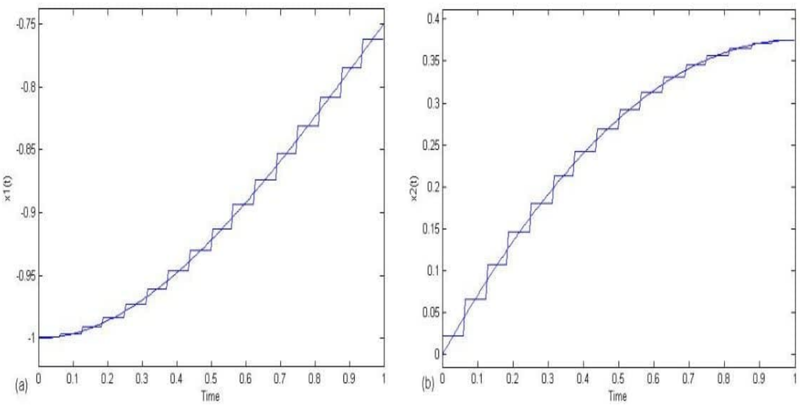

Figure 2: Comparison of exact and approximation solutions for the state signals, $x_1(t)$ (a) and $x_2(t)$ (b), to Example 5.2, $m = 16$.

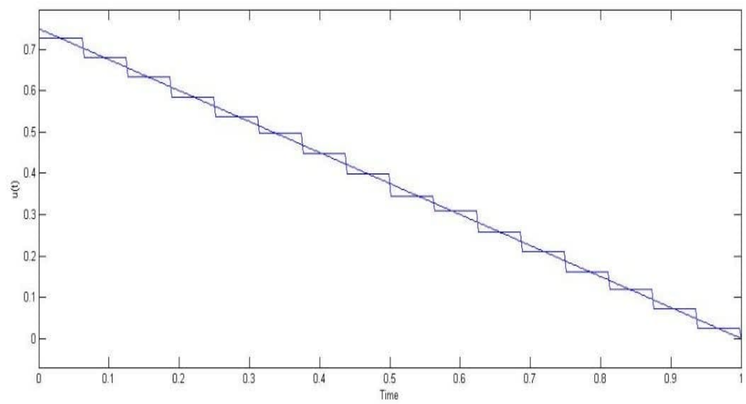

Figure 3: Comparison of exact and approximation solutions for the control signal to Example 5.2, $m = 16$.

6 Conclusion

In this paper, a direct method based on a hybrid genetic algorithm (HGA) and Haar wavelet (HW) was proposed for solving NOCPs. At first, using HWs, a sequence of control curves was constructed as an initial population for HGA, where each individual was a linear combination of them with unknown coefficients. HWs converted the NOCP to an algebraic problem. Next, HGA was applied to achieve the optimal coefficients. The method was applied for solving two test problems.

Declarations

Availability of supporting data

All data generated or analyzed during this study are included in this published paper.

Competing interests

The authors declare no competing interests are relevant to the content of this paper.

Authors' contributions

The main manuscript text is collectively written by all authors.

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